Contents

Chapter 0

Preliminaries

In Sections 0.1, 0.2, 0.3 and 0.4, we give a summary of definitions and notations that will be used in the rest of the book. In Section 0.5, we give some results on continuous functions that will be used in Chapter 2.

0.1 Set Concepts and Notations

In this section, we give a summary of definitions, notations and terminologies on sets without bothering about the axioms that guarantee the existence of some of the sets. Some concepts, for example, the concepts of "equipotent sets" and "union of a family of sets" involve the concept of "bijective function" and "family" respectively that will be discussed in Section 0.2.

The concepts of "sets" and "belongs to" are undefined. We will also use the term "object" (which means "set" in axiomatic set theory).

Notation Given an object x and a set A, either x belongs to A or x does not belong to A.

- If x belongs to A, then we say that "x is *an element of* A" and we write $x \in A$;
- If x does not belong to A, then we say "x is *not an element of* A" and we write $x \notin A$.

Notation and Terminology

- Let X and Y be sets. If X and Y have the same elements (that is, every element of A is also an element of B and vice versa), then we say that X and Y are *equal* and we write $X = Y$; otherwise, we say that X and Y are *not equal* and we write $X \neq Y$.
- The set that has no element is called the *empty set* and is denoted by $\emptyset$.
  - ◇ A set X is *empty* means that $X = \emptyset$.
  - ◇ A set X is *non-empty* means that $X \neq \emptyset$.
- A set that has exactly one element is called a *singleton.*

Set Relations Let A and B be sets.

- If every element of A is also an element of B, then we say that A is a *subset* of B and we write $A \subseteq B$; we also say that B is a *superset* of A and we write $B \supseteq A$.
- If $A \subseteq B$ and $A \neq B$, then we say that A is a *proper subset* of B and we write $A \subsetneqq B$.

Note that sets X and Y are equal if and only if $X \subseteq Y$ and $Y \subseteq X$.

Set Operations Let A and B be sets.

- The *intersection* of A and B, denoted by $A \cap B$, is the set whose elements are those belonging to both A and B, that is, an object x belongs to $A \cap B$ if and only if $x \in A$ and $x \in B$.
- The the union of A and B, denoted by $A \cup B$, is the set whose elements are those belonging to A or B, that is, an object x belongs to $A \cup B$ if and only if $x \in A$ or $x \in B$.
- The *relative complement* of B *in* A, denoted by $A \setminus B$, is the set whose elements are those belonging to A but not B, that is, an object x belongs to $A \setminus B$ if and only if $x \in A$ and $x \notin B$.

  If $B \subseteq A$, then $A \setminus B$ is simply called the *complement* of B *in* A.
- The *symmetric difference* of A and B, denoted by $A \triangle B$, is the set whose elements are those belonging to A or B but not both, that is, an object x belongs to $A \triangle B$ if and only if ($x \in A$ or $x \in B$) and $x \notin A \cap B$.

  Note that $A \triangle B = (A \setminus B) \cup (B \setminus A)$.

Specifying Sets Let A be a set. Suppose $P(x)$ is a statement with a free variable x such that for each $a \in A$, either $P(a)$ is true or $P(a)$ is false. Then we denote $\{x \in A : P(x)\}$ to be the set determined by the following:

(S) an object a belongs to $\{x \in A : P(x)\}$ if and only if $a \in A$ and $P(a)$ is true.

For example, we have $A \cap B = \{x \in A : x \in B\}$ and $A \setminus B = \{x \in A : x \notin B\}$.

The variable x is called a dummy variable. It can be replaced by any symbol. For example, we have $A \cap B = \{i \in A : i \in B\}$.

Some Special Sets

- The set of all real numbers is denoted by $\mathbb{R}$.
- The set of all rational numbers is denoted by $\mathbb{Q}$.
- The set of all integers is denoted by $\mathbb{Z}$.
- The set of all positive integers is denoted by $\mathbb{Z}^+$.
- The set of all natural numbers is denoted by $\mathbb{N}$, that is, $\mathbb{N} = \mathbb{Z}^+ \cup \{0\}$.

Equipotent Sets Two sets A and B are said to be *equipotent* if there exists a bijective mapping having domain A and codomain B.

Bernstein Theorem Let A and B be sets. Suppose that A is equipotent to a subset of B and B is equipotent to a subset of A. Then A and B are equipotent.

Countable and Uncountable Sets A set X is said to be

- *finite* if $X = \emptyset$ or there exists $n \in \mathbb{Z}^+$ such that X and the set $\{i \in \mathbb{Z}^+ : 1 \le i \le n\}$ are equipotent (in this case, we say that X has n elements);
- *infinite* if it is not finite;
- *countable* if it is equipotent to a subset of $\mathbb{Z}^+$;
- *countably infinite* if it is countable and infinite;
- *uncountable* if it is not countable.

Remark A countable set means a finite set or a countably infinite set.

Fact The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countably infinite and the set $\mathbb{R}$ is uncountable.

Families of Sets A *family of sets* is a family $\{A_i\}_{i\in\Lambda}$ where all the A_i's are sets.

- If $\Lambda = \{x \in \mathbb{Z} : x \ge n_0\}$ where n_0 is an integer, then $\{A_i\}_{i\in\Lambda}$ is also denoted by $(A_i)_{i=n_0}^{\infty}$ and is called a *sequence of sets*.
- If $\Lambda = \{x \in \mathbb{Z} : n_0 \le x \le n_1\}$ where n_0 and n_1 are integers with $n_0 \le n_1$, then $\{A_i\}_{i\in\Lambda}$ is also denoted by $(A_i)_{i=n_0}^{n_1}$ and is called a *finite sequence of sets*.

Moreover alternative notations for finite sequences and sequences can be found in Section 0.2.

Operations on Families of Sets Let $\{A_i\}_{i\in\Lambda}$ be a family of sets.

- The *intersection* of the sets in the family $\{A_i\}_{i\in\Lambda}$, denoted by $\bigcap\limits_{i\in\Lambda} A_i$, is the set whose elements are those belonging to all the A_i's, that is, an object x belongs to $\bigcap\limits_{i\in\Lambda} A_i$ if and only if for every $i \in \Lambda$, the object x belongs to A_i.
  - ◇ If $\Lambda = \{x \in \mathbb{Z} : x \ge n_0\}$ where n_0 is an integer, then $\bigcap\limits_{i\in\Lambda} A_i$ is also denoted by $\bigcap\limits_{i=n_0}^{\infty} A_i$ or $\bigcap\limits_{i\ge n_0} A_i$.
  - ◇ If $\Lambda = \{x \in \mathbb{Z} : n_0 \le x \le n_1\}$ where n_0 and n_1 are integers with $n_0 \le n_1$, then $\bigcap\limits_{i\in\Lambda} A_i$ is also denoted by $\bigcap\limits_{i=n_0}^{n_1} A_i$ or $\bigcap\limits_{n_0\le i\le n_1} A_i$.
- The *union* of the sets in the family $\{A_i\}_{i\in\Lambda}$, denoted by $\bigcup\limits_{i\in\Lambda} A_i$ is the set whose elements are those belonging to some A_i's, that is, an object x belongs to $\bigcup\limits_{i\in\Lambda} A_i$ if and only if there exists $i \in \Lambda$ such that the object x belongs to A_i.

◇ If $\Lambda = \{x \in \mathbb{Z} : x \geq n_0\}$ where n_0 is an integer, then $\bigcup_{i \in \Lambda} A_i$ is also denoted by $\bigcup_{i=n_0}^{\infty} A_i$ or $\bigcup_{i \geq n_0} A_i$.

◇ If $\Lambda = \{x \in \mathbb{Z} : n_0 \leq x \leq n_1\}$ where n_0 and n_1 are integers with $n_0 \leq n_1$, then $\bigcup_{i \in \Lambda} A_i$ is also denoted by $\bigcup_{i=n_0}^{n_1} A_i$ or $\bigcup_{n_0 \leq i \leq n_1} A_i$.

Fact The union of every countable family of countable sets is countable, that is, if $\{A_i\}_{i \in \Lambda}$ is a countable family of sets such that for every $i \in \Lambda$, the set A_i is countable, then the set $\bigcup_{i \in \Lambda} A_i$ is countable.

Operations on Sequences of Sets Let $(A_i)_{i=n_0}^{\infty}$ be a sequence of sets.

- We denote $\limsup_{n \to \infty} A_n = \bigcap_{n=n_0}^{\infty} \bigcup_{i=n}^{\infty} A_i$, that is, $\limsup_{n \to \infty} A_n = \bigcap_{n=n_0}^{\infty} B_n$ where $(B_n)_{n=n_0}^{\infty}$ is the sequence of sets given by $B_n = \bigcup_{i=n}^{\infty} A_i$ $(n \geq n_0)$.
- We denote $\liminf_{n \to \infty} A_n = \bigcup_{n=n_0}^{\infty} \bigcap_{i=n}^{\infty} A_i$, that is, $\liminf_{n \to \infty} A_n = \bigcup_{n=n_0}^{\infty} B_n$ where $(B_n)_{n=n_0}^{\infty}$ is the sequence of sets given by $B_n = \bigcap_{i=n}^{\infty} A_i$ $(n \geq n_0)$.

Set of sets Let X be a set.

- The *power set* of X, denoted by $\mathcal{P}(X)$, is the set whose elements are the subsets of X, that is, an object A belongs to $\mathcal{P}(X)$ if and only if $A \subseteq X$.
- A *partition* of a non-empty set X is a subset $\mathcal{S}$ of $\mathcal{P}(X)$ satisfying the following two conditions.

  (P1) For every $A \in \mathcal{S}$, we have $A \neq \emptyset$.

  (P2) For every $x \in X$, there exists one and only one $A \in \mathcal{S}$ such that $x \in A$.

  Note If $A, B \in \mathcal{S}$ and $A \neq B$, then by (P2), we have $A \cap B = \emptyset$.

Axiom of Choice Let X be a non-empty set and let $\mathcal{S}$ be a partition of X. Then there exists a subset Y of X such that for every $A \in \mathcal{S}$, the set $A \cap Y$ is a singleton.

Product of Sets Let A and B be sets. The *Cartesian product* of A and B, denoted by $A \times B$, is the set whose elements are the ordered pairs (a, b) where $a \in A$ and $b \in B$, that is, an object x belongs to $A \times B$ if and only if there exist $a \in A$ and $b \in B$ such that $x = (a, b)$.

Remark

- An ordered pair (a, b) where a and b are objects, is defined to be a set in a way such that $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

- An *ordered triple*, denoted by (a, b, c) where a, b and c are objects, is defined to be a set in a way such that $(a, b, c) = (d, e, f)$ if and only if $a = d$, $b = e$ and $c = f$.

Remark The notation (a, b) has another meaning when a and b are extended real numbers with $a < b$. Readers can determine the meaning from the context.

Quantifiers The symbols $\forall$ and $\exists$ are called the *universal quantifier* and *existential quantifier*.

- In words, the quantifier $\forall$ is written as "*for all*" or "*for every*".
- In words, the quantifier $\exists$ is written as "*there exist(s)*".

When we use a quantifier, we always specify the set of objects that is applied to. For example,

◇ $\forall\, x \in (0, \infty)$, $x + 1 > 0$;

◇ $\exists\, x \in (0, \infty)$, $x - 1 > 0$.

For simplicity,

◇ $x \in (0, \infty)$ is also written as (real number) $x > 0$;

◇ $x \in \{i \in \mathbb{Z} : i \geq n\}$, where $n \in \mathbb{Z}^+$, is also written as (integer) $x \geq n$;

◇ $x \in \{i \in \mathbb{Z} : n_0 \leq i \leq n_1\}$, where $n_0, n_1 \in \mathbb{Z}$ and $n_0 \leq n_1$, is also written as $x = n_0, \ldots, n_1$.

Although some notations may be ambiguous, readers can determine their meanings from the contexts.

0.2 Relations, Functions and Families

In this section, we give a summary of definitions, notations and results on relations, functions and families. We will also give the definition of "equivalence relation" and discuss related results that will be used in the construction of a non-measurable set in Chapter 1. Some concepts on families involve concepts that will be discussed in later sections. For example, the concepts of "families of extended real numbers" and "families of real-valued functions" involve the concepts of "extended real numbers" and "real-valued functions" that will be discussed in Section 0.3 and Section 0.4 respectively.

Definition A *relation* is an ordered triple (X, Y, G) where X and Y are non-empty sets and G is a subset of $X \times Y$.

- The set X and Y are called the *domain* and *codomain* respectively of the relation.
- The set G is called the *graph* of the relation.

Terminology

- A *relation between a set X and a set Y* means a relation having domain X and codomain Y.
- A *relation on a set X* means a relation having domain and codomain equal to X.

Remark In many books, the domain and codomain of a relation are not assumed to be non-empty. If $X = \emptyset$ or $Y = \emptyset$, then G must be the empty set. This case is not interesting. Under the convention that both X and Y are non-empty, some concepts that required special treatment, for example, intersection of an empty family of sets, can be avoided (note that a family is a function and a function is a relation).

Example 0.2.1 The ordered triple $(\mathbb{R}, \mathbb{R}, G)$ where

$$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y\}$$

is a relation on $\mathbb{R}$. It gives the "order relation" between two real numbers, whether one is less than or equal to the other.

Notation Let R be a relation, that is $R = (X, Y, G)$ where X and Y are non-empty sets and $G \subseteq X \times Y$. If $(a, b) \in G$, then we write ${}_aR_b$.

To specify a relation R between a set X and a set Y (or a relation on a set X), it suffices to specify what the subset G of $X \times Y$ is, that is, it suffices to tell which elements $(a, b) \in X \times Y$ satisfy ${}_aR_b$. For example, the relation R on $\mathbb{R}$ given by

$${}_aR_b \quad \text{if and only if} \quad a \leq b, \qquad (a, b) \in \mathbb{R} \times \mathbb{R}$$

is the relation given in Example 0.2.1.

Definition A relation R on a non-empty set X is said to be an *equivalence relation* if the following three conditions are satisfied:

(E1) for every $a \in X$, we have ${}_aR_a$;

(E2) if ${}_aR_b$, then ${}_bR_a$;

(E3) if ${}_aR_b$ and ${}_bR_c$, then ${}_aR_c$.

The relation on $\mathbb{R}$ given in Example 0.2.1 is not an equivalence relation. Although (E1) and (E3) are satisfied, condition (E2) is not satisfied.

Example 0.2.2 Denote R to be the relation on $\mathbb{R}$ given by

$${}_aR_b \quad \text{if and only if} \quad a - b \in \mathbb{Q}, \qquad (a, b) \in \mathbb{R} \times \mathbb{R}.$$

Then R is an equivalence relation.

Proof

- For every $a \in \mathbb{R}$, we have $a - a = 0 \in \mathbb{Q}$ and so ${}_aR_a$.
- Suppose ${}_aR_b$, that is, $a - b \in \mathbb{Q}$. Then we have $b - a \in \mathbb{Q}$, that is, ${}_bR_a$.
- Suppose ${}_aR_b$ and ${}_bR_c$, that is, $a - b \in \mathbb{Q}$ and $b - c \in \mathbb{Q}$. Then we have $a - c = (a - b) + (b - c) \in \mathbb{Q}$ and so ${}_aR_c$.

$\square$

Notation Let R be an equivalence relation on a set X. For every $a \in X$, we denote a/R to be the subset of X given by

$$a/R = \{x \in X : {}_xR_a\}.$$

Note that by (E1), we have $a \in a/R$ and so $a/R \neq \emptyset$.

Theorem 0.2.1 *Let R be an equivalence relation on a set X. Then for every $(a, b) \in X \times X$, we have*

$$\textit{either} \quad a/R = b/R \quad \textit{or} \quad a/R \cap b/R = \emptyset.$$

Remark There does not exist $(a, b) \in X \times X$ such that $a/R = b/R$ and $a/R \cap b/R = \emptyset$. This is because for every $a \in X$, the set a/R is non-empty.

Proof Let $(a, b) \in X \times X$. Suppose that $a/R \cap b/R \neq \emptyset$, that is, there exists $x_0 \in a/R \cap b/R$. We want to show that

$$a/R = b/R,$$

or equivalently that

$$a/R \subseteq b/R \quad \text{and} \quad a/R \supseteq b/R.$$

($\subseteq$) Let $x \in a/R$. We want to show that $x \in b/R$.
By definition, we have

$${}_xR_a. \tag{1}$$

Since $x_0 \in a/R \cap b/R$, it follows that

$${}_{x_0}R_a \tag{2}$$

$${}_{x_0}R_b \tag{3}$$

In view of (2), by (E2), we have

$${}_aR_{x_0}. \tag{4}$$

In view of (1) and (4), by (E3), we have

$${}_xR_{x_0}. \tag{5}$$

In view of (5) and (3), by (E3), we have

$${}_xR_b,$$

that is, $x \in b/R$.

($\supseteq$) Interchanging the role of a and b in the proof of the ($\subseteq$) part, we get $b/R \subseteq a/R$.

□

Definition Let R be an equivalence relation on a set X. A (non-empty) subset S of X is said to be an *R-equivalence class* if there exists $a \in X$ such that $a/R = S$.

Example 0.2.3 Let R be the relation on $\mathbb{Z}$ given by

$$_mR_n \quad \text{if and only if} \quad m - n \text{ is divisible by } 2, \qquad (m, n) \in \mathbb{Z} \times \mathbb{Z}.$$

It is straightforward to check that R is an equivalence relation. Note that $0/R$ is the set of all even integers and $1/R$ is the set of all odd integers. Hence there are only two R-equivalence classes, namely, $0/R$ and $1/R$.

Note that $\{0/R, 1/R\}$ is a partition of $\mathbb{Z}$. In fact, we have the following general result.

Theorem 0.2.2 *Let R be an equivalence relation on a set X. Denote by $\mathcal{S}$ to be the subset of $\mathcal{P}(X)$ given by*

$$\mathcal{S} = \{S \in \mathcal{P}(X) : S \textit{ is an } R\textit{-equivalence class}\}.$$

Then $\mathcal{S}$ is a partition of X.

Proof We want to show that (P1) and (P2) given in the definition of partition are satisfied.

(P1) For every $S \in \mathcal{S}$, by construction, there exists $a \in X$ such that $a/R = S$. Since $a \in a/R$, it follows that $S \neq \emptyset$.

(P2) Let $a \in X$. We want to show that there exists one and only one $S \in \mathcal{S}$ such that $x \in S$.

(*Existence*) Since a/R is an R-equivalence class and $a \in a/R$, it follows that there exists $S \in \mathcal{S}$ such that $a \in S$.

(*Uniqueness*) Suppose $a \in S_1$ and $a \in S_2$, where $S_1, S_2 \in \mathcal{S}$. Then we have $S_1 \cap S_2 \neq \emptyset$. Hence by Theorem 1, we have $S_1 = S_2$. □

Definition A *function* is a relation such that the graph G of the relation satisfies the following condition:

(F) For every x in the domain of the relation, there exists a unique y in the codomain of the relation such that $(x, y) \in G$.

Example 0.2.4 The ordered triple $(\mathbb{R}, \mathbb{R}, G)$ where

$$G = \{(x, y) \in \mathbb{R} \times \mathbb{R} : y = x^2\}$$

is a function having domain $\mathbb{R}$ and codomain $\mathbb{R}$.

Terminology

- A *mapping* means a function.

- A *function from* a set X *into* a set Y means a function having domain X and codomain Y.
- $f : X \longrightarrow Y$ is a function means f is a function from X into Y.
- A function *defined on* a set X means a function having domain X.

Notation and Terminology Let f be a function, that is, $f = (X, Y, G)$ where X and Y are non-empty sets and G is a subset of $X \times Y$ satisfying (F).

- For every $x \in X$, the unique $y \in Y$ such that $(x, y) \in G$ is denoted by $f(x)$ and is called the *value of f at x*.
- The domain of f is denoted by $\mathrm{dom}\,(f)$, that is, $\mathrm{dom}\,(f) = X$.
- The *range* of f, denoted by $\mathrm{range}\,(f)$, is the subset of Y given by

$$\mathrm{range}\,(f) = \{y \in Y : \text{there exists } x \in X \text{ such that } f(x) = y\}.$$

- For every $A \subseteq \mathrm{dom}\,(f)$, we denote $f(A)$ to be the subset of Y given by

$$f(A) = \{y \in Y : \text{there exists } x \in A \text{ such that } f(x) = y\}.$$

  The set $f(A)$ is called the *image of A under f*.

  Note that the range of f is the image of the domain of f under f.

  For convenience, the set $f(A)$ is also written as $\{f(x) : x \in A\}$.
- For every $B \subseteq Y$, we denote $f^{-1}(B)$ to be the subset of $\mathrm{dom}\,(f)$ given by

$$f^{-1}(B) = \{x \in \mathrm{dom}\,(f) : f(x) \in B\}.$$

  The set $f^{-1}(B)$ is called the *inverse image of B under f*.

To specify a function from a set X into a set Y, it suffices to tell what the value $f(x)$ is for x belonging to X. For example, the function f from $\mathbb{R}$ into $\mathbb{R}$ given by

$$f(x) = x^2 \qquad \text{for } x \in X \tag{6}$$

is the function given in Example 0.2.4.

Instead of writing (6), sometimes we just write

$$f(x) = x^2 \qquad (x \in X),$$

or simply

$$f(x) = x^2$$

since it is understood that x belongs to the domain X. The symbol x, which represents an element of X, is called a dummy variable. It can be replaced by any symbol. For example, we can also write $f(i) = i^2$.

Compositions of Functions Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be functions. We denote $g \circ f$ to be the function from X into Z given by

$$(g \circ f)(x) = g\big(f(x)\big).$$

Restrictions and Extensions of Functions

- Let $f : X \longrightarrow Y$ be a function. Let A be a non-empty subset of X. The *restriction of f on A*, denoted by $f\big|_A$ is the function from A into Y given by

$$f\big|_A(x) = f(x) \quad \text{for } x \in A.$$

- Let $f : X \longrightarrow Y$ be a function. Suppose X_1 is a set with $X \subseteq X_1$ and $g : X_1 \longrightarrow Y$ is a function such that

$$\text{for every } x \in X, \qquad g(x) = f(x).$$

  Then we say that g is an *extension of f to X_1*.

Injective, Surjective and Bijective Functions Let f be a function from a set X into a set Y.

- Suppose that for every $b \in Y$, there exists at most one $a \in X$ such that $f(a) = b$. Then we say that the function f is *injective*.
- Suppose that for every $b \in Y$, there exists at least one $a \in X$ such that $f(a) = b$. Then we say that the function f is *surjective*.
- Suppose that for every $b \in Y$, there exists one and only one $a \in X$ such that $f(a) = b$. Then we say that the function f is *bijective*.

Note that a function is bijective means that it is both injective and surjective.

Definition A *family* is a surjective function.

Example 0.2.5 The ordered triple $(\mathbb{Z}, Y, G)$ where

$$Y = \{n \in \mathbb{Z} : \text{there exists } i \in \mathbb{Z} \text{ such that } i^2 = n\} \quad \text{and} \quad G = \{(x, y) \in \mathbb{Z} \times Y : y = x^2\}$$

is a surjective function, that is, a family.

Terminology

- The domain of a family is called the *index set* of the family.
- Let φ be a family having index set Λ. For every $i \in \Lambda$, the value $\varphi(i)$ of the family φ at i is called a *term* of the family.

- A *family in* Y or *a family of elements of* Y, where Y is a non-empty set, means a family whose range is a subset of Y.

  For example, a family of real numbers means a family whose range is a subset of $\mathbb{R}$, a family of extended real numbers means a family whose range is a subset of $\mathbb{R}^*$.
  Note that every family of real numbers is also a family of extended real numbers.

- A *family of subsets of* Y, where Y is a set, means a family whose range is a subset of $\mathcal{P}(Y)$.

  For example, a family of subsets of $\mathbb{R}$ means a family whose range is a subset of $\mathcal{P}(\mathbb{R})$.

- A *family of sets* means a family of subsets of Y for some set Y.

- A *family of real-valued functions having the same domain* X means a family whose range is a subset of the set of all real-valued functions defined on X.

Notation To specify a family φ having index set Λ, it suffices to tell what the value $\varphi(i)$ is for $i \in \Lambda$. Usually, when we consider a family with index set Λ, we don't give a symbol for the family; instead, we write

$$\{x_i\}_{i\in\Lambda},$$

where x_i (or any letter with a subscript i) denotes the value of the family at i (for $i \in \Lambda$). For example, the family given in Example 0.2.5 is also written as

$$\{i^2\}_{i\in\mathbb{Z}}.$$

Terminology

- A family $\{x_i\}_{i\in\Lambda}$ is said to be *countable* if the index set Λ is countable. Similar terminologies apply to *finite family*, *countably infinite family* and *uncountable family.*
- A family $\{x_i\}_{i\in\Lambda}$ is said to be *distinct* if $x_i \neq x_j$ whenever $i \neq j$ $(i, j \in \Lambda)$.
- A family of sets $\{A_i\}_{i\in\Lambda}$ is said to be *disjoint* if $A_i \cap A_j = \emptyset$ whenever $i \neq j$ $(i, j \in \Lambda)$.
  - ◇ Sets X and Y are said to be *disjoint* if the family $\{A_i\}_{i\in\{1,2\}}$, where $A_1 = X$ and $A_2 = Y$, is a disjoint family.
  - ◇ A set A is said to be the *disjoint union* of sets X and Y if X and Y are disjoint and $A = X \cup Y$.

Sequences

- A family $\{x_i\}_{i\in\Lambda}$ having index set $\Lambda = \{n \in \mathbb{Z} : n \geq n_0\}$, where n_0 is an integer, is called a *sequence* and is also denoted by $(x_i)_{i=n_0}^{\infty}$ or $(x_i)_{i\geq n_0}$ or $\{x_i\}_{i=n_0}^{\infty}$ or $\{x_i\}_{i\geq n_0}$.

- A family $\{x_i\}_{i\in\Lambda}$ having index set $\Lambda = \{n \in \mathbb{Z} : n_0 \leq n \leq n_1\}$, where n_0 and n_1 are integers with $n_0 \leq n_1$, is called a *finite sequence* and is also denoted by $(x_i)_{i=n_0}^{n_1}$ or $(x_i)_{n_0\leq i\leq n_1}$ or $\{x_i\}_{i=n_0}^{n_1}$ or $\{x_i\}_{n_0\leq i\leq n_1}$.

If $n_0 = 1$ and $n_1 = 2$, that is, $\Lambda = \{1, 2\}$, then $(x_i)_{i=1}^{2}$ is called a *pair.*

Terminology A pair a, b of elements of a set X means a pair $(x_i)_{i=1}^{2}$ (a family) in X such that $x_1 = a$ and $x_2 = b$.

Terminology A sequence $(S_n)_{n=n_0}^{\infty}$ of sets is said to be

- *increasing* if for every $n \geq n_0$, we have $S_n \subseteq S_{n+1}$;
- *decreasing* if for every $n \geq n_0$, we have $S_n \supseteq S_{n+1}$.

Subfamilies Let $\{x_i\}_{i\in\Lambda}$ be a family and let Γ be a non-empty subset of Λ. Then the family $\{x_i\}_{i\in\Gamma}$ is called a *subfamily* of $\{x_i\}_{i\in\Lambda}$

For example, if $\{A_i\}_{i=1}^{\infty}$ is a sequence of sets, then for every $n \in \mathbb{Z}^+$, the set $\bigcup\limits_{i=n}^{\infty} A_i$ is the union of the sets in the family $\{A_i\}_{i=n}^{\infty}$ which is a subfamily of the family $\{A_i\}_{i=1}^{\infty}$

Subsequences Let φ be a sequence with index set $\{n \in \mathbb{Z} : n \geq n_0\}$, where $n_0 \in \mathbb{Z}$. Let   be a strictly increasing sequence of elements in $\{n \in \mathbb{Z} : n \geq n_0\}$. Then the composition $\varphi \circ$ is called a *subsequence* of the sequence φ. If we write $\varphi = (x_n)_{n=n_0}^{\infty}$ and write $= (n_k)_{k=k_0}^{\infty}$, then the subsequence $\varphi \circ$ is written as $(x_{n_k})_{k=k_0}^{\infty}$.

Sequences of Families Let $\left(\{x_{n,i}\}_{i\in\Lambda_n}\right)_{n=1}^{\infty}$ be a sequence of families such that there exists a set X satisfying condition (i) and there exists a set Y satisfying condition (ii) given below:

(i) for every $n \in \mathbb{Z}^+$, the family $\{x_{n,i}\}_{i\in\Lambda_n}$ is a family of elements of X;

(ii) for every $n \in \mathbb{Z}^+$, the set Λ_n is a subset of Y,

where $x_{n,i}$ means $x_n(i)$. Denote $\Gamma = \{(n, i) \in \mathbb{Z}^+ \times Y : i \in \Lambda_n\}$. Note that the set Γ is independent of the choice of Y satisfying (ii). The family $\{x_{n,i}\}_{(n,i)\in\Gamma}$ is a family of elements of X and we denote this family by $\{x_{n,i}\}_{i\in\Lambda_n, n\in\mathbb{Z}^+}$.

- Suppose that for every $n \in \mathbb{Z}^+$, the family $\{x_{n,i}\}_{i\in\Lambda_n}$ is a family of sets. Then we denote $\bigcup\limits_{n=1}^{\infty} \bigcup\limits_{i\in\Lambda} x_{n,i}$ to be the union of the family $\{x_{n,i}\}_{(n,i)\in\Gamma}$ of sets, that is, $\bigcup\limits_{n=1}^{\infty} \bigcup\limits_{i\in\Lambda} x_{n,i} = \bigcup\limits_{(n,i)\in\Gamma} x_{n,i}$.
- Suppose that for every $n \in \mathbb{Z}^+$, the family $\{x_{n,i}\}_{i\in\Lambda_n}$ is a countable family of non-negative extended real numbers. Then $\{x_{n,i}\}_{(n,i)\in\Gamma}$ is a countable family of non-negative extended real numbers and we denote $\sum\limits_{n=1}^{\infty} \sum\limits_{i\in\Lambda} x_{n,i}$ to be the sum of the terms of the family, that is, $\sum\limits_{n=1}^{\infty} \sum\limits_{i\in\Lambda} x_{n,i} = \sum\limits_{(n,i)\in\Gamma} x_{n,i}$.

0.3 Number Systems

The following result is a consequence of the fact that any non-empty subset of $\mathbb{Z}^+$ has a smallest element.

Principle of Mathematical Induction Let $P(n)$ be a statement with a free variable n. Suppose the following two conditions hold:

(1) $P(1)$ is true.

(2) $P(k)$ is true, where $k \in \mathbb{Z}^+$, implies that $P(k+1)$ is true.

Then for every $n \in \mathbb{Z}^+$, the statement $P(n)$ is true.

Remark Other versions of the Principle of Mathematical Induction can be deduced from the above version.

We can define sequences by induction. The following example is an illustration.

Define by Induction Let $(A_n)_{n=1}^{\infty}$ be a sequence of sets. Then there exists a disjoint sequence of sets $(B_n)_{n=1}^{\infty}$ such that for every $n \in \mathbb{Z}^+$, we have $\bigcup\limits_{k=1}^{n} A_k = \bigcup\limits_{k=1}^{n} B_k$. Such a sequence is defined inductively as follows:

$$\begin{aligned} B_1 &= A_1, \\ B_n &= A_n \setminus \bigcup_{k=1}^{n-1} B_k \quad (n \geq 2). \end{aligned}$$

Real Number System Readers are assumed to be familiar the arithmetic operations $+$, $-$, $\times$ and $\div$ on $\mathbb{R}$ as well as the absolute value $|\cdot|$ operation on $\mathbb{R}$.

Sums of Finite Families of Real Numbers We use induction to define the sum $\sum\limits_{i\in\Lambda} x_i$ of a finite family $\{x_i\}_{i\in\Lambda}$ of real numbers:

(1) If Λ is a singleton, we define $\sum\limits_{i\in\Lambda} x_i$ to be the unique real number belonging to the range of the family $\{x_i\}_{i\in\Lambda}$.

(2) If Λ has n elements, where $n \geq 2$, we define $\sum\limits_{i\in\Lambda} x_i$ to be $\sum\limits_{i\in\Lambda\setminus\{i_0\}} x_i + x_{i_0}$ where $i_0 \in \Lambda$.

Note that the definition is independent of the choice of $i_0 \in \Lambda$.

Notation If $\Lambda = \{n \in \mathbb{Z} : n_0 \leq n \leq n_1\}$, where $n_0, n_1 \in \mathbb{Z}$ and $n_0 \leq n_1$, then $\sum\limits_{i\in\Lambda}$ is also written as $\sum\limits_{i=n_0}^{n_1} x_i$ or $\sum\limits_{n_0\leq i\leq n_1} x_i$.

Ordering In $\mathbb{R}$, we have the order relations following order relations:

$$<, \quad \leq, \quad > \quad \text{and} \quad \geq .$$

Terminology A real number r is said to be

- *positive* if $r > 0$;
- *non-negative* if $r \geq 0$.

Triangle Inequality Let $\{x_i\}_{i=1}^n$ be a finite family of real numbers. Then we have $\Big|\sum_{i=1}^{n} x_i\Big| \leq \sum_{i=1}^{n} |x_i|$.

Fact The set $\mathbb{Q}$ is order dense in $\mathbb{R}$ in the sense that for every pair a, b of real numbers with $a < b$, there exists $q \in \mathbb{Q}$ such that $a < q < b$.

Interval A non-empty subset I of $\mathbb{R}$ is called an *interval* if the following condition is satisfied:

(I) for every pair a, b of elements in I with $a < b$ and for every real number r with $a < r < b$, we have $r \in I$.

Terminology An interval is said to be

- *degenerate* if it is a singleton;
- *non-degenerate* if it is not degenerate.

Boundedness A subset S of $\mathbb{R}$ is said to be

- *bounded above* if there exists a real number b such that for every $s \in S$, we have $s \leq b$; otherwise, it is said to be *not bounded above*;
- *bounded below* if there exists a real number a such that for every $s \in S$, we have $s \geq a$; otherwise, it is said to be *not bounded below*;
- *bounded* if it is bounded above and bounded below; otherwise, it is said to be *unbounded.*

Open Subsets and Closed Subsets of $\mathbb{R}$

- A subset G of $\mathbb{R}$ is said to be *open* if for every $a \in G$, there exists a positive real number r such that $\{x \in \mathbb{R} : a - r < x < a + r\} \subseteq G$.
- A subset S of $\mathbb{R}$ is said to be a G_δ-set if there exists a sequence $(G_n)_{n=1}^\infty$ of open subsets of $\mathbb{R}$ such that $\bigcap_{n=1}^{\infty} G_n = S$.
- A subset F of $\mathbb{R}$ is said to be *closed* if $\mathbb{R} \setminus F$ is open.
- A subset S of $\mathbb{R}$ is said to be an F_δ-set if there exists a sequence $(F_n)_{n=1}^\infty$ of closed subsets of $\mathbb{R}$ such that $\bigcup_{n=1}^{\infty} F_n = S$.

Fact The union of every family of open sets is open and the union of every finite family of closed sets is closed.

Types of Intervals

(*Bounded*) There are four types of bounded intervals:

(i) $\{x \in \mathbb{R} : a < x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$;

(ii) $\{x \in \mathbb{R} : a \leq x \leq b\}$, where $a, b \in \mathbb{R}$ and $a \leq b$;

(iii) $\{x \in \mathbb{R} : a < x \leq b\}$, where $a, b \in \mathbb{R}$ and $a < b$;

(iv) $\{x \in \mathbb{R} : a \leq x < b\}$, where $a, b \in \mathbb{R}$ and $a < b$.

(*Bounded above but not bounded below*) There are two types of intervals that are bounded above but not bounded below:

(v) $\{x \in \mathbb{R} : x > a\}$, where $a \in \mathbb{R}$;

(vi) $\{x \in \mathbb{R} : x \geq a\}$, where $a \in \mathbb{R}$.

(*Bounded below but not bounded above*) There are two types of intervals that are bounded below but not bounded above:

(vii) $\{x \in \mathbb{R} : x < a\}$, where $a \in \mathbb{R}$;

(viii) $\{x \in \mathbb{R} : x \leq a\}$, where $a \in \mathbb{R}$.

(*Not bounded above and not bounded below*) There is only one such interval:

(ix) $\mathbb{R}$

- Intervals in the form given in (ii) with $a = b$ are degenerate intervals; the rest are non-degenerate intervals.
- Intervals in the form given in (i), (v), (vii) and (ix) are open intervals and intervals in the form given in (ii), (vi), (viii) and (ix) are closed intervals.

Component Intervals Let G be a non-empty open subset of $\mathbb{R}$. An open interval I is called a *component interval* of G if the following two conditions are satisfied:

(i) $I \subseteq G$.

(ii) If J is an open interval and $I \subseteq J \subseteq G$, then $J = I$.

Fact Let G be a non-empty open subset of $\mathbb{R}$. Denote Λ to be the set of all component intervals of G, that is,

$$\Lambda = \{I \in \mathcal{P}(\mathbb{R}) : I \text{ is a component interval of } G\}.$$

Then the set Λ is non-empty and countable. Moreover, the set Λ is a partition of G, that is, the following three conditions are satisfied:

(i) For every $I \in \Lambda$, we have $I \neq \emptyset$.

(ii) If $I, J \in \Lambda$ and $I \neq J$, then $I \cap J = \emptyset$.

(iii) $G = \bigcup_{I \in \Lambda} I$.

The family $\{I\}_{I\in\Lambda}$ is called the *canonical family of component intervals of* G.

Remark Every $I \in \Lambda$ can be written in the form (a, b) where $a, b \in \mathbb{R}^*$. Therefore, the canonical family of component intervals of G can be written in the form $\{(a_i, b_i)\}_{i\in\Lambda}$.

Extended Real Numbers We denote $\mathbb{R}^*$ to be the set given by

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\},$$

where $-\infty$ and ∞ and are not real numbers and $-\infty \neq \infty$.

Terminology Elements in $\mathbb{R}^*$ are called *extended real numbers.*

The extended real number ∞ is also written as $+\infty$. An extended real number x is said to be finite if $x \in \mathbb{R}$.

Arithmetic in $\mathbb{R}^*$ The operations $+$ and $\cdot$ on $\mathbb{R}^*$ are the commutative operations, in the sense described below in (C1) and (C2), that are extensions of the operations $+$ and $\cdot$ in $\mathbb{R}$ given by the following:

- If $a \in \mathbb{R}$, then $a + \infty = \infty$ and $a + (-\infty) = -\infty$.
- If $a \in \mathbb{R}$ and $a > 0$, then $a \cdot \infty = \infty$ and $a \cdot (-\infty) = -\infty$.

  If $a \in \mathbb{R}$ and $a < 0$, then $a \cdot \infty = -\infty$ and $a \cdot (-\infty) = \infty$.
- We also have

$$\begin{aligned} \infty + \infty &= \infty \\ (-\infty) + (-\infty) &= -\infty \end{aligned} \qquad \begin{aligned} \infty \cdot \infty &= \infty \\ (-\infty) \cdot (-\infty) &= \infty \\ \infty \cdot (-\infty) &= -\infty \end{aligned}$$

- Each of the following is undefined:

$$\infty + (-\infty), \qquad 0 \cdot \infty, \qquad 0 \cdot (-\infty).$$

(C1) if $a + b$ is defined, then $b + a$ is also defined and $a + b = b + a$;

(C2) if $a \cdot b$ is defined, then $b \cdot a$ is also defined and $a \cdot b = b \cdot a$.

Notation For $a, b \in \mathbb{R}^*$,

◇ if $a \cdot b$ is defined, then $a \cdot b$ is also written as $a \times b$ or ab as usual;

◇ we denote $-a = (-1) \cdot a$;

◇ we denote $a - b$ to be the extended real number $a + (-b)$.

Translation and Dilation of Subsets of $\mathbb{R}^*$ Let S be a non-empty subset of $\mathbb{R}^*$.

- For every $a \in \mathbb{R}$, we denote $S + a = \{s + a : s \in S\}$.

  Remark The set $\{s + a : s \in S\}$ is the image of S under f where $f : \mathbb{R} \longrightarrow \mathbb{R}$ is the function given by $f(x) = x + a$. Similar meaning applies to the set $\{as : s \in S\}$ given below.

- For every $a \in \mathbb{R}$ with $a \neq 0$, we denote $aS = \{as : s \in S\}$.

  We denote $-S = (-1)S$.

Ordering in $\mathbb{R}^*$ The order relation $<$ on $\mathbb{R}^*$ is the extension of the relation $<$ on $\mathbb{R}$ given by the following:

- For every $a \in \mathbb{R}$, we have $-\infty < a$ and $a < \infty$.
- We also have $-\infty < \infty$.

In $\mathbb{R}^*$, the meanings of $\leq$, $>$ and $\geq$ are the same as that in $\mathbb{R}$.

Terminology An extended real number a is said to be *non-negative* if $a \geq 0$, that is, it is a non-negative real number or it is ∞.

Notations Let a and b be extended real numbers with $a < b$. We denote

- $(a, b) = \{x \in \mathbb{R}^* : a < x < b\}$;
- $[a, b] = \{x \in \mathbb{R}^* : a \leq x \leq b\}$;
- $(a, b] = \{x \in \mathbb{R}^* : a < x \leq b\}$;
- $[a, b) = \{x \in \mathbb{R}^* : a \leq x < b\}$;
- $[a, a] = \{x \in \mathbb{R}^* : a \leq x \leq a\}$, that is, $[a, a] = \{a\}$.

Remark Using the above notations, the types of interval listed on Page 17 can be written as follows:

(i) (a, b) where $a, b \in \mathbb{R}$ and $a < b$;

(ii) $[a, b]$ where $a, b \in \mathbb{R}$ and $a \leq b$;

(iii) $(a, b]$ where $a, b \in \mathbb{R}$ and $a < b$;

(iv) $[a, b)$ where $a, b \in \mathbb{R}$ and $a < b$;

(v) (a, ∞) where $a \in \mathbb{R}$;

(vi) $[a, \infty)$ where $a \in \mathbb{R}$;

(vii) $(-\infty, a)$ where $a \in \mathbb{R}$;

(viii) $(-\infty, a]$ where $a \in \mathbb{R}$;

(ix) $(-\infty, \infty)$

Sequences of Extended Real Numbers A sequence $(x_n)_{n=n_0}^{\infty}$ of extended real numbers (respectively a finite sequence $(x_n)_{n=n_0}^{n_1}$ of extended real numbers) is said to be

- *increasing* if for every integer with $n \geq n_0$ (respectively for every integer n with $n_0 \leq n < n_1$), we have $x_n \leq x_{n+1}$;
- *decreasing* if the sequence $(-x_n)_{n=n_0}^{\infty}$ of extended real numbers (respectively the finite sequence $(-x_n)_{n=n_0}^{n_1}$ of extended real numbers) is increasing;
- *strictly increasing* if for every integer with $n \geq n_0$ (respectively for every integer n with $n_0 \leq n < n_1$), we have $x_n < x_{n+1}$;
- *strictly decreasing* if the sequence $(-x_n)_{n=n_0}^{\infty}$ of extended real numbers (respectively the finite sequence $(-x_n)_{n=n_0}^{n_1}$ of extended real numbers) is strictly increasing.

Suprema and Infima of Sets of Extended Real Numbers Let S be a non-empty subset of $\mathbb{R}^*$.

- We denote $\sup S$ to be the extended real number satisfying the following two conditions:
  - ◇ for every $s \in S$, we have $\sup S \geq s$;
  - ◇ if a is an upper bound for S, then $a \geq \sup S$.

  An extended real number a is called an *upper bound for S* if for every $s \in S$, we have $a \geq s$.

  If the extended real number $\sup S$ belongs to S, then it is called the *maximum of S* and is denoted by $\max S$.
- We denote $\inf S$ to be the extended real number $-\sup(-S)$.

  If the extended real number $\inf S$ belongs to S, then it is called the *minimum of S* and is denoted by $\min S$.

Note that if S is a non-empty subset of $\mathbb{R}$ that is bounded above (respectively bounded below), then we have $\sup S \in \mathbb{R}$ (respectively $\inf S \in \mathbb{R}$).

Integral Parts of Real Numbers Let x be a real number. We denote $[x]$ to be the integer given by

$$[x] = \max\{n \in \mathbb{Z} : n \leq x\}.$$

Endpoints of Intervals Let I be an interval (which is a subset of $\mathbb{R}$). The extended real numbers $\inf I$ and $\sup I$ are called the *left endpoint* and *right endpoint* of I respectively.

For example,

- ◇ if $I = [a, b)$, where a and b are real numbers with $a < b$, then the left endpoint and right endpoint of I are a and b respectively;
- ◇ if $I = (a, \infty)$, where a is a real number, then the left endpoint and right endpoint of I are a and ∞ respectively;
- ◇ if $I = \{a\}$, where $a \in \mathbb{R}$, then a is the left endpoint as well as the right endpoint of I.

Suprema and Infima of Families of Extended Real Numbers Let $\{x_i\}_{i\in\Lambda}$ be a family of extended real numbers.

- We denote $\sup\limits_{i\in\Lambda} x_i$ to be the supremum of the range of the family.
- We denote $\inf\limits_{i\in\Lambda} x_i$ to be the infimum of the range of the family.

Note that the range of the family $\{x_i\}_{i\in\Lambda}$ is also denoted by $\{x_i : i \in \Lambda\}$.

Notation

- If $\Lambda = \{n \in \mathbb{Z} : n_0 \le n\}$ where n_0 is an integer, then $\sup\limits_{i\in\Lambda} x_i$ and $\inf\limits_{i\in\Lambda} x_i$ are also written as $\sup\limits_{i\ge n_0} x_i$ and $\inf\limits_{i\ge n_0} x_i$ respectively.
- If $\Lambda = \{n \in \mathbb{Z} : n_0 \le n \le n_1\}$ where n_0 and n_1 are integers with $n_0 \le n_1$, then both $\sup\limits_{i\in\Lambda} x_i$ and $\inf\limits_{i\in\Lambda} x_i$ belong to the range of the family $\{x_i\}_{i\in\Lambda}$ and they are denoted by $\max\limits_{n_0\le i\le n_1} x_i$ and $\min\limits_{n_0\le i\le n_1} x_i$ respectively.

Sums of Infinite Families of Non-negative Extended Real Numbers Let $\{x_i\}_{i\in\Lambda}$ be a family of non-negative extended real numbers. To define $\sum\limits_{i\in\Lambda} x_i$, we consider the following cases:

(*Case 1*) $\exists\, i \in \Lambda$ such that $x_i = \infty$

In this case, we define $\sum\limits_{i\in\Lambda} x_i$ to be ∞.

(*Case 2*) $\forall\, i \in \Lambda$, $-\infty < x_i < \infty$

In this case, we define $\sum\limits_{i\in\Lambda} x_i$ to be the extended real number given by

$$\sum_{i\in\Lambda} x_i = \sup\Big\{\sum_{i\in\Gamma} x_i : \{x_i\}_{i\in\Gamma} \text{ is a finite subfamily of } \{x_i\}_{i\in\Lambda}\Big\}.$$

Notation If $\Lambda = \{n \in \mathbb{Z}^+ : n \ge n_0\}$ where $n_0 \in \mathbb{Z}$, then $\sum\limits_{i\in\Lambda} x_i$ is also written as $\sum\limits_{i=n_0}^{\infty} x_i$.

Limsup and Liminf of Sequences of Extended Real Numbers Let $(x_n)_{n=n_0}^{\infty}$ be a sequence of extended real numbers.

- We denote $\limsup\limits_{n\to\infty} x_n$ to be the extended real number $\inf\limits_{n\ge 1}\Big(\sup\limits_{k\ge n} x_k\Big)$, that is, $\limsup\limits_{n\to\infty} x_n = \inf\limits_{n\ge n_0} y_n$ where $(y_n)_{n=n_0}^{\infty}$ is the sequence of extended real numbers given by $y_n = \sup\limits_{k\ge n} x_k$.
- We denote $\liminf\limits_{n\to\infty} x_n$ to be the extended real number $-\limsup\limits_{n\to\infty}(-x_n)$.
- Suppose that $\limsup\limits_{n\to\infty} x_n = \liminf\limits_{n\to\infty} x_n$. Then we denote the extended real number $\limsup\limits_{n\to\infty} x_n$ by $\lim\limits_{n\to\infty} x_n$ and we say that the *limit of the sequence* $(x_n)_{n=n_0}^{\infty}$ *exists in* $\mathbb{R}^*$, or simply, $\lim\limits_{n\to\infty} x_n$ *exists in* $\mathbb{R}^*$.

Remark To be accurate, the concept of limit should be discussed in the context of metric space. We can define a metric on $\mathbb{R}^*$ so that the limit of a sequence $(x_n)_{n=n_0}^{\infty}$ exists and equals α if and only if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \alpha$.

Note

◇ If $(x_n)_{n=n_0}^{\infty}$ is an increasing sequence of extended real numbers, then $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}^*$.

◇ If $(x_n)_{n=n_0}^{\infty}$ is a sequence of non-negative extended real numbers, then $\sum_{i=n_0}^{\infty} x_i = \lim_{n\to\infty} \sum_{i=n_0}^{n} x_i$.

Limits of Sequences of Real Numbers Let $(x_n)_{n=n_0}^{\infty}$ be a sequence of real numbers. If the limit of the sequence $(x_n)_{n=n_0}^{\infty}$ exists in $\mathbb{R}^*$ and $\lim_{n\to\infty} x_n \in \mathbb{R}$, then we say that

◇ *the sequence $(x_n)_{n=n_0}^{\infty}$ is convergent*, or

◇ *the sequence $(x_n)_{n=n_0}^{\infty}$ converges to $\lim_{n\to\infty} x_n$*, or

◇ *the limit of the sequence $(x_n)_{n=n_0}^{\infty}$ exists in* $\mathbb{R}$, or

◇ *the limit of the sequence $(x_n)_{n=n_0}^{\infty}$ exists*;

otherwise, we say that

◇ *the sequence $(x_n)_{n=n_0}^{\infty}$ is divergent*, or

◇ *the sequence $(x_n)_{n=n_0}^{\infty}$ diverges*, or

◇ *the limit of the sequence $(x_n)_{n=n_0}^{\infty}$ does not exist in* $\mathbb{R}$.

Note

- The notation $\lim_{n\to\infty} x_n = a$, where $a \in \mathbb{R}$, means that for every $\epsilon \in (0, \infty)$, there exists an integer $N \geq n_0$ such that for every $n \in \mathbb{Z}$ with $n \geq N$, we have $|x_n - a| < \epsilon$.

  Remark For simplicity, "*for every $\epsilon \in (0, \infty)$*" is also written as "*for every $\epsilon > 0$*" and "*for every $n \in \mathbb{Z}$ with $n \geq N$*" is also written as "*for every $n \geq N$*".

- The notation $\lim_{n\to\infty} x_n = \infty$ means that for every $r \in \mathbb{R}$, there exists (an integer) $N \geq n_0$ such that for every $n \geq N$, we have $x_n > r$.

  The notation $\lim_{n\to\infty} x_n = -\infty$ means that for every $r \in \mathbb{R}$, there exists (an integer) $N \geq n_0$ such that for every $n \geq N$, we have $x_n < r$.

0.4 Real-Valued Functions

Terminology A function having codomain $\mathbb{R}$ is called a *real-valued function.*

- A real-valued function f is said to be *non-negative* if range $(f) \subseteq [0, \infty)$.
- A real-valued function f is said to be *identically* 0 if range $(f) = \{0\}$; otherwise, it is said to be *not identically* 0.

Let f and g be real-valued functions having the same domain X. Then we define $f+g$, $f-g$, $f\cdot g$ and f/g pointwisely (for f/g, provided it is defined). For example, $f+g$ is the function from X into $\mathbb{R}$ given by

$$(f+g)(x) = f(x) + g(x) \qquad \text{for } x \in X.$$

Similarly, the notations $|f|$ (where f is a real-valued function) and $\sum\limits_{i=n_0}^{n_1} f_i$ (where $\{f_i\}_{i=n_0}^{n_1}$ is a finite family of real-valued functions having the same domain) are also defined pointwisely.

Bounded Functions A real-valued function f is said to be

- *bounded* if its range is a bounded subset of $\mathbb{R}$;
- *unbounded* if it is not bounded.

Monotone Functions A real-valued function f with $\operatorname{dom}(f) \subseteq \mathbb{R}$ is said to be

- *(monotonic) increasing* if for every $(x, y) \in \operatorname{dom}(f) \times \operatorname{dom}(f)$ with $x < y$, we have $f(x) \leq f(y)$;
- *(monotonic) decreasing* if the function $-f$ is increasing;
- *monotone* if it is increasing or decreasing.

Sequences of Real-valued Functions A sequence of real-valued functions $(f_n)_{n=n_0}^{\infty}$ having the same domain X is said to be

- *increasing* if for every $x \in X$ and for every $n \geq n_0$, we have $f_n(x) \leq f_{n+1}(x)$;
- *decreasing* if the sequence of functions $(-f_n)_{n=n_0}^{\infty}$ is increasing.

Suprema and Infima of Families of Real-Valued Functions Let $\{f_i\}_{i\in\Lambda}$ be a family of real-valued functions having the same domain X.

- Suppose that for every $x \in X$, the set $\{f_i(x) : i \in \Lambda\}$ is bounded above. Then we denote $\sup\limits_{i\in\Lambda} f_i$ to be the function from X into $\mathbb{R}$ given by

$$\Big(\sup_{i\in\Lambda} f_i\Big)(x) = \sup_{i\in\Lambda}\Big(f_i(x)\Big).$$

- Suppose that for every $x \in X$, the set $\{f_i(x) : i \in \Lambda\}$ is bounded below. Then we denote $\inf\limits_{i\in\Lambda} f_i$ to be the function from X into $\mathbb{R}$ given by

$$\inf_{i\in\Lambda} f_i = -\Big(\sup_{i\in\Lambda}(-f_i)\Big)$$

- If the index set Λ is finite, then the functions $\sup\limits_{i\in\Lambda} f_i$ and $\inf\limits_{i\in\Lambda} f_i$ are also denoted by $\max\limits_{i\in\Lambda} f_i$ and $\min\limits_{i\in\Lambda} f_i$ respectively.

  Let f and g be real-valued functions having the same domain X. We denote $\max\{f, g\} = \max\limits_{i\in\{1,2\}}\{h_i\}$ and $\min\{f, g\} = \min\limits_{i\in\{1,2\}}\{h_i\}$ where $\{h_i\}_{i\in\{1,2\}}$ is the family of functions given by $h_1 = f$ and $h_2 = g$.

Limsup and Liminf of Sequences of Real-valued Functions Let $(f_n)_{n=n_0}^{\infty}$ be a sequence of real-valued functions having the same domain X.

- Suppose that for every $x \in X$, the set $\{f_n(x) : n \geq n_0\}$ is bounded above and $\lim_{n\to\infty} f_n(x) \neq -\infty$. We denote $\limsup_{n\to\infty} f_n$ to be the function from X into $\mathbb{R}$ given by

$$\limsup_{n\to\infty} f_n = \inf_{n\geq n_0} \Big(\sup_{k\geq n} f_k\Big),$$

  that is, $\limsup_{n\to\infty} f_n = \inf_{n\geq n_0} g_n$ where $(g_n)_{n=n_0}^{\infty}$ is the sequence of functions from X into $\mathbb{R}$ given by $g_n = \sup_{k\geq n} f_k$ for $n \geq n_0$.

- Suppose that for every $x \in X$, the set $\{f_n(x) : n \geq n_0\}$ is bounded below and $\lim_{n\to\infty} f_n(x) \neq \infty$. We denote $\liminf_{n\to\infty} f_n$ to be the function from X into $\mathbb{R}$ given by

$$\liminf_{n\to\infty} f_n = -\Big(\limsup_{n\to\infty}(-f_n)\Big).$$

Remark The notation $\lim_{n\to\infty} f_n(x) \neq \infty$ (respectively $-\infty$) means that $\lim_{n\to\infty} f_n(x)$ does not exist in $\mathbb{R}^*$ or $\lim_{n\to\infty} f_n(x)$ exists in $\mathbb{R}^*$ but it is not equal to ∞ (respectively $-\infty$).

Limits of Sequences of Real-valued Functions Let $(f_n)_{n=n_0}^{\infty}$ be a sequence of real-valued functions having the same domain X. Suppose that

$(*)$ for every $x \in X$, the sequence of real numbers $\big(f_n(x)\big)_{n=n_0}^{\infty}$ is convergent.

Then we define $\lim_{n\to\infty} f_n$ to be the function from X into $\mathbb{R}$ given by

$$\Big(\lim_{n\to\infty} f_n\Big)(x) = \lim_{n\to\infty} f_n(x), \quad (x \in X).$$

We say that the function $\lim_{n\to\infty} f_n$ is the *pointwise limit of* $(f_n)_{n=1}^{\infty}$. We also say that $(f_n)_{n=1}^{\infty}$ *converges pointwise to* $\lim_{n\to\infty} f_n$.

Terminology A sequence $(f_n)_{n=n_0}^{\infty}$ of real-valued functions having the same domain X is said to *converges pointwise to* g, where g is a function from X into $\mathbb{R}$, means that Condition $(*)$ holds and the functions $\lim_{n\to\infty} f_n$ and g are equal.

Uniform Convergence of Sequences of Real-Valued Functions Let $(f_n)_{n=n_0}^{\infty}$ be a sequence of real-valued functions having the same domain X and let g be a real-valued function having domain X. We say that

- $(f_n)_{n=n_0}^{\infty}$ *converges uniformly to* g if for every $\epsilon > 0$, there exists $N \geq n_0$ such that for every $x \in X$ and for every $n \geq N$, we have $|f_n(x) - g(x)| < \epsilon$;
- $(f_n)_{n=n_0}^{\infty}$ *converges uniformly in* A *to* g, where A is a non-empty subset of X, if $\big(f_n\big|_A\big)_{n=n_0}^{\infty}$ converges uniformly to $g\big|_A$.

Series of Real-Valued Functions Let $(f_n)_{n=n_0}^{\infty}$ be a sequence of real-valued functions having the same domain X such that the sequence of functions $\left(\sum\limits_{i=n_0}^{n} f_i\right)_{n=n_0}^{\infty}$ converges pointwise to a function g. Then the function g is denoted by $\sum\limits_{i=n_0}^{\infty} f_i$

0.5 Continuous Functions

In this section, we give the definition of continuous functions and state some standard results on continuous functions. We also state and prove a few more results that will be used in Chapter 2. Although, the results are valid in the context of metric spaces, we only give the versions for real-valued functions defined on (non-empty) subsets of $\mathbb{R}$.

Definition Let f be a real-valued function with $\operatorname{dom}(f) \subseteq \mathbb{R}$.

- Suppose a is an element of $\operatorname{dom}(f)$ and the following condition is satisfied:

  (†) For every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x \in \operatorname{dom}(f) \cap (a - \delta, a + \delta)$, we have $|f(x) - f(a)| < \epsilon$.

  Then we say that the function f is *continuous at* a.

- If for every $a \in \operatorname{dom}(f)$, the function f is continuous at a, then we say that the function f is *continuous.*

Definition Let f be a real-valued function with $\operatorname{dom}(f) \subseteq \mathbb{R}$. Let a be a real number such that there exists an open interval I with $a \in I \subseteq \operatorname{dom}(f)$. We say that

- f is *left continuous at* a if the following condition is satisfied:

  ($*$) For every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x \in \operatorname{dom}(f) \cap (a - \delta, a)$, we have $|f(x) - f(a)| < \epsilon$.

- f is *right continuous at* a if the following condition is satisfied:

  ($**$) For every $\epsilon > 0$, there exists $\delta > 0$ such that for every $x \in \operatorname{dom}(f) \cap (a, a + \delta)$, we have $|f(x) - f(a)| < \epsilon$.

Results

(1) Let f be a real-valued function with $\operatorname{dom}(f) \subseteq \mathbb{R}$ and let a be a real number such that there exists an open interval I with $a \in I \subseteq \operatorname{dom}(f)$. Then f is continuous at a if and only if f is left continuous and right continuous at a.

(2) Let f be a continuous function and let A be a non-empty subset of $\operatorname{dom}(f)$. Then the restriction $f\big|_A$ of f on A is continuous.

(3) Let $(f_n)_{n=n_0}^{\infty}$ be a sequence of continuous functions having the same domain X and let g be a real-valued function having domain X. Suppose $(f_n)_{n=n_0}^{\infty}$ converges uniformly to g. Then the function g is continuous.

Proposition 0.5.1 *Let f be a continuous function. Then for every open subset V of $\mathbb{R}$, there exists an open subset G of $\mathbb{R}$ such that*

$$f^{-1}(V) = G \cap \operatorname{dom}(f).$$

Proof It suffices to consider the case where $V \neq \emptyset$. This is because $f^{-1}(\emptyset) = \emptyset = \emptyset \cap \operatorname{dom}(f)$ and $\emptyset$ is an open subset of $\mathbb{R}$.

Let V be a non-empty open subset of $\mathbb{R}$. Since $\{f(a)\}_{a \in f^{-1}(V)}$ is a family of elements of the open set V, it follows that there exists a family $\{\epsilon_a\}_{a \in f^{-1}(V)}$ of positive real numbers such that

$$\text{for every } a \in f^{-1}(V), \quad \big(f(a) - \epsilon_a, f(a) + \epsilon_a\big) \subseteq V. \tag{7}$$

Since f is continuous, it follows that there exists a family $\{\delta_a\}_{a \in f^{-1}(V)}$ of positive real numbers such that

$$\text{for every } a \in f^{-1}(V), \quad \text{for every } x \in \operatorname{dom}(f) \cap (a - \delta_a, a + \delta_a), \quad |f(x) - f(a)| < \epsilon_a. \tag{8}$$

In view of (7) and (8), we see that

$$\text{for every } a \in f^{-1}(V), \quad \operatorname{dom}(f) \cap (a - \delta_a, a + \delta_a) \subseteq f^{-1}(V). \tag{9}$$

Denote $G = \bigcup_{a \in f^{-1}(V)} (a - \delta_a, a + \delta_a)$. Then G is an open subset of $\mathbb{R}$. Moreover, by the construction of G and (9), we see that $\operatorname{dom}(f) \cap G = f^{-1}(V)$. □

Proposition 0.5.2 *Let f be a real-valued function with* $\operatorname{dom}(f) \subseteq \mathbb{R}$. *Suppose there exists a disjoint finite family $\{X_i\}_{i=1}^n$ of non-empty closed subsets of $\mathbb{R}$ with $\bigcup_{i=1}^n X_i = \operatorname{dom}(f)$ such that for every $i = 1, \ldots, n$, the restriction $f\big|_{X_i}$ of f on X_i is continuous. Then the function f is continuous.*

Proof Let $a \in \operatorname{dom}(f)$. We want to show that f is continuous at a.

Let $\epsilon > 0$. By the assumption on the family $\{X_i\}_{i=1}^n$, there exists a unique $i \in \{1, \ldots, n\}$ such that $x \in X_i$. Since $f\big|_{X_i}$ is continuous at x, there exists $\delta_1 > 0$ such that

$$\text{for every } x \in X_i \cap (a - \delta_1, a + \delta_1), \quad |f(x) - f(a)| < \epsilon. \tag{10}$$

Denote $F_i = \bigcup_{k \in \Lambda_i} X_k$ where $\Lambda_i = \{1, \ldots, n\} \setminus \{i\}$. Since F_i is a closed subset of $\mathbb{R}$ and $x \notin F_i$, it follows that there exists $\delta_2 > 0$ such that

$$\text{for every } x \in F_i, \quad |x - a| > \delta_2. \tag{11}$$

Denote $\delta = \min\{\delta_1, \delta_2\}$. Then we have $\delta > 0$. Moreover, by (10) and (11), we have

$$\text{for every } x \in \operatorname{dom}(f) \cap (a - \delta, a + \delta), \quad |f(x) - f(a)| < \epsilon.$$

This is because if $x \in \operatorname{dom}(f)$ and $|x - a| < \delta$, then $x \in X_i$. □

The above result does not hold if the family of closed subsets is infinite. However, if the subsets (not necessarily closed) are "separated nicely", then the result holds. For this, we introduce the following

Notation Let A and B be non-empty subsets of $\mathbb{R}$. We denote $d(A, B)$ to be the real number given by

$$d(A, B) = \inf\{|a - b| : a \in A \text{ and } b \in B\}.$$

The non-negative real number $d(A, B)$ can be considered as the distance between A and B.

Proposition 0.5.3 *Let f be a real-valued function with* $\mathrm{dom}\,(f) \subseteq \mathbb{R}$. *Suppose there exists a family $\{X_i\}_{i\in\Lambda}$ of non-empty subsets of* $\mathrm{dom}\,(f)$ *with* $\bigcup_{i\in\Lambda} X_i = \mathrm{dom}\,(f)$ *such that*

$$\textit{for every } i \in \Lambda, \qquad d\Big(X_i, \bigcup_{k\in\Lambda\setminus\{i\}} X_k\Big) > 0 \quad \textit{and} \quad f\big|_{X_i} \textit{ is continuous.}$$

Then the function f is continuous.

Proof The proof is similar to that for Proposition 0.5.2. Instead of using that the condition that F_i is closed, where $F_i = \bigcup_{k\in\Lambda\setminus\{i\}} X_k$, we can take $\delta_2 = d(X_i, F_i)$. □

The result below is a special case of the Tietz Extension Theorem. We give a direct proof using the structure of the real number line.

Proposition 0.5.4 *Let f be a continuous function with* $\mathrm{dom}\,(f)$ *a closed subset of* $\mathbb{R}$. *Then for every subset X of $\mathbb{R}$ with $X \supseteq \mathrm{dom}\,(f)$, there exists a continuous function g defined on X such that for every $x \in \mathrm{dom}\,(f)$, we have $f(x) = g(x)$.*

Proof We divide the proof into two steps. In Step 1, we prove the result for the case where $X = \mathbb{R}$ and in Step 2, we prove the result in general.

(*Step 1*) We want to show that there exists a continuous function g defined on $\mathbb{R}$ such that $f(x) = g(x)$ for all $x \in \mathrm{dom}\,(f)$.

We may assume that $\mathrm{dom}\,(f)$ is a proper subset of $\mathbb{R}$, for otherwise, there is nothing to prove. Since $\mathrm{dom}\,(f)$ is a closed (proper) subset of $\mathbb{R}$, it follows that the complement of $\mathrm{dom}\,(f)$ in $\mathbb{R}$, denoted by G, is a non-empty open subset of $\mathbb{R}$. Hence there exists a disjoint countable family $\{(a_n, b_n)\}_{n\in\Lambda}$ of open intervals such that $G = \bigcup_{n\in\Lambda} (a_n, b_n)$. Note that for $n \in \Lambda$, if the interval (a_n, b_n) is bounded, then both a_n and b_n belong to $\mathrm{dom}\,(f)$. Moreover, since $\mathrm{dom}\,(f) \neq \emptyset$, it follows that G is a proper subset of $\mathbb{R}$ and so for $n \in \Lambda$, if $a_n = -\infty$, then $b_n \in \mathrm{dom}\,(f)$ and if

$b_n = \infty$, then $a_n \in \operatorname{dom}(f)$. Denote g to be the function from $\mathbb{R}$ into itself given by

$$g(x) = \begin{cases} f(x) & \text{if } x \in \operatorname{dom}(f) \\ f(a_n) + \frac{x-a_n}{b_n-a_n}\big(f(b_n) - f(a_n)\big) & \text{if } x \in (a_n, b_n) \text{ where } a_n \neq -\infty,\ b_n \neq \infty \quad (n \in \Lambda) \\ f(a_n) & \text{if } x \in (a_n, b_n) \text{ where } b_n = \infty \quad (n \in \Lambda) \\ f(b_n) & \text{if } x \in (a_n, b_n) \text{ where } a_n = -\infty \quad (n \in \Lambda). \end{cases}$$

It is clear from the construction of g that for every $x \in \operatorname{dom}(f)$, we have $g(x) = f(x)$. We want to show that g is continuous on $\mathbb{R}$.

Let $x \in \mathbb{R}$. To show that g is continuous at x, we consider the following two cases:

(*Case 1*) $x \notin \operatorname{dom}(f)$

In this case, there exists $n \in \Lambda$ such that $x \in (a_n, b_n)$. Since g is linear in the open interval (a_n, b_n), it follows that g is continuous at x.

(*Case 2*) $x \in \operatorname{dom}(f)$

For this case, we consider the following four subcases:

(*Subcase 2a*) $\forall\, n \in \Lambda$, $x \neq a_n$ and $x \neq b_n$

In this subcase, we apply definition to show that g is continuous at x.

Let $\epsilon > 0$. By the continuity of f at x, there exists $\delta > 0$ such that

$$\text{for every } y \in (x-\delta, x+\delta) \cap \operatorname{dom}(f), \quad |f(y) - f(x)| < \epsilon.$$

The assumptions on x imply that there exists $u \in (x-\delta, x) \cap \operatorname{dom}(f)$ and there exists $v \in (x, x+\delta) \cap \operatorname{dom}(f)$. Denote $\delta_1 = \min\{\delta, x-u, v-x\}$. Then $\delta_1 > 0$ and we have

$$\text{for every } y \in (x-\delta_1, x+\delta_1), \quad |g(y) - g(x)| < \epsilon.$$

This is because if $y \in (x-\delta_1, x+\delta_1) \cap G$, then $y \in (a_n, b_n)$ for some $n \in \Lambda$ with $u \leq a_n < b_n < x$ or $x < a_n < b_n \leq v$. Hence the value $g(y)$ belongs to the interval with endpoints $f(a_n)$ and $f(b_n)$.

(*Subcase 2b*) $\exists\, i \in \Lambda$, $x = a_i$ and $\forall\, k \in \Lambda$, $x \neq b_k$

In this case, the right continuity of g at x follows from the condition that g is linear in the interval $[a_i, b_i)$ and the proof for the left continuity is similar to that for (Subcase 2a).

(*Subcase 2c*) $\exists\, i \in \Lambda$, $x = b_i$ and $\forall\, k \in \Lambda$, $x \neq a_k$

In this case, the left continuity of g at x follows from the condition that g is linear in the interval $(a_i, b_i]$ and the proof for the right continuity is similar to that for (Subcase 2a).

(*Subcase 2d*) $\exists\, i \in \Lambda$, $x = a_i$ and $\exists\, j \in \Lambda$, $x = b_j$

In this case, the right continuity of g at x follows from the condition that g is linear in the interval $[a_i, b_i)$ and the left continuity follows from that g is linear in the interval $(a_j, b_j]$.

(*Step 2*) Let X be a subset of $\mathbb{R}$ with $X \supseteq \operatorname{dom}(f)$. We want to show that there exists a continuous function g defined on X such that for every $x \in \operatorname{dom}(f)$, we have $f(x) = g(x)$.

By Step 1, there exists a continuous function h from $\mathbb{R}$ into itself such that for every $x \in \operatorname{dom}(f)$, we have $f(x) = h(x)$. Denote g to be the restriction of h on X. Then g is a continuous function defined on X and for every $x \in \operatorname{dom}(f)$, we have $f(x) = g(x)$.

□

Chapter 1

Lebesgue Measure

1.1 Introduction

The aim of this chapter is to introduce the concept of *measurable subsets* of $\mathbb{R}$ and to study their properties. The Lebesgue measure defined on the collection of all measurable subsets of $\mathbb{R}$ is a generalization of the concept of length defined on the collection of all intervals. In Chapter 2, we will introduce the concept of *measurable functions* using the idea of measurable sets. In Chapter 3, we will introduce the concept of *Lebesgue integral* for integrable functions. The concept of Lebesgue integral is a generalization of that of Riemann integral. First we recall the following terminology and definition.

Terminology Let I be a non-degenerate interval with left endpoint a (can be $-\infty$) and right endpoint b (can be ∞). A *subdivision* of I is a strictly increasing finite sequence $(x_i)_{i=0}^n$, where $n \in \mathbb{Z}^+$, of extended real numbers such that $x_0 = a$ and $x_n = b$.

Definition A real-valued function f is said to be *Riemann integrable over a non-degenerate closed and bounded interval* $[a, b]$, where $[a, b] \subseteq \operatorname{dom}(f)$, if there exists a real number I satisfying the following condition:

(†) For every $\epsilon > 0$, there exists $\delta > 0$ such that for every subdivision $(x_i)_{i=0}^n$ of $[a, b]$ with

$$\max_{1 \le i \le n} (x_i - x_{i-1}) < \delta,$$

and for every finite sequence $(t_i)_{i=1}^n$ of real numbers with $t_i \in [x_{i-1}, x_i]$ for every $i = 1, \ldots, n$, we have

$$\left| I - \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \right| < \epsilon.$$

The number I is unique if it exists. It is called the *Riemann integral of f over the interval* $[a, b]$ and is denoted by $\int_a^b f(x)\,\mathrm{d}x$.

Terminology A real-valued function f is said to be *Riemann integrable* if its domain is a non-degenerate closed and bounded interval and f is Riemann integrable over its domain.

Denote $f : [0,1] \longrightarrow \mathbb{R}$ to be the function (called the *Dirichlet function on* $[0,1]$) given by

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \setminus \mathbb{Q}. \end{cases}$$

It is well-known that f is not Riemann integrable over $[0,1]$. This is because in every non-degenerate interval contained in $[0,1]$, the Dirichlet function takes both values 1 and 0 and so the sum $\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ can take both values 1 and 0 if the t_i's are chosen suitably.

Below we describe how to extend the meaning of integral for the Dirichlet function. The idea is to divide $[0,1]$ into finitely many disjoint subsets so that the oscillation of the function in each of these subsets is small. From the construction of the Dirichlet function, it is clear that we can divide $[0,1]$ into the following two disjoint subsets:

$$S_1 = [0,1] \cap \mathbb{Q} \quad \text{and} \quad S_2 = [0,1] \setminus \mathbb{Q}.$$

In each of the two subsets, the oscillation of the Dirichlet function is 0. If we can measure the "lengths" of S_1 and S_2, then it is natural to define the integral of the Dirichlet function to be

$$1 \times \text{length of } S_1 + 0 \times \text{length of } S_2.$$

Recall that an interval I is non-empty subset of $\mathbb{R}$ satisfying Condition (I) given on Page 16. The length of an interval I, denoted by $\ell(I)$, is the difference between the right endpoint and left endpoint of I. For example,

- if $I = [a,a]$, where $a \in \mathbb{R}$, then $\ell(I) = 0$;
- if $I = (a,b)$, where $a, b \in \mathbb{R}$ with $a < b$, then $\ell(I) = b - a$;
- if $I = (a,\infty)$ where $a \in \mathbb{R}$, then $\ell(I) = \infty - a = \infty$;
- if $I = (-\infty,\infty)$, then $\ell(I) = \infty - (-\infty) = \infty$.

The task of this chapter is to extend the concept of length to a wider class of subsets of $\mathbb{R}$. The new concept that we will define is called the *Lebesgue measure* or simply the *measure*. We will write $\mathsf{m}(S)$ to denote the measure of S, where S is a subset of $\mathbb{R}$ such that its measure can be defined. By an extension, we mean

(i) for every interval I, we have $\mathsf{m}(I) = \ell(I)$.

Remark The equality $\mathsf{m}(I) = \ell(I)$ means that $\mathsf{m}(I)$ is defined and that the equality holds. Similar meanings apply to the equalities in (ii), (iii), (iii$'$) and (iii$''$) below.

Since length is translation invariant, it is natural to require m be translation invariant also, that is,

(ii) for every $S \subseteq \mathbb{R}$ such that $\mathsf{m}(S)$ is defined and for every real number x, we have $\mathsf{m}(S + x) = \mathsf{m}(S)$.

It is also natural to require m be *additive*, that is,

(iii) for every disjoint pair $(S_i)_{i=1}^{2}$ of subsets of $\mathbb{R}$ such that both $\mathsf{m}(S_1)$ and $\mathsf{m}(S_2)$ are defined, we have $\mathsf{m}(S_1 \cup S_2) = \mathsf{m}(S_1) + \mathsf{m}(S_2)$.

Note that if m is additive, then it is *finitely additive* (see exercise), that is,

(iii′) for every disjoint finite family $(S_i)_{i=1}^{n}$ of subsets of $\mathbb{R}$ such that for every $i = 1, \ldots, n$, the value $\mathsf{m}(S_i)$ is defined, we have $\mathsf{m}\Big(\bigcup\limits_{i=1}^{n} S_i\Big) = \sum\limits_{i=1}^{n} \mathsf{m}(S_i)$.

It would be better if m is *countably additive*, that is,

(iii″) for every disjoint sequence $(S_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$ such that for every $n \in \mathbb{Z}^+$, the value $\mathsf{m}(S_n)$ is defined, we have $\mathsf{m}\Big(\bigcup\limits_{n=1}^{\infty} S_n\Big) = \sum\limits_{n=1}^{\infty} \mathsf{m}(S_n)$.

Ideally, we would like m to have the following property also:

(iv) for every $S \subseteq \mathbb{R}$, $\mathsf{m}(S)$ is defined.

Terminology Let X be a set. A function whose domain is a (non-empty) subset of $\mathcal{P}(X)$ is called a *set function on* X.

The set function on $\mathbb{R}$, called the *(Lebesgue) outer measure on* $\mathbb{R}$, that we will construct in the next section satisfies (i), (ii) and (iv) but not (iii). Instead of (iv), we restrict the outer measure to a smaller class of subsets of $\mathbb{R}$ so that (iii) is satisfied. It turns out that (iii″) is also satisfied. This property is important in the theory of Lebesgue integration. It ensures that the order of taking limit and integration can be interchanged.

Remark It is impossible to construct a set function satisfying (i), (ii), (iii″) and (iv) and it is not known whether there is a set function satisfying (i), (ii), (iii) and (iv).

Exercise 1.1

1. Denote $\mu : \mathcal{P}(\mathbb{R}) \longrightarrow \mathbb{R}^*$ to be the set function on $\mathbb{R}$ given by

$$\mu(A) = \begin{cases} |A| & \text{if } A \text{ is finite,} \\ \infty & \text{otherwise,} \end{cases}$$

where $|A|$ denotes the number of elements of A. Show that μ satisfies (ii), (iii″) and (iv).

2. (*a*) Let $\mu : \operatorname{dom}(\mu) \longrightarrow [0,\infty]$ be a set function on a set X. Show that if μ is additive, then it is finitely additive.
   (*b*) Give an example of a set function $\mu : \mathcal{P}(X) \longrightarrow [0,\infty]$ on a set X such that μ is additive but not countably additive.
3. (*a*) Let $\mu : \operatorname{dom}(\mu) \longrightarrow [0,\infty]$ be a set function on a set X. Suppose μ is countably additive and $\emptyset \in \operatorname{dom}(\mu)$. Show that μ is additive.
   (*b*) Give an example of a set function $\mu : \mathcal{P}(X) \setminus \{\emptyset\} \longrightarrow [0,\infty]$ on an infinite set X such that μ is countably additive but not additive.

1.2 The Lebesgue Outer Measure

The length of an interval I, with left and right endpoints a and b respectively, is defined to be $b-a$. Since we want "length" to be additive or even countably additivity, it is natural to define the length of a non-empty open subset G of $\mathbb{R}$, denoted by $\ell(G)$, to be the sum of the lengths of its component intervals, that is,

$$\ell(G) = \sum_{i\in\Lambda} (b_i - a_i), \tag{1.1}$$

where $\{(a_i, b_i)\}_{i\in\Lambda}$ is the canonical family of component intervals of G (which is a disjoint family of open intervals with $\bigcup_{i\in\Lambda} (a_i, b_i) = G$). Note that the sum in (1.1) is well-defined since $\{b_i - a_i\}_{i\in\Lambda}$ is a countable family of non-negative extended real numbers.

Let S be a subset of $\mathbb{R}$. Let G be a non-empty open subset of $\mathbb{R}$ with $S \subseteq G$. If the "length" of S can be defined, its value must not exceed $\sum_{n\in\Lambda} (b_n - a_n)$, where $\{(a_n, b_n)\}_{n\in\Lambda}$ is a disjoint countable family of open intervals such that $G = \bigcup_{n\in\Lambda} (a_n, b_n)$. For convenience, in the definition below, we do not require the family $\{(a_n, b_n)\}_{n\in\Lambda}$ be disjoint. There are many ways to cover S by countable families $\{(a_n, b_n)\}_{n\in\Lambda}$ of open intervals. The outer measure of S, which will be denoted by $\mathsf{m}^*(S)$, is defined to be the infimum of all such sums $\sum_{n\in\Lambda} (b_n - a_n)$. In this way, we get a function m^* whose domain is $\mathcal{P}(\mathbb{R})$.

Definition The *Lebesgue outer measure on* $\mathbb{R}$, denoted by m^*, is the function from $\mathcal{P}(\mathbb{R})$ into $\mathbb{R}^*$ given by

$$\mathsf{m}^*(S) = \inf\Big\{ \sum_{n\in\Lambda} (b_n - a_n) : S \subseteq \bigcup_{n\in\Lambda} (a_n, b_n) \text{ where } \Lambda \text{ is countable} \Big\}.$$

For simplicity, the Lebesgue outer measure on $\mathbb{R}$ is also called *the (Lebesgue) outer measure.* For every $S \subseteq \mathbb{R}$, the extended real number $\mathsf{m}^*(S)$ is called the *(Lebesgue) outer measure of* S. For convenience, $\mathsf{m}^*(S)$ is sometimes written as m^*S.

Remark It can be shown that (see Exercise 1.4) for every $S \subseteq \mathbb{R}$, the value of $\mathsf{m}^*(S)$ is the same as

$$\inf\Big\{ \sum_{n\in\Lambda} (b_n - a_n) : S \subseteq \bigcup_{n\in\Lambda} (a_n, b_n) \text{ where } \Lambda \text{ is countable and } (a_i, b_i) \cap (a_j, b_j) = \emptyset \text{ whenever } i \neq j \Big\},$$

that is, $\mathsf{m}^*(S) = \inf\big\{ \ell(G) : G \text{ is a non-empty open subset of } \mathbb{R} \text{ and } S \subseteq G \big\}$.

Since length is translation invariant, it is natural that the outer measure is also translation invariant.

Proposition 1.2.1 *The Lebesgue outer measure* m^* *is translation invariant, that is, for every subset* S *of* $\mathbb{R}$ *and for every real number* r*, we have* $\mathsf{m}^*(S) = \mathsf{m}^*(S+r)$.

Idea Apply definition. If $S \subseteq \bigcup_{n\in\Lambda}(a_n, b_n)$, then $S + r \subseteq \bigcup_{n\in\Lambda}(a_n + r, b_n + r)$.

Proof We divide the proof into two steps. In Step 1, we prove that for every $A \subseteq \mathbb{R}$ and for every $s \in \mathbb{R}$, we have $\mathsf{m}^*(A) \leq \mathsf{m}^*(A+s)$. In Step 2, we prove the required result.

(*Step 1*) Let A be a subset of $\mathbb{R}$ and let s be a real number. We want to show that

$$\mathsf{m}^*(A+s) \leq \mathsf{m}^*(A). \tag{1.2}$$

For this we consider the following two cases:

(*Case 1*) $\mathsf{m}^*(A) = \infty$

The required inequality holds trivially in this case.

(*Case 2*) $\mathsf{m}^*(A) < \infty$

To show that $\mathsf{m}^*(A+s) \leq \mathsf{m}^*(A)$, it is sufficient (and also necessary) to show that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}^*(A+s) < \mathsf{m}^*(A) + \epsilon.$$

Let $\epsilon > 0$. By the definition of $\mathsf{m}^*(A)$, there exists a countable family $\{(a_n, b_n)\}_{n\in\Lambda}$ of open intervals such that $A \subseteq \bigcup_{n\in\Lambda}(a_n, b_n)$ and that

$$\sum_{n\in\Lambda}(b_n - a_n) < \mathsf{m}^*(A) + \epsilon. \tag{1.3}$$

Since $A + s \subseteq \bigcup_{n\in\Lambda}(a_n + s, b_n + s)$, it follows from definition that

$$\mathsf{m}^*(A+s) \leq \sum_{n\in\Lambda}\big((b_n + s) - (a_n + s)\big) = \sum_{n\in\Lambda}(b_n - a_n). \tag{1.4}$$

Combining (1.3) and (1.4), we get

$$\mathsf{m}^*(A+s) < \mathsf{m}^*(A) + \epsilon.$$

(*Step 2*) Let $S \subseteq \mathbb{R}$ and let $r \in \mathbb{R}$. We want to show that

$$\mathsf{m}^*(S+r) = \mathsf{m}^*(S).$$

By Step 1, it suffices to show that $\mathsf{m}^*(S+r) \geq \mathsf{m}^*(S)$.

Since (1.2) is valid for every subset A of $\mathbb{R}$ and for every real number s, by putting $A = S + r$ and putting $s = -r$, we get

$$\mathsf{m}^*\big((S+r)+(-r)\big) \leq \mathsf{m}^*(S+r),$$

that is, $\mathsf{m}^*(S) \leq \mathsf{m}^*(S+r)$.

□

Since length is monotone, it is natural that the outer measure is also monotone. The result is quite obvious since smaller subsets can be covered by more families of open intervals.

Proposition 1.2.2 *The Lebesgue outer measure* m^* *is monotone, that is, if* A *and* B *are subsets of* $\mathbb{R}$ *and* $A \subseteq B$, *then* $\mathsf{m}^*(A) \leq \mathsf{m}^*(B)$.

Idea Apply definition. If $B \subseteq \bigcup\limits_{n\in\Lambda}(a_n, b_n)$, then $A \subseteq \bigcup\limits_{n\in\Lambda}(a_n, b_n)$.

Proof Let A and B be subsets of $\mathbb{R}$ with $A \subseteq B$. We want to show that

$$\mathsf{m}^*(A) \leq \mathsf{m}^*(B).$$

For this we consider the following two cases:

(*Case 1*) $\mathsf{m}^*(B) = \infty$

The required inequality holds trivially in this case.

(*Case 2*) $\mathsf{m}^*(B) < \infty$

To show that $\mathsf{m}^*(A) \leq \mathsf{m}^*(B)$, it is sufficient (and also necessary) to show that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}^*(A) < \mathsf{m}^*(B) + \epsilon.$$

Let $\epsilon > 0$. By the definition of $\mathsf{m}^*(B)$, there exists a countable family of open intervals $\{(a_n, b_n)\}_{n\in\Lambda}$ such that $B \subseteq \bigcup\limits_{n\in\Lambda}(a_n, b_n)$ and that

$$\sum_{n\in\Lambda}(b_n - a_n) < \mathsf{m}^*(B) + \epsilon. \tag{1.5}$$

Since $A \subseteq B$, it follows that $A \subseteq \bigcup\limits_{n\in\Lambda}(a_n, b_n)$. Hence by definition and by (1.5), we get

$$\mathsf{m}^*(A) \leq \sum_{n\in\Lambda}(b_n - a_n) < \mathsf{m}^*(B) + \epsilon.$$

□

Next we want to show that the Lebesgue outer measure of an interval I is the length of I. The idea is to prove the result for the special case where I is a closed and bounded interval (Lemma 1.2.4) and then use the special case to prove the result in general (Theorem 1.2.5). To prove the special case, we need the following result which seems to be obvious.

Lemma 1.2.3 *If I is a closed and bounded interval and $\{J_i\}_{i\in\Lambda}$ is a finite family of open intervals such that $I \subseteq \bigcup_{i\in\Lambda} J_i$, then we have $\ell(I) < \sum_{i\in\Lambda} \ell(J_i)$.*

Idea Apply induction on the number of elements of Λ.

Proof We prove the result by induction on n, where n is the number of elements of Λ,

(1) It is clear that the result is true when $n = 1$. Indeed, if $[a, b] \subseteq (\alpha, \beta)$, then $\alpha < a$ and $\beta > b$ and so $b - a < \beta - \alpha$.

(2) Suppose the result is true for n.

Let I be a closed and bounded interval and let $\{J_i\}_{i\in\Lambda}$ be a family of open intervals such that $I \subseteq \bigcup_{i\in\Lambda} J_i$, where Λ has $n + 1$ elements. Denote $I = [a, b]$ and for each $i \in \Lambda$, denote $J_i = (\alpha_i, \beta_i)$. We want to show that

$$b - a = \sum_{i\in\Lambda} (\beta_i - \alpha_i).$$

Since $[a, b] \subseteq \bigcup_{i\in\Lambda} (\alpha_i, \beta_i)$, it follows that there exists $i_0 \in \Lambda$ such that $b \in (\alpha_{i_0}, \beta_{i_0})$.

- If $\alpha_{i_0} < a$, then we have

$$b - a < \beta_{i_0} - \alpha_{i_0} < \sum_{i\in\Lambda} (\beta_i - \alpha_i).$$

- If $\alpha_{i_0} \geq a$, then we have $[a, \alpha_{i_0}] \subseteq \bigcup_{i\in\Lambda\setminus\{i_0\}} (\alpha_i, \beta_i)$. Induction assumption yields

$$\alpha_{i_0} - a < \sum_{i\in\Lambda\setminus\{i_0\}} (\beta_i - \alpha_i)$$

from which we obtain

$$\begin{aligned} b - a &= (b - \alpha_{i_0}) + (\alpha_{i_0} - a) \\ &< (\beta_{i_0} - \alpha_{i_0}) + \sum_{i\in\Lambda\setminus\{i_0\}} (\beta_i - \alpha_i) \\ &= \sum_{i\in\Lambda} (\beta_i - \alpha_i). \end{aligned}$$

Thus the result is true for $n + 1$.

□

Lemma 1.2.4 *The Lebesgue outer measure of a closed and bounded interval is the length of the interval, that is, if a and b are real numbers with $a \leq b$, then $\mathsf{m}^*[a, b] = b - a$.*

Idea To show that $\mathsf{m}^*[a, b] \geq b - a$, apply the Heine-Borel Theorem and Lemma 1.2.3.

Proof Let a and b be real numbers with $a \le b$. We want to show that $\mathsf{m}^*[a,b] = b-a$. For this, it is sufficient (and also necessary) to show that

$$\mathsf{m}^*[a,b] \le b-a \quad \text{and} \quad \mathsf{m}^*[a,b] \ge b-a.$$

- First, we want to show that $\mathsf{m}^*[a,b] \le b-a$. For this, it is sufficient (and also necessary) to show that

  $$\text{for every } \epsilon > 0, \quad \mathsf{m}^*[a,b] \le b-a+\epsilon.$$

  Let $\epsilon > 0$. Since $[a,b] \subseteq \left(a-\frac{\epsilon}{2}, b+\frac{\epsilon}{2}\right)$, by definition, we have

  $$\mathsf{m}^*[a,b] \le \left(b+\tfrac{\epsilon}{2}\right) - \left(a-\tfrac{\epsilon}{2}\right) = b-a+\epsilon.$$

- Next, we want to show that $\mathsf{m}^*[a,b] \ge b-a$. For this, we apply definition.
  Suppose $\{(a_n,b_n)\}_{n\in\Lambda}$ is a countable family of open intervals such that $[a,b] \subseteq \bigcup\limits_{n\in\Lambda}(a_n,b_n)$, we want to show that

  $$\sum_{n\in\Lambda}(b_n-a_n) \ge b-a.$$

  By the Heine-Borel Theorem, there exists a finite subset Λ_1 of Λ such that $[a,b] \subseteq \bigcup\limits_{n\in\Lambda_1}(a_n,b_n)$. Hence by Lemma 1.2.3, we have

  $$b-a < \sum_{n\in\Lambda_1}(b_n-a_n) \le \sum_{n\in\Lambda}(b_n-a_n).$$

□

Theorem 1.2.5 *The Lebesgue outer measure of an interval is the length of the interval, that is, for every interval I, we have $\mathsf{m}^*(I) = \ell(I)$.*

Idea Use the monotonicity of the Lebesgue outer measure and Lemma 1.2.4. If I is bounded and non-degenerate, then $(a,b) \subseteq I \subseteq [a,b]$ for some $a,b \in \mathbb{R}$. If I is unbounded, then $I \supseteq [c,d]$ for some $c,d \in \mathbb{R}$, where $d-c$ can be arbitrarily large.

Proof Let I be an interval. We want to show that $\mathsf{m}^*(I) = \ell(I)$, that is, $\mathsf{m}^*(I) = b-a$ where a and b are the left endpoint and right endpoint of I respectively. For this, we consider the following two cases:

(*Case 1*) I is a bounded interval

In this case, both a and b are real numbers. To prove that $\mathsf{m}^*(I) = b-a$, we consider the following two subcases:

(*Subcase 1.1*) $a = b$

In this subcase, $I = [a,a]$ is a degenerate interval. By Lemma 1.2.4, we have $\mathsf{m}^*(I) = 0 = b-a$.

(*Subcase 1.2*) $a < b$

In this subcase, we have

$$(a, b) \subseteq I \subseteq [a, b].$$

By the monotonicity of the Lebesgue outer measure (Proposition 1.2.2), we have

$$\mathsf{m}^*(a, b) \leq \mathsf{m}^*(I) \leq \mathsf{m}^*[a, b]. \tag{1.6}$$

By Lemma 1.2.4, we have

$$\mathsf{m}^*[a, b] = b - a. \tag{1.7}$$

In view of (1.6) and (1.7), to show that $\mathsf{m}^*(I) = b-a$, it suffices to show that $\mathsf{m}^*(a, b) \geq b-a$, or equivalently, that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}^*(a, b) \geq b - a - \epsilon.$$

Let $\epsilon > 0$. Denote $\epsilon' = \frac{1}{2}\min\{\epsilon, b - a\}$. Since $[a + \epsilon', b - \epsilon'] \subseteq (a, b)$, by the monotonicity of the Lebesgue outer measure and Lemma 1.2.4, we have

$$\begin{aligned} \mathsf{m}^*(a, b) &\geq \mathsf{m}^*[a + \epsilon', b - \epsilon'] \\ &= b - a - 2\epsilon' \\ &\geq b - a - \epsilon. \end{aligned}$$

(*Case 2*) I is an unbounded interval.

In this case, we have $a = -\infty$ or $b = \infty$. We want to show that $\mathsf{m}^*(I) = \infty$. For this, we consider the following two (not mutually exclusive) subcases:

(*Subcase 2.1*) $b = \infty$

In this subcase, there exists $c \in \mathbb{R}$ such that

$$\text{for every } r \in (0, \infty), \quad [c, c + r] \subseteq I.$$

By the monotonicity of the Lebesgue outer measure and Lemma 1.2.4, we have

$$\text{for every } r \in (0, \infty), \quad \mathsf{m}^*(I) \geq \mathsf{m}^*[c, c + r] = r$$

which yields $\mathsf{m}^*(I) = \infty$.

(*Subcase 2.2*) $a = -\infty$

In this subcase, there exists $c \in \mathbb{R}$ such that

$$\text{for every } r \in (0, \infty), \quad [c - r, c] \subseteq I.$$

Using the same argument as in Subcase 2.1, we get $\mathsf{m}^*(I) = \infty$.

□

The next result gives the outer measure of some "small" subsets of $\mathbb{R}$. The result follows easily from the definition of the outer measure.

Proposition 1.2.6 *For every countable subset S of $\mathbb{R}$, we have $\mathsf{m}^*(S) = 0$.*

Idea For every $\epsilon > 0$, there exists a family of open intervals $\{(a_n, b_n)\}_{n=1}^{\infty}$ such that $S \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$ and that $\sum_{n=1}^{\infty}(b_n - a_n) < \epsilon$.

Proof Let S be a countable subset of $\mathbb{R}$. We want to show that $\mathsf{m}^*(S) = 0$. By the non-negativity of the Lebesgue outer measure, it suffices to show that $\mathsf{m}^*(S) \leq 0$, or equivalently that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}^*(S) \leq \epsilon.$$

Let $\epsilon > 0$. Since S is a countable subset of $\mathbb{R}$, there exists a countably infinite family $\{s_n\}_{n=1}^{\infty}$ of real numbers such that $S \subseteq \bigcup_{n=1}^{\infty}\{s_n\}$. Denote $\{(a_n, b_n)\}_{n\in\mathbb{Z}^+}$ to be the countable family of open intervals given by

$$a_n = s_n - 2^{-n-1}\epsilon \quad \text{and} \quad b_n = s_n + 2^{-n-1}\epsilon \qquad (n \in \mathbb{Z}^+). \tag{1.8}$$

Since $S \subseteq \bigcup_{n=1}^{\infty}(a_n, b_n)$, it follows from the definition of the Lebesgue outer measure, together with the construction in (1.8), that

$$\begin{aligned} \mathsf{m}^*(S) &\leq \sum_{n=1}^{\infty}(b_n - a_n) \\ &= \sum_{n=1}^{\infty}\frac{2\epsilon}{2^{n+1}} = \epsilon. \end{aligned}$$

□

Definition A subset S of $\mathbb{R}$ with $\mathsf{m}^*(S) = 0$ is called a *null set.*

Proposition 1.2.6 means that countable subsets of $\mathbb{R}$ are null sets. However, the converse is not true. In Section 1.5, we will give an example of an uncountable subset of $\mathbb{R}$ with outer measure zero.

The idea in the proof (the $\leq$ part) of Proposition 1.2.6 can be used in the proof of the following:

Theorem 1.2.7 *The Lebesgue outer measure m^* is countably subadditive, that is, for every sequence $(S_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$, we have*

$$\mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) \leq \sum_{n=1}^{\infty}\mathsf{m}^*(S_n).$$

Idea If for every $n \in \mathbb{Z}^+$, $\{(a_{n,k}, b_{n,k})\}_{k\in\Lambda_n}$ is a countable family of open intervals such that $S_n \subseteq \bigcup_{k\in\Lambda_n} (a_{n,k}, b_{n,k})$ and $\sum_{k\in\Lambda_n} (b_{n,k} - a_{n,k}) < \mathsf{m}^*(S_n) + \epsilon_n$, then $\{(a_{n,k}, b_{n,k})\}_{k\in\Lambda_n, n\in\mathbb{Z}^+}$ is a countable family of open intervals satisfying $\bigcup_{n=1}^{\infty} S_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k\in\Lambda_n} (a_{n,k}, b_{n,k})$ and $\mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) < \sum_{n=1}^{\infty} \Big(\mathsf{m}^*(S_n) + \epsilon_n\Big)$

Proof Let $(S_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$. We want to show that $\mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) \leq \sum_{n=1}^{\infty} \mathsf{m}^*(S_n)$. For this, we consider the following two cases:

(*Case 1*) $\sum_{n=1}^{\infty} \mathsf{m}^*(S_n) = \infty$

The required inequality holds trivially in this case.

(*Case 2*) $\sum_{n=1}^{\infty} \mathsf{m}^*(S_n) < \infty$

In this case, for every $n \in \mathbb{Z}^+$, we have $\mathsf{m}^*(S_n) < \infty$. To proof the required inequality, it is sufficient (and also necessary) to prove that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) < \sum_{n=1}^{\infty} \mathsf{m}^*(S_n) + \epsilon.$$

Let $\epsilon > 0$. It follows from definition that there exists a sequence $\big(\{(a_{n,k}, b_{n,k})\}_{k\in\Lambda_n}\big)_{n=1}^{\infty}$ of countable families of open intervals such that for every $n \in \mathbb{Z}^+$, we have $S_n \subseteq \bigcup_{k\in\Lambda_n} (a_{n,k}, b_{n,k})$ and that

$$\text{for every } n \in \mathbb{Z}, \quad \sum_{k\in\Lambda_n} (b_{n,k} - a_{n,k}) < \mathsf{m}^*(S_n) + 2^{-n}\epsilon. \tag{1.9}$$

Since $\{(a_{n,k}, b_{n,k})\}_{k\in\Lambda_n, n\in\mathbb{Z}^+}$ is a countable family of open intervals and

$$\bigcup_{n=1}^{\infty} S_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{k\in\Lambda_n} (a_{n,k}, b_{n,k}),$$

it follows from definition that

$$\mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) \leq \sum_{n=1}^{\infty} \sum_{k\in\Lambda_n} (b_{n,k} - a_{n,k}). \tag{1.10}$$

Combining (1.9) and (1.10), we have

$$\begin{aligned} \mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) &< \sum_{n=1}^{\infty} \big(\mathsf{m}^*(S_n) + 2^{-n}\epsilon\big) \\ &= \sum_{n=1}^{\infty} \mathsf{m}^*(S_n) + \epsilon. \end{aligned}$$

□

Although countable subadditivity does not imply finite subadditivity (see exercise), the Lebesgue outer measure is finitely subadditive. This is because the Lebesgue outer measure of the empty set is zero.

Corollary 1.2.8 *The Lebesgue outer measure* m^* *is finitely subadditive, that is, for every finite family* $\{S_i\}_{i=1}^n$ *of subsets of* $\mathbb{R}$*, we have*

$$\mathsf{m}^*\Big(\bigcup_{i=1}^n S_i\Big) \le \sum_{i=1}^n \mathsf{m}^*(S_i).$$

Idea Apply Theorem 1.2.7 by taking $S_i = \emptyset$ for $i > n$.

Proof Let $\{A_i\}_{i=1}^n$ be a finite family of subsets of $\mathbb{R}$. We want to show that $\mathsf{m}^*\Big(\bigcup_{i=1}^n A_i\Big) \le \sum_{i=1}^n \mathsf{m}^*(A_i)$.

Denote $(S_i)_{i=1}^\infty$ to be the sequence of subsets of $\mathbb{R}$ given by

$$S_i = \begin{cases} A_i & \text{if } 1 \le i \le n, \\ \emptyset & \text{if } i > n. \end{cases}$$

By Theorem 1.2.7, we have

$$\mathsf{m}^*\Big(\bigcup_{i=1}^\infty S_i\Big) \le \sum_{i=1}^\infty \mathsf{m}^*(S_i). \tag{1.11}$$

Since for every $i > n$, the set S_i is empty, it follows from Proposition 1.2.6 that for every $i > n$, we have $\mathsf{m}^*(S_i) = 0$. Hence (1.11) can be written as

$$\mathsf{m}^*\Big(\bigcup_{i=1}^n S_i\Big) \le \sum_{i=1}^n \mathsf{m}^*(S_i)$$

which is the same as the inequality that we want to prove. □

Remark In the definition of $\mathsf{m}^*(S)$, if we replace countably many open intervals by finitely many open intervals, then we can obtain finite subadditivity but not countable subadditivity (see exercise).

Exercise 1.2

1. For each of the following, prove or disprove it.
   (a) If $S \subseteq \mathbb{R}$ is bounded, then $\mathsf{m}^*(S) < \infty$.
   (b) If $S \subseteq \mathbb{R}$ is unbounded, then $\mathsf{m}^*(S) = \infty$.
2. Let $S \subseteq \mathbb{R}$. Suppose $\mathsf{m}^*(S) < \infty$. Show that for every $\epsilon > 0$, there exists an open subset G of $\mathbb{R}$ such that $S \subseteq G$ and that $\mathsf{m}^*(G) < \mathsf{m}^*(S) + \epsilon$.
3. Let $(A_n)_{n=1}^\infty$ be an increasing sequence of subsets of $\mathbb{R}$. Show that $\lim\limits_{n\to\infty} \mathsf{m}^*(A_n) \le \mathsf{m}^*\big(\bigcup_{n=1}^\infty A_n\big)$.
4. Let A and B be subsets of $\mathbb{R}$. Show that $\mathsf{m}^*(A \cup B) = \mathsf{m}^*(A) + \mathsf{m}^*(B)$ if A or B is a null set.
5. For each of the following subsets of $\mathbb{R}$, find its outer measure.
   (a) $[0, 1] \setminus \mathbb{Q}$
   (b) $\mathbb{R} \setminus \mathbb{Q}$

6. Use the results in this section to show that $\mathbb{R}$ is uncountable.
7. Let $S \subseteq \mathbb{R}$. Denote f to be the function from $[0,\infty)$ into $\mathbb{R}$ given by $f(x) = \mathsf{m}^*\big([-x,x] \cap S\big)$. Show that f is continuous.
8. Let $S \subseteq \mathbb{R}$. Show that for every $c \in [0, \mathsf{m}^*(S)]$, there exists $A \subseteq S$ such that $\mathsf{m}^*(A) = c$.
9. Let $\{I_j\}_{j\in\Lambda}$ be a finite family of open intervals such that $\mathbb{Q} \cap [0,1] \subseteq \bigcup\limits_{j\in\Lambda} I_j$. Show that $\sum\limits_{j\in\Lambda} \ell(I_j) \geq 1$.
10. Denote $\mathsf{m}^{\#}$ to be the function from $\mathcal{P}(\mathbb{R})$ into $\mathbb{R}^*$ given by
$$m^{\#}(S) = \inf\big\{ \sum_{n\in\Lambda} (b_n - a_n) : S \subseteq \bigcup_{n\in\Lambda} (a_n, b_n) \text{ where } \Lambda \text{ is finite}\big\}.$$
Show that $m^{\#}$ is subadditive but not countably subadditive.
11. Give an example of a set function $\mu : \mathcal{P}(X) \longrightarrow [0,\infty]$ such that μ is countably subadditive but not finitely subadditive.

1.3 Nonadditivity of the Lebesgue Outer Measure

In the last section, we have seen that the Lebesgue outer measure m^* is countably subadditive. It is natural to ask whether it is additive or even countably additive. In this section, we will give an example to show that m^* is not countably additive. The example also shows that m^* is not additive.

First we describe the idea of the example. Our aim is to "construct" a disjoint sequence $(S_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$ such that $\mathsf{m}^*\big(\bigcup\limits_{n=1}^{\infty} S_n\big) < \sum\limits_{n=1}^{\infty} \mathsf{m}^*(S_n)$. The definition of outer measure involves taking infimum; it is not easy to compute $\mathsf{m}^*(S)$ if S is "complicated". We know the outer measure of intervals; we try to construct $(S_n)_{n=1}^{\infty}$ such that $\bigcup\limits_{n=1}^{\infty} S_n$ is a *bounded (non-degenerate) interval*. If the outer measures of all the S_n's are the same, then we have $\sum\limits_{n=1}^{\infty} \mathsf{m}^*(S_n) = \infty > \mathsf{m}^*\big(\bigcup\limits_{n=1}^{\infty} S_n\big)$ as we desire, for otherwise we would have $\sum\limits_{n=1}^{\infty} \mathsf{m}^*(S_n) = 0 < \mathsf{m}^*\big(\bigcup\limits_{n=1}^{\infty} S_n\big)$ which contradicts the fact that m^* is countably subadditive.

Below, we will "construct" a disjoint sequence $(S_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$ such that $\bigcup\limits_{n=1}^{\infty} S_n = [0,1)$. Suppose we have a set X contained in $[0,1)$. Since the Lebesgue outer measure is translation invariant, it follows that for every real number r (between 0 and 1), we have $\mathsf{m}^*(X) = \mathsf{m}^*(X+r)$. However, part of $X+r$ may not be contained in $[0,1)$. To remedy this, we wrap it around. The new set obtained is denoted by $X \overset{\circ}{+} r$. In this way, we get a family $\{X \overset{\circ}{+} r\}_{0\leq r<1}$ of subsets of $[0,1)$. To get a countable family, we take $r \in \mathbb{Q} \cap [0,1)$. The set X should be constructed so that $X \overset{\circ}{+} r$ and $X \overset{\circ}{+} s$ are disjoint whenever r and s are distinct rational numbers in $[0,1)$.

Notation Let x and y be real numbers in $[0,1)$. We define $x \overset{\circ}{+} y$ to be the real number in $[0,1)$ given as follows:

$$x \overset{\circ}{+} y = \begin{cases} x+y & \text{if } x+y<1, \\ x+y-1 & \text{if } x+y \geq 1. \end{cases}$$

For every $S \subseteq [0,1)$ and for every $y \in [0,1)$, we define $S \overset{\circ}{+} y$ to be the subset of $\mathbb{R}$ given by

$$S \overset{\circ}{+} y = \{x \overset{\circ}{+} y : x \in S\}.$$

Below, we will show that S and $S \overset{\circ}{+} y$ have the same outer measure. First, we prove part of this result.

Lemma 1.3.1 *For every $S \subseteq [0,1)$ and for every $y \in [0,1)$, we have $\mathsf{m}^*(S \overset{\circ}{+} y) \leq \mathsf{m}^*(S)$.*

Idea If $(a,b) \subseteq [0,1)$, then

$$(a,b) \overset{\circ}{+} y = \begin{cases} (a+y, b+y) & \text{if } b+y \leq 1, \\ (a+y-1, b+y-1) & \text{if } a+y \geq 1, \\ (a+y,1) \cup [0, b+y-1) & \text{if } a < 1-y < b. \end{cases}$$

If $1-y \notin S$, then $0 \notin S \overset{\circ}{+} y$ and so we may replace $[0, b+y-1)$ by $(0, b+y-1)$.

Proof We divide the proof into two steps. In Step 1, we prove the result for the case where $S \cap \{0, 1-y\} = \emptyset$. In Step 2, we prove the result in general.

(*Step 1*) Let $A \subseteq [0,1)$ and let $y \in [0,1)$. Suppose $A \cap \{0, 1-y\} = \emptyset$. We want to show that

$$\mathsf{m}^*(A \overset{\circ}{+} y) \leq \mathsf{m}^*(A).$$

Denote A_1 and A_2 to be the subsets of A given by

$$A_1 = A \cap (0, 1-y) \quad \text{and} \quad A_2 = A \cap (1-y, 1).$$

Since $A \cap \{0, 1-y\} = \emptyset$, it follows from the construction of A_1 and A_2 that

$$A = A_1 \cup A_2 \qquad \text{(disjoint union)}$$

which yields

$$A \overset{\circ}{+} y = (A_1 \overset{\circ}{+} y) \cup (A_2 \overset{\circ}{+} y) \qquad \text{(disjoint union)}.$$

By the subadditivity of the Lebesgue outer measure, we have

$$\mathsf{m}^*(A \overset{\circ}{+} y) \leq \mathsf{m}^*(A_1 \overset{\circ}{+} y) + \mathsf{m}^*(A_2 \overset{\circ}{+} y).$$

Hence to get the required inequality, it suffices to show that

$$\mathsf{m}^*(A_1 \overset{\circ}{+} y) + \mathsf{m}^*(A_2 \overset{\circ}{+} y) \le \mathsf{m}^*(A),$$

or equivalently that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}^*(A_1 \overset{\circ}{+} y) + \mathsf{m}^*(A_2 \overset{\circ}{+} y) < \mathsf{m}^*(A) + \epsilon.$$

Let $\epsilon > 0$. By the definition of the outer measure of A, there exists a countable family $\{(\alpha_n, \beta_n)\}_{n\in\Lambda}$ of open intervals such that $A \subseteq \bigcup_{n\in\Lambda} (\alpha_n, \beta_n)$ and that

$$\sum_{n\in\Lambda} (\beta_n - \alpha_n) < \mathsf{m}^*(A) + \epsilon. \tag{1.12}$$

Denote $\{(a_n, b_n)\}_{n\in\Lambda}$ to be the family of open intervals given by

$$a_n = \max\{0, \alpha_n\}, \quad b_n = \min\{1, \beta_n\}, \qquad (n \in \Lambda).$$

By construction, we have

$$\text{for every } n \in \Lambda, \quad b_n - a_n \le \beta_n - \alpha_n. \tag{1.13}$$

Moreover, since $0 \notin A \subseteq [0, 1)$, it follows that $A \subseteq \bigcup_{n\in\Lambda} (a_n, b_n)$.

Denote $\{\Lambda_i\}_{i=1}^3$ to be the family of subsets of Λ given by

$$\begin{aligned} \Lambda_1 &= \{n \in \Lambda : b_n \le 1 - y\}, \\ \Lambda_2 &= \{n \in \Lambda : a_n \ge 1 - y\}, \\ \Lambda_3 &= \{n \in \Lambda : a_n < 1 - y < b_n\}, \end{aligned}$$

and denote $\{(c_n, d_n)\}_{n\in\Lambda_1}$, $\{(c'_n, d'_n)\}_{n\in\Lambda_2}$, $\{(c_n, d_n)\}_{n\in\Lambda_3}$ and $\{(c'_n, d'_n)\}_{n\in\Lambda_3}$ to be the families of open intervals given by

$$\begin{aligned} (c_n, d_n) &= (a_n + y, b_n + y) && (n \in \Lambda_1), \\ (c'_n, d'_n) &= (a_n + y - 1, b_n + y - 1) && (n \in \Lambda_2), \\ (c_n, d_n) &= (a_n + y, 1) && (n \in \Lambda_3), \\ (c'_n, d'_n) &= (0, b_n + y - 1) && (n \in \Lambda_3). \end{aligned}$$

It follows from the above constructions that

$$\sum_{n\in\Lambda} (b_n - a_n) = \sum_{n\in\Lambda_1\cup\Lambda_3} (d_n - c_n) + \sum_{n\in\Lambda_2\cup\Lambda_3} (d'_n - c'_n) \tag{1.14}$$

Moreover, since $A_1 \subseteq (0, 1 - y)$ and $A_2 \subseteq (1 - y, 1)$, it follows that

$$A_1 \overset{\circ}{+} y \subseteq \bigcup_{n\in\Lambda_1\cup\Lambda_3} (c_n, d_n) \qquad \text{and} \qquad A_2 \overset{\circ}{+} y \subseteq \bigcup_{n\in\Lambda_2\cup\Lambda_3} (c'_n, d'_n).$$

Hence by definition, we have

$$\mathsf{m}^*(A_1 \overset{\circ}{+} y) \le \sum_{n \in \Lambda_1 \cup \Lambda_3} (d_n - c_n) \quad \text{and} \quad \mathsf{m}^*(A_2 \overset{\circ}{+} y) \le \sum_{n \in \Lambda_2 \cup \Lambda_3} (d'_n - c'_n). \tag{1.15}$$

Adding the two inequalities in (1.15) and using (1.14), (1.13) and (1.12), we get

$$\begin{aligned} \mathsf{m}^*(A_1 \overset{\circ}{+} y) + \mathsf{m}^*(A_2 \overset{\circ}{+} y) &\le \sum_{n \in \Lambda_1 \cup \Lambda_3} (d_n - c_n) + \sum_{n \in \Lambda_2 \cup \Lambda_3} (d'_n - c'_n) \\ &= \sum_{n \in \Lambda} (b_n - a_n) \\ &\le \sum_{n \in \Lambda} (\beta_n - \alpha_n) \\ &< \mathsf{m}^*(A) + \epsilon. \end{aligned}$$

(*Step 2*) Let $S \subseteq [0,1)$ and let $y \in [0,1)$. We want to show that

$$\mathsf{m}^*(S \overset{\circ}{+} y) \le \mathsf{m}^*(S).$$

Denote $A = S \setminus \{0, 1-y\}$. Since $A \cap \{0, 1-y\} = \emptyset$, by Step 1, we have

$$\mathsf{m}^*(A \overset{\circ}{+} y) \le \mathsf{m}^*(A). \tag{1.16}$$

Since $S \subseteq A \cup \{0, 1-y\}$, it follows that

$$S \overset{\circ}{+} y \subseteq (A \overset{\circ}{+} y) \cup \{y, 0\}.$$

Hence by the subadditivity of the Lebesgue outer measure, together with Proposition 1.2.6, we get

$$\mathsf{m}^*(S \overset{\circ}{+} y) \le \mathsf{m}^*(A \overset{\circ}{+} y) + \mathsf{m}^*\{y, 0\} = \mathsf{m}^*(A \overset{\circ}{+} y). \tag{1.17}$$

Combining (1.16) and (1.17), we get

$$\begin{aligned} \mathsf{m}^*(S \overset{\circ}{+} y) &\le \mathsf{m}^*(A) \\ &\le \mathsf{m}^*(S) \end{aligned}$$

where the last inequality follows from the monotonicity of the Lebesgue outer measure since $A \subseteq S$.

□

For convenience, we introduce a binary operation on $[0,1)$, denoted by $\overset{\circ}{-}$, as follows:

Notation Let x and y be real numbers in $[0,1)$. We define $x \overset{\circ}{-} y$ to be the real number in $[0,1)$ given as follows:

$$x \overset{\circ}{-} y = \begin{cases} x - y & \text{if } x - y \ge 0, \\ x - y + 1 & \text{if } x - y < 0 \end{cases}$$

For every $S \subseteq [0, 1)$ and for every $y \in [0, 1)$, we define $S \overset{\circ}{-} y$ to be the subset of $\mathbb{R}$ given by

$$S \overset{\circ}{-} y = \{x \overset{\circ}{-} y : x \in S\}.$$

The following result gives the relation between the two operations $\overset{\circ}{+}$ and $\overset{\circ}{-}$.

Lemma 1.3.2 *For every $x \in [0, 1)$ and for every $y \in [0, 1)$, we have $(x \overset{\circ}{+} y) \overset{\circ}{-} y = x$.*

Idea Apply definition. Consider two cases: (i) $x < 1 - y$; (ii) $x \geq 1 - y$.

Proof Let $x, y \in [0, 1)$. We want to show that $(x \overset{\circ}{+} y) \overset{\circ}{-} y = x$. For this, we consider the following two cases:

(*Case 1*) $x < 1 - y$

In this case, we have $x \overset{\circ}{+} y = x + y$, from which we get

$$(x \overset{\circ}{+} y) \overset{\circ}{-} y = (x + y) - y = x$$

since $(x + y) - y \geq 0$.

(*Case 2*) $x \geq 1 - y$

In this case, we have $x \overset{\circ}{+} y = x + y - 1$, from which we get

$$(x \overset{\circ}{+} y) \overset{\circ}{-} y = (x + y - 1) \overset{\circ}{-} y = (x + y - 1) - y + 1 = x$$

since $(x + y - 1) - y < 0$.

□

Lemma 1.3.3 *For every $S \subseteq [0, 1)$ and for every $y \in [0, 1)$, we have $\mathsf{m}^*(S \overset{\circ}{-} y) \leq \mathsf{m}^*(S)$.*

Proof The proof is similar to that for Lemma 1.3.1. □

Proposition 1.3.4 *For every $S \subseteq [0, 1)$ and for every $y \in [0, 1)$, we have $\mathsf{m}^*(S \overset{\circ}{+} y) = \mathsf{m}^*(S)$.*

Idea Apply the above three lemmas.

Proof Let $S \subseteq [0, 1)$ and let $y \in [0, 1)$. We want to show that $\mathsf{m}^*(S \overset{\circ}{+} y) = \mathsf{m}^*(S)$.

- By Lemma 1.3.1, we have $\mathsf{m}^*(S \overset{\circ}{+} y) \leq \mathsf{m}^*(S)$.
- By Lemma 1.3.2, we have $S = (S \overset{\circ}{+} y) \overset{\circ}{-} y$. Hence by Lemma 1.3.3, we get

$$\mathsf{m}^*(S) \leq \mathsf{m}^*(S + y).$$

□

Remark The above result can be proved easily if we apply results for measurable subsets of $\mathbb{R}$ in the next section.

Below we give an example to show that m^* is not countably additive. As we have mentioned, we want to have a subset X of $[0,1)$ such that whenever r and s are different rational numbers in $[0,1)$, the sets $X \overset{\circ}{+} r$ and $X \overset{\circ}{+} s$ are disjoint. This is done by requiring the difference of every pair of numbers in X be irrational.

Example Denote R to be the relation on $[0,1)$ given by

$$ {}_xR_y \quad \text{if and only if} \quad x-y \in \mathbb{Q}, \qquad \text{for } (x,y) \in [0,1) \times [0,1). $$

It is clear that R is an equivalence relation. Since the set of all equivalence classes of R is a partition of $[0,1)$, it follows from the Axiom of Choice that there exists a subset X of $[0,1)$ such that the following condition is satisfied:

(†) For every equivalence class E of R, the set $X \cap E$ has exactly one element.

Since $[0,1)\cap\mathbb{Q}$ is equipotent to $\mathbb{Z}^+$, it follows that there exists a sequence $(r_n)_{n=1}^{\infty}$ of rational numbers in $[0,1)$ such that the following two conditions are satisfied:

(C1) $[0,1) \cap \mathbb{Q} = \bigcup_{n=1}^{\infty} \{r_n\}$.

(C2) $r_i \neq r_j$ whenever $i \neq j$ $(i,j \in \mathbb{Z}^+)$.

Denote $(X_n)_{n=1}^{\infty}$ to be the sequence of subsets of $[0,1)$ given by

$$ X_n = X \overset{\circ}{+} r_n \quad (n \in \mathbb{Z}^+). $$

Claim 1 $\bigcup_{n=1}^{\infty} X_n = [0,1)$.

Idea For every $s \in [0,1)$, there exists $x \in X$ such that ${}_xR_s$.

Proof It suffices to show that $[0,1) \subseteq \bigcup_{n=1}^{\infty} X_n$.

Let $s \in [0,1)$. We want to show that $s \in \bigcup_{n=1}^{\infty} X_n$. By the construction of $(X_n)_{n=1}^{\infty}$ and using (C1), it suffices to show that there exists $r \in \mathbb{Q} \cap [0,1)$ such that $s \in X \overset{\circ}{+} r$.

Since the set s/R is an equivalence class of R, it follows from the construction of X that there exists a unique $x \in X$ such that $x \in s/R$, that is, ${}_xR_s$ which, by the construction of R, implies that

$$ x - s \in \mathbb{Q}. $$

To obtain an element $r \in \mathbb{Q} \cap [0,1)$ such that $s \in X \overset{\circ}{+} r$, we consider the following two cases:

(*Case 1*) $s \geq x$

In this case, the number $s - x$, denoted by r, belongs to $[0,1) \cap \mathbb{Q}$. Note that $x \overset{\circ}{+} r = x + r$. Hence we have $s = x + r \in X \overset{\circ}{+} r$.

(*Case 2*) $s < x$

In this case, the number $s-x+1$, denoted by r, belongs to $[0,1)\cap\mathbb{Q}$. Note that $x \overset{\circ}{+} r = x+r-1$. Hence we have $s = x+r-1 \in X \overset{\circ}{+} r$.

□

Claim 2 $X_i \cap X_j = \emptyset$ whenever $i \neq j$ $(i, j \in \mathbb{Z}^+)$.

Idea The intersection of X and each equivalence class of R is a singleton.

Proof Suppose $(X \overset{\circ}{+} r_i) \cap (X \overset{\circ}{+} r_j) \neq \emptyset$ (where $i, j \in \mathbb{Z}^+$). We want to show that $i = j$. By (C2), it suffices to show that $r_i = r_j$.

Since $(X \overset{\circ}{+} r_i) \cap (X \overset{\circ}{+} r_j) \neq \emptyset$, there exist $x_1, x_2 \in X$ such that

$$x_1 \overset{\circ}{+} r_i = x_2 \overset{\circ}{+} r_j.$$

By the definition of $\overset{\circ}{+}$, the above equality implies that

$$x_1 - x_2 = r_j - r_i \quad \text{or} \quad x_1 - x_2 = r_j - r_i - 1 \quad \text{or} \quad x_1 - x_2 = r_j - r_i + 1. \tag{1.18}$$

Hence we have $x_1 - x_2 \in \mathbb{Q}$. Since $x_1, x_2 \in [0,1)$, it follows that

$${}_{x_1}R_{x_2}.$$

Note that the set x_2/R is an equivalent class of R and that

$$x_1 \in x_2/R \quad \text{and} \quad x_2 \in x_2/R.$$

Since both x_1 and x_2 belong to X, it follows from (†) that

$$x_1 = x_2.$$

Hence (1.18) reduces to

$$0 = r_j - r_i \quad \text{or} \quad 0 = r_j - r_i - 1 \quad \text{or} \quad 0 = r_j - r_i + 1.$$

Since $r_i, r_j \in [0,1)$, it follows that $r_j - r_i \neq 1$ and $r_j - r_i \neq -1$. Hence we have $r_i = r_j$. □

Result m^* is not countably additive.

Idea Consider the disjoint sequence $(X_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$.

Proof By Claim 2, the sequence $(X_n)_{n=1}^{\infty}$ is a disjoint sequence of subsets of $\mathbb{R}$. We want to show that

$$\mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} X_n\Big) < \sum_{n=1}^{\infty} \mathsf{m}^*(X_n).$$

(1) By Claim 1, we have $\mathsf{m}^*\big(\bigcup_{n=1}^{\infty} X_n\big) = 1$.

(2) By the countable subadditivity of the outer measure, together with (1), we have

$$\sum_{n=1}^{\infty} \mathsf{m}^*(X_n) \geq 1 \tag{1.19}$$

By Lemma 1.3.1, we have

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathsf{m}^*(X_n) = \mathsf{m}^*(X). \tag{1.20}$$

From (1.19) and (1.20), we see that $\mathsf{m}^*(X) > 0$ from which we obtain $\sum_{n=1}^{\infty} \mathsf{m}^*(X_n) = \infty$. □

Result$^+$ m^* is not additive.

Idea When N is large, $\sum_{n=1}^{N} \mathsf{m}^*(X_n) > 1$.

Proof It suffices to show that m^* is not finitely additive (see Exercise 1.1).

Denote $N = [\mathsf{m}^*(X)^{-1}] + 1$. We want to show that

$$\mathsf{m}^*\Big(\bigcup_{n=1}^{N} X_n\Big) < \sum_{n=1}^{N} \mathsf{m}^*(X_n).$$

- Since $\bigcup_{n=1}^{N} X_n \subseteq [0,1)$, it follows that $\mathsf{m}^*\Big(\bigcup_{n=1}^{N} X_n\Big) \leq 1$.
- By (1.20) together with the construction of N, we have $\sum_{n=1}^{N} \mathsf{m}^*(X_n) = N \times \mathsf{m}^*(X) > 1$. □

Exercise 1.3

1. Use the idea discussed in this section to prove the following:
   Suppose that S_1 and S_2 are subsets of $\mathbb{R}$ and that there exists a disjoint pair $(I_k)_{k=1}^2$ of intervals such that $S_1 \subseteq I_1$ and $S_2 \subseteq I_2$. Then we have $\mathsf{m}^*(S_1 \cup S_2) = \mathsf{m}^*(S_1) + \mathsf{m}^*(S_2)$.
2. Give an example of a decreasing sequence $(A_n)_{n=1}^{\infty}$ of subsets of $\mathbb{R}$ with $\mathsf{m}^*(A_1) < \infty$ such that $\lim_{n\to\infty} \mathsf{m}^*(A_n) > \mathsf{m}^*\big(\bigcap_{n=1}^{\infty} A_n\big)$.

$3^{\#}$. Do we have $\mathsf{m}^*(X_i \cup X_j) < \mathsf{m}^*(X_i) + \mathsf{m}^*(X_j)$ whenever $i \neq j$, where the X_n's are the subsets of $[0,1)$ given in the example in this section?

1.4 Measurable Sets and the Lebesgue Measure

In the last section, we see that the Lebesgue outer measure m^* on $\mathbb{R}$ is not additive. Instead of considering m^* on $\mathcal{P}(\mathbb{R})$, we try to find a smaller collection of subsets of $\mathbb{R}$ such that when m^* is restricted to this smaller collection, it is additive. The collection that we want to find should be large enough so that

(i) every interval belongs to the collection.

In order that m^* be additive, the collection must be closed under taking disjoint union:

(ii°) if A and B are sets belonging to the collection and $A \cap B = \emptyset$, then $A \cup B$ belongs to the collection also.

Preferably, we would like the collection to be closed under taking union:

(ii) if A and B are sets belonging to the collection, then $A \cup B$ belongs to the collection also.

It is natural to require the collection be closed under taking complement:

(iii) if A belongs to the collection, then $\mathbb{R} \setminus A$ belongs to the collection also.

Now suppose we have a collection of subsets of $\mathbb{R}$ satisfying (ii) and (iii). If the Lebesgue outer measure, restricted to the collection, is additive, then for every pair of sets A and B belonging to the collection, we have

$$\mathsf{m}^*(B) = \mathsf{m}^*(B \cap A) + \mathsf{m}^*(B \setminus A). \tag{1.21}$$

The right side in (1.21) is well-defined because by (ii) and (iii), the collection is closed under taking intersection.

Convention For every subset X of $\mathbb{R}$, we use X^c to denote the complement of X in $\mathbb{R}$, that is,

$$X^c = \mathbb{R} \setminus X.$$

There are two ways to extract "nice" sets:

(1) Take all sets A such that for every subset B of $\mathbb{R}$, the equality in (1.21) holds;

(2) Take all sets B such that for every subset A of $\mathbb{R}$, the equality in (1.21) holds.

Option (1) seems better because by symmetry, if A is "nice", then so is A^c. The following definition is due to Carathéodory.

Definition A subset A of $\mathbb{R}$ is said to be *Lebesgue measurable*, or simply *measurable*, if the following condition is satisfied:

(†) For every subset S of $\mathbb{R}$, we have $\mathsf{m}^*(S) = \mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c)$.

We give an example of measurable subset of $\mathbb{R}$ in the following lemma. More examples will be given later in the form of lemma, corollary etc.

Lemma 1.4.1 *The set $\mathbb{R}$ is measurable.*

Idea Apply definition.

Proof We want to show that

$$\text{for every } S \subseteq \mathbb{R}, \quad \mathsf{m}^*(S) = \mathsf{m}^*(S \cap \mathbb{R}) + \mathsf{m}^*(S \cap \mathbb{R}^c).$$

Let S be a subset of $\mathbb{R}$. Note that

$$S \cap \mathbb{R} = S \quad \text{and} \quad S \cap \mathbb{R}^c = \emptyset. \tag{1.22}$$

By (1.22) and using the fact that the Lebesgue outer measure of the empty set is zero, we get

$$\begin{aligned} \mathsf{m}^*(S \cap \mathbb{R}) + \mathsf{m}^*(S \cap \mathbb{R}^c) &= \mathsf{m}^*(S) + \mathsf{m}^*(\emptyset) \\ &= \mathsf{m}^*(S). \end{aligned}$$

□

Proposition 1.4.2 *If A is a measurable subset of $\mathbb{R}$, then so is A^c.*

Idea Apply definition and use $(A^c)^c = A$.

Proof Let A be a measurable subset of $\mathbb{R}$. We want to show that A^c is measurable.

Since the set A is measurable, we have

$$\text{for every } S \subseteq \mathbb{R}, \quad \mathsf{m}^*(S) = \mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c). \tag{1.23}$$

Since $(A^c)^c = A$, the condition in (1.23) can be written as

$$\text{for every } S \subseteq \mathbb{R}, \quad \mathsf{m}^*(S) = \mathsf{m}^*(S \cap A^c) + \mathsf{m}^*\big(S \cap (A^c)^c\big).$$

Therefore, by definition, the set A^c is measurable. □

Corollary 1.4.3 *The empty set is measurable.*

Proof This is an immediate consequence of Lemma 1.4.1 and Proposition 1.4.2. □

To check whether a subset A of $\mathbb{R}$ is measurable or not, instead of checking the equality in (†), it suffices to check the inequality $\geq$. Moreover, it suffices to check it for S with finite outer measure.

Proposition 1.4.4 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent:*

(1) *The set A is measurable.*

(2) *For every $S \subseteq \mathbb{R}$, we have $\mathsf{m}^*(S) \geq \mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c)$.*

(3) *For every $S \subseteq \mathbb{R}$ with $\mathsf{m}^*(S) < \infty$, we have $\mathsf{m}^*(S) \geq \mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c)$.*

Idea The Lebesgue outer measure is subadditive. Use $S = (S \cap A) \cup (S \cap A^c)$.

Proof

(1) $\Longrightarrow$ (3) Obvious.

(3) $\Longrightarrow$ (2) If $\mathsf{m}^*(S) = \infty$, the inequality $\mathsf{m}^*(S) \geq \mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c)$ holds trivially.

(2) $\Longrightarrow$ (1) For every $S \subseteq \mathbb{R}$, we have

$$S = (S \cap A) \cup (S \cap A^c).$$

Since the Lebesgue outer measure is subadditive, it follows that

$$\mathsf{m}^*(S) \leq \mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c).$$

Hence we get the required implication.

□

Remark To check whether a bounded subset A of $\mathbb{R}$ is measurable, instead of checking the equality in (†) for every subset S of $\mathbb{R}$, it suffices to check it for only one S, for example, for $S = [\inf A, \sup A]$ or any bounded interval containing A. In fact, one can define a subset A of $\mathbb{R}$ to be measurable if for every positive integer n, we have $\mathsf{m}^*\big(A \cap [-n, n]\big) = \mathsf{m}_*\big(A \cap [-n, n]\big)$. More details can be found in the exercise.

Theorem 1.4.5 *Null sets are measurable, that is, if $A \subseteq \mathbb{R}$ and $\mathsf{m}^*(A) = 0$, then A is measurable.*

Idea Apply definition and use the monotonicity of m^*. Subsets of null sets are null sets.

Proof Suppose $A \subseteq \mathbb{R}$ and $\mathsf{m}^*(A) = 0$. We want to show that A is measurable. By Proposition 1.4.4, it suffices to show that

$$\text{for every } S \subseteq \mathbb{R}, \quad \mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c) \leq \mathsf{m}^*(S).$$

Let S be a subset of $\mathbb{R}$. Note that

$$S \cap A \subseteq A \quad \text{and} \quad S \cap A^c \subseteq S.$$

It follows from the assumption on A and the monotonicity of the Lebesgue outer measure that

$$\mathsf{m}^*(S \cap A) \leq \mathsf{m}^*(A) = 0 \quad \text{and} \quad \mathsf{m}^*(S \cap A^c) \leq \mathsf{m}^*(S)$$

which implies that

$$\mathsf{m}^*(S \cap A) + \mathsf{m}^*(S \cap A^c) \leq \mathsf{m}^*(S).$$

□

Corollary 1.4.6 *Countable subsets of $\mathbb{R}$ are measurable.*

Proof This is an immediate consequence of Theorem 1.4.5 and Proposition 1.2.6. □

Next we will show that the collection of all measurable subsets of $\mathbb{R}$ is closed under taking countable union, that is, if $\{A_i\}_{i\in\Lambda}$ is a countable family of measurable subsets of $\mathbb{R}$, then the set $\bigcup_{i\in\Lambda} A_i$ is measurable. First we prove a special case in the following lemma.

Lemma 1.4.7 *Let A_1 and A_2 be measurable subsets of $\mathbb{R}$. Then the set $A_1 \cup A_2$ is measurable.*

Idea The measurability of A_1 implies that for every $S \subseteq \mathbb{R}$, we have $\mathsf{m}^*(S) = \mathsf{m}^*(S\cap A_1)+\mathsf{m}^*(S\cap A_1^c)$. Use the measurability of A_2 to consider the set $S \cap A_1^c$.

Proof To show that $A_1 \cup A_2$ is measurable, by Proposition 1.4.4, it suffices to show that

$$\text{for every } S \subseteq \mathbb{R}, \quad \mathsf{m}^*(S) \geq \mathsf{m}^*\big(S \cap (A_1 \cup A_2)\big) + \mathsf{m}^*\big(S \cap (A_1 \cup A_2)^c\big). \tag{1.24}$$

Let S be a subset of $\mathbb{R}$. Since the set A_1 is measurable, we have

$$\mathsf{m}^*(S) = \mathsf{m}^*(S \cap A_1) + \mathsf{m}^*(S \cap A_1^c). \tag{1.25}$$

Since the set A_2 is measurable, by considering the set $S \cap A_1^c$ in the condition for measurability, we have

$$\mathsf{m}^*(S \cap A_1^c) = \mathsf{m}^*(S \cap A_1^c \cap A_2) + \mathsf{m}^*(S \cap A_1^c \cap A_2^c). \tag{1.26}$$

By (1.26) and using $A_1^c \cap A_2^c = (A_1 \cup A_2)^c$, the equality in (1.25) can be written as

$$\mathsf{m}^*(S) = \mathsf{m}^*(S \cap A_1) + \mathsf{m}^*(S \cap A_1^c \cap A_2) + \mathsf{m}^*\big(S \cap (A_1 \cup A_2)^c\big). \tag{1.27}$$

In view of (1.27), to show that the inequality in (1.24) holds is equivalent to show that

$$\mathsf{m}^*\big(S \cap (A_1 \cup A_2)\big) \leq \mathsf{m}^*(S \cap A_1) + \mathsf{m}^*(S \cap A_1^c \cap A_2). \tag{1.28}$$

For this, note that

$$S \cap (A_1 \cup A_2) = (S \cap A_1) \cup (S \cap A_2) = (S \cap A_1) \cup (S \cap A_2 \cap A_1^c).$$

The required inequality in (1.28) then follows since the Lebesgue outer measure is subadditive. □

Lemma 1.4.8 *Let A_1 and A_2 be measurable subsets of $\mathbb{R}$. Then the set $A_1 \cap A_2$ is measurable.*

Idea Use $A_1 \cap A_2 = (A_1^c \cup A_2^c)^c$.

Proof Since A_1 and A_2 are measurable, by Proposition 1.4.2, the sets

$$A_1^c \text{ and } A_2^c \text{ are measurable.}$$

Hence by Lemma 1.4.7, the set

$$A_1^c \cup A_2^c \text{ is measurable.}$$

Therefore, by Proposition 1.4.2 again, the set

$$(A_1^c \cup A_2^c)^c \text{ is measurable.}$$

The required result then follows since $A_1 \cap A_2 = (A_1^c \cup A_2^c)^c$. □

Corollary 1.4.9 *Let A and B be measurable subsets of $\mathbb{R}$. Then the set $A \setminus B$ is measurable.*

Proof Since $A \setminus B = A \cap B^c$, the result follows from Lemma 1.4.8 and Proposition 1.4.2 □

Before proving that the collection of all measurable subsets of $\mathbb{R}$ is closed under taking countable union, we first establish the following lemma which means that the outer measure is finitely additive for sets that are "nicely separated".

Lemma 1.4.10 *For every disjoint finite family $\{A_i\}_{i=1}^n$ of measurable subsets of $\mathbb{R}$ and for every subset S of $\mathbb{R}$, we have*

$$\mathsf{m}^*\Big(S \cap \Big(\bigcup_{i=1}^n A_i\Big)\Big) = \sum_{i=1}^n \mathsf{m}^*(S \cap A_i).$$

Idea Use induction.

Proof We prove the result by induction.

(1) It is clear that the result is true for $n = 1$.

(2) Suppose the result is true for n.

Let $\{A_i\}_{i=1}^{n+1}$ be a disjoint family of measurable subsets of $\mathbb{R}$ and let S be a subset of $\mathbb{R}$. We want to show that

$$\mathsf{m}^*\Big(S \cap \Big(\bigcup_{i=1}^{n+1} A_i\Big)\Big) = \sum_{i=1}^{n+1} \mathsf{m}^*(S \cap A_i).$$

By considering the set $S \cap \big(\bigcup_{i=1}^{n+1} A_i\big)$ and using the measurability of A_{n+1}, we get

$$\mathsf{m}^*\Big(S \cap \Big(\bigcup_{i=1}^{n+1} A_i\Big)\Big) = \mathsf{m}^*\Big(S \cap \Big(\bigcup_{i=1}^{n+1} A_i\Big) \cap A_{n+1}\Big) + \mathsf{m}^*\Big(S \cap \Big(\bigcup_{i=1}^{n+1} A_i\Big) \cap A_{n+1}^c\Big). \quad (1.29)$$

Since the family $\{A_i\}_{i=1}^{n+1}$ is disjoint, it follows that the sets A_{n+1} and $\bigcup_{i=1}^{n} A_i$ are disjoint. Hence we have

$$\Big(\bigcup_{i=1}^{n+1} A_i\Big) \cap A_{n+1} = A_{n+1} \quad \text{and} \quad \Big(\bigcup_{i=1}^{n+1} A_i\Big) \cap A_{n+1}^c = \bigcup_{i=1}^{n} A_i$$

and so (1.29) reduces to

$$\mathsf{m}^*\Big(S \cap \Big(\bigcup_{i=1}^{n+1} A_i\Big)\Big) = \mathsf{m}^*\big(S \cap A_{n+1}\big) + \mathsf{m}^*\Big(S \cap \Big(\bigcup_{i=1}^{n} A_i\Big)\Big). \quad (1.30)$$

Since $\{A_i\}_{i=1}^n$ is a disjoint family, it follows from the induction assumption that

$$\mathsf{m}^*\Big(S \cap \Big(\bigcup_{k=1}^{n} A_k\Big)\Big) = \sum_{k=1}^{n} \mathsf{m}^*\big(S \cap A_k\big). \quad (1.31)$$

Combining (1.30) and (1.31), we get

$$\mathsf{m}^*\Big(S \cap \Big(\bigcup_{k=1}^{n+1} A_k\Big)\Big) = \sum_{k=1}^{n+1} \mathsf{m}^*(S \cap A_k).$$

Therefore the result is true for $n+1$.

□

Theorem 1.4.11 *Let $\{A_n\}_{n\in\Lambda}$ be a countable family of measurable subsets of $\mathbb{R}$. Then the set $\bigcup_{n\in\Lambda} A_n$ is measurable.*

Idea For the case where Λ is countably infinite, apply Lemma 1.4.10 by replacing $(A_n)_{n\in\Lambda}$ by a family $(B_n)_{n\in\Lambda}$ of disjoint measurable sets such that $\bigcup_{n\in\Lambda} A_n = \bigcup_{n\in\Lambda} B_n$.

Proof Note that the index set of a family is non-empty. To prove the result, we consider the following two cases:

(*Case 1*) Λ is finite

In this case, the result follows from Lemma 1.4.7 and induction.

(*Case 2*) Λ is infinite

In this case, to prove the result, we may assume that $\Lambda = \mathbb{Z}^+$. Let $(A_n)_{n=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}$. We want to show that the set $\bigcup_{n=1}^{\infty} A_n$ is measurable.

Denote $(B_n)_{n=1}^{\infty}$ to be the sequence of subsets of $\mathbb{R}$ defined inductively by

$$\begin{aligned} B_1 &= A_1 \\ B_n &= A_n \cap \Big(\bigcup_{i=1}^{n-1} B_i\Big)^c \quad (n \geq 2). \end{aligned}$$

By what we obtain in Case 1, together with Proposition 1.4.2 and Lemma 1.4.8, we see that $(B_n)_{n=1}^{\infty}$ is a sequence of measurable sets. Moreover, by construction, the sequence of sets $(B_n)_{n=1}^{\infty}$ is disjoint and

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n. \tag{1.32}$$

In view of (1.32), to show that the required result holds is to show that $\bigcup_{n=1}^{\infty} B_n$ is measurable. For this, we apply Proposition 1.4.4.

Let S be a subset of $\mathbb{R}$. We want to show that

$$\mathsf{m}^*(S) \geq \mathsf{m}^*\Big(S \cap \bigcup_{n=1}^{\infty} B_n\Big) + \mathsf{m}^*\Big(S \cap \Big(\bigcup_{n=1}^{\infty} B_n\Big)^c\Big). \tag{1.33}$$

By what we obtain in Case 1, we see that for every $k \in \mathbb{Z}^+$, the set $\bigcup_{n=1}^{k} B_n$ is measurable. Hence we have

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}^*(S) = \mathsf{m}^*\Big(S \cap \bigcup_{n=1}^{k} B_n\Big) + \mathsf{m}^*\Big(S \cap \Big(\bigcup_{n=1}^{k} B_n\Big)^c\Big). \tag{1.34}$$

Since $(B_n)_{n=1}^{\infty}$ is a disjoint sequence of measurable sets, by Lemma 1.4.10, we have

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}^*\Big(S \cap \bigcup_{n=1}^{k} B_n\Big) = \sum_{n=1}^{k} \mathsf{m}^*(S \cap B_n). \tag{1.35}$$

Note that for every $k \in \mathbb{Z}^+$, we have $S \cap \Big(\bigcup_{n=1}^{k} B_n\Big)^c \supseteq S \cap \Big(\bigcup_{n=1}^{\infty} B_n\Big)^c$. Hence by the monotonicity of the Lebesgue outer measure, we have

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}^*\Big(S \cap \Big(\bigcup_{n=1}^{k} B_n\Big)^c\Big) \geq \mathsf{m}^*\Big(S \cap \Big(\bigcup_{n=1}^{\infty} B_n\Big)^c\Big). \tag{1.36}$$

Combining (1.34), (1.35) and (1.36), we get

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}^*(S) \geq \sum_{n=1}^{k} \mathsf{m}^*(S \cap B_n) + \mathsf{m}^*\Big(S \cap \Big(\bigcup_{n=1}^{\infty} B_n\Big)^c\Big),$$

from which we obtain

$$\mathsf{m}^*(S) \geq \sum_{n=1}^{\infty} \mathsf{m}^*(S \cap B_n) + \mathsf{m}^*\Big(S \cap \Big(\bigcup_{n=1}^{\infty} B_n\Big)^c\Big). \tag{1.37}$$

By the countable subadditivity of the Lebesgue outer measure, we have

$$\sum_{n=1}^{\infty} \mathsf{m}^*(S \cap B_n) \geq \mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} (S \cap B_n)\Big) = \mathsf{m}^*\Big(S \cap \bigcup_{n=1}^{\infty} B_n\Big). \tag{1.38}$$

The required inequality in (1.33) then follows from (1.37) and (1.38). □

Corollary 1.4.12 *Let $\{A_n\}_{n\in\Lambda}$ be a countable family of measurable subsets of $\mathbb{R}$. Then the set $\bigcap_{n\in\Lambda} A_n$ is measurable.*

Proof The proof is similar to that for Lemma 1.4.8. Instead of applying Lemma 1.4.7, we apply Theorem 1.4.11. □

Below we will show that open subsets of $\mathbb{R}$ are measurable. First we prove a special case in the following lemma.

Lemma 1.4.13 *Every interval in the form (a, ∞), where $a \in \mathbb{R}$, is measurable.*

Idea If G is an open interval, then the sum of the Lebesgue outer measures of $G \cap (a, \infty)$ and $G \cap (a, \infty)^c$ is equal to the length of G.

Proof Let $I = (a, \infty)$ where $a \in \mathbb{R}$. We want to show that I is measurable. For this, we apply Proposition 1.4.4.

Let $S \subseteq \mathbb{R}$ with $\mathsf{m}^*(S) < \infty$. We want to show that

$$\mathsf{m}^*\big(S \cap (a,\infty)\big) + \mathsf{m}^*\big(S \cap (a,\infty)^c\big) \le \mathsf{m}^*(S)$$

or equivalently that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}^*\big(S \cap (a,\infty)\big) + \mathsf{m}^*\big(S \cap (a,\infty)^c\big) < \mathsf{m}^*(S) + \epsilon.$$

Let $\epsilon > 0$. By the definition of $\mathsf{m}^*(S)$, there exists a countable family $\{(\alpha_n, \beta_n)\}_{n\in\Lambda}$ of open intervals such that $S \subseteq \bigcup_{n\in\Lambda} (\alpha_n, \beta_n)$ and that

$$\sum_{n\in\Lambda} (\beta_n - \alpha_n) < \mathsf{m}^*(S) + \epsilon. \tag{1.39}$$

Denote $\{I_n\}_{n\in\Lambda}$ and $\{J_n\}_{n\in\Lambda}$ to be the families of subsets of $\mathbb{R}$ given by

$$I_n = (\alpha_n, \beta_n) \cap (a, \infty) \quad \text{and} \quad J_n = (\alpha_n, \beta_n) \cap (-\infty, a] \qquad (n \in \Lambda). \tag{1.40}$$

It is easily seen from the construction in (1.40) that

$$\text{for every } n \in \Lambda, \quad \mathsf{m}^*(I_n) + \mathsf{m}^*(J_n) = \beta_n - \alpha_n. \tag{1.41}$$

Since $S \cap (a,\infty) \subseteq \bigcup_{n\in\Lambda} I_n$, it follows from the monotonicity and the countable subadditivity of the Lebesgue outer measure that

$$\mathsf{m}^*\big(S \cap (a,\infty)\big) \le \mathsf{m}^*\Big(\bigcup_{n\in\Lambda} I_n\Big) \le \sum_{n\in\Lambda} \mathsf{m}^*(I_n). \tag{1.42}$$

Similarly, since $S \cap (-\infty, a] \subseteq \bigcup_{n\in\Lambda} J_n$, it follows that

$$\mathsf{m}^*\big(S \cap (-\infty, a]\big) \le \sum_{n\in\Lambda} \mathsf{m}^*(J_n). \tag{1.43}$$

Combining (1.42), (1.43), (1.41) and (1.39), we get

$$\begin{aligned} \mathsf{m}^*\big(S \cap (a,\infty)\big) + \mathsf{m}^*\big(S \cap (-\infty, a]\big) &\le \sum_{n\in\Lambda} \big(\mathsf{m}^*(I_n) + \mathsf{m}^*(J_n)\big) \\ &= \sum_{n\in\Lambda} (\beta_n - \alpha_n) \\ &< \mathsf{m}^*(S) + \epsilon. \end{aligned}$$

□

Theorem 1.4.14 *All open subsets of* $\mathbb{R}$ *are measurable.*

Idea Use Lemma 1.4.13. The collection of all measurable sets is closed under taking complement, intersection and countable union.

Proof We divide the proof into two steps. In Step 1, we prove the result for open intervals and in Step 2, we prove the result in general.

(*Step 1*) Let I be an open interval. We want to show that I is measurable. For this, we divide the proof into the following four cases, where a and b are the left and right endpoints of I:

(*Case 1*) $a = -\infty$ and $b = \infty$

In this case, $I = \mathbb{R}$ which is measurable by Lemma 1.4.1

(*Case 2*) $a \in \mathbb{R}$ and $b = \infty$

This case has been dealt with in Lemma 1.4.13.

(*Case 3*) $a = -\infty$ and $b \in \mathbb{R}$

In this case, we have

$$I = (-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \tfrac{1}{n}] = \bigcup_{n=1}^{\infty} (b - \tfrac{1}{n}, \infty)^c. \tag{1.44}$$

By Lemma 1.4.13, together with Proposition 1.4.2, we see that

$$\text{for every } n \in \mathbb{Z}^+, \quad (b - \tfrac{1}{n}, \infty)^c \text{ are measurable.}$$

In view of (1.44), by Theorem 1.4.11, we see that the open interval I is measurable.

(*Case 4*) $a \in \mathbb{R}$ and $b \in \mathbb{R}$ $(a < b)$

In this case, we have

$$I = (a, \infty) \cap (-\infty, b).$$

Hence by what we obtain in Case 2 and Case 3, together with Lemma 1.4.8, we see that the open interval I is measurable.

(*Step 2*) Let G be an open subset of $\mathbb{R}$. We want to show that G is measurable. For this, we consider the following two cases:

(*Case 1*) $G = \emptyset$

This case has been dealt with in Corollary 1.4.3.

(*Case 2*) $G \neq \emptyset$

In this case, there exists a countable family of open intervals $\{(a_n, b_n)\}_{n\in\Lambda}$ such that

$$G = \bigcup_{n\in\Lambda} (a_n, b_n).$$

By Step 1, the (countable) family $\{(a_n, b_n)\}_{n\in\Lambda}$ is a family of measurable subsets of $\mathbb{R}$. Hence by Theorem 1.4.11, the set G is measurable.

□

Since every closed subset of $\mathbb{R}$ is the complement of an open subset of $\mathbb{R}$, the following is an immediate consequence of Theorem 1.4.14 and Proposition 1.4.2.

Corollary 1.4.15 *All closed subsets of $\mathbb{R}$ are measurable.*

Notation and Terminology We denote by $\mathcal{M}$ the collection of all measurable subsets of $\mathbb{R}$ and denote the restriction of m^* to $\mathcal{M}$ by m, that is, m is the function from $\mathcal{M}$ into $\mathbb{R}^*$ given by

$$\mathsf{m}(A) = \mathsf{m}^*(A).$$

- The function m is called the *Lebesgue measure on* $\mathbb{R}$, or simply the *Lebesgue measure.*
- For every $A \in \mathcal{M}$, the value $\mathsf{m}(A)$ is called the *Lebesgue measure of* A, or simply the *measure of* A.

Since the Lebesgue measure is a restriction of m^*, it is

(i) monotone, that is, if $A, B \in \mathcal{M}$ and $A \subseteq B$, then $\mathsf{m}(A) \leq \mathsf{m}(B)$.

It is straightforward to check that (see exercise) the Lebesgue measure is

(ii) translation invariant, that is, for every $A \in \mathcal{M}$ and for every real number r, we have $A+r \in \mathcal{M}$ and $\mathsf{m}(A+r) = \mathsf{m}(A)$.

Also, by Theorem 1.4.11, the Lebesgue measure is

(iii) countably subadditive, that is, if $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{M}$, then $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{M}$ and $\mathsf{m}\Big(\bigcup\limits_{n=1}^{\infty} A_n\Big) \leq \sum\limits_{n=1}^{\infty} \mathsf{m}(A_n)$.

Moreover, it follows from the definition of measurability that the Lebesgue measure is additive (and hence it is finitely additive). In fact, it is countably additive.

Theorem 1.4.16 *The Lebesgue measure is countably additive, that is, if* $(A_n)_{n=1}^{\infty}$ *is a disjoint sequence of measurable subsets of* $\mathbb{R}$*, then*

$$\mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mathsf{m}(A_n).$$

Idea For every $k \in \mathbb{Z}^+$, apply Lemma 1.4.10 to the disjoint family $\{A_n\}_{n=1}^{k}$ of measurable subsets of $\mathbb{R}$, using $S = \mathbb{R}$.

Proof Let $(A_n)_{n=1}^{\infty}$ be a sequence of disjoint measurable subsets of $\mathbb{R}$. We want to show that

$$\mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \sum_{n=1}^{\infty} \mathsf{m}(A_n).$$

By the countable subadditivity of the Lebesgue measure, it suffices to show that

$$\mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big) \geq \sum_{n=1}^{\infty} \mathsf{m}(A_n). \tag{1.45}$$

By the monotonicity of the Lebesgue measure, we have

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big) \geq \mathsf{m}\Big(\bigcup_{n=1}^{k} A_n\Big). \tag{1.46}$$

Since for every $k \in \mathbb{Z}^+$, the finite family $\{A_n\}_{n=1}^k$ is a disjoint family of measurable subsets of $\mathbb{R}$, it follows from Lemma 1.4.10 (by putting $S = \mathbb{R}$) that

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}\Big(\bigcup_{n=1}^{k} A_n\Big) = \sum_{n=1}^{k} \mathsf{m}(A_n). \tag{1.47}$$

Combining (1.46) and (1.47), we get

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big) \geq \sum_{n=1}^{k} \mathsf{m}(A_n)$$

from which we obtain the required inequality in (1.45). □

Remark Using the idea of Corollary 1.2.8, we see that the Lebesgue measure is finitely additive.

Theorem 1.4.17 *Let $(A_n)_{n=1}^\infty$ be an increasing sequence of measurable subsets of $\mathbb{R}$. Then we have*

$$\mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mathsf{m}(A_n).$$

Idea Apply Theorem 1.4.16 by constructing a disjoint sequence $(B_n)_{n=1}^\infty$ of measurable subsets of $\mathbb{R}$ such that for every $k \in \mathbb{Z}^+$, we have $\bigcup_{n=1}^{k} B_n = A_k$.

Proof Denote $(B_n)_{n=1}^\infty$ to be the sequence of subsets of $\mathbb{R}$ given by

$$\begin{aligned} B_1 &= A_1, \\ B_n &= A_n \setminus A_{n-1} \quad (n \geq 2). \end{aligned}$$

Since the sequence of sets $(A_n)_{n=1}^\infty$ is increasing, it follows from construction that $(B_n)_{n=1}^\infty$ is a disjoint sequence of subsets of $\mathbb{R}$. Moreover, by Proposition 1.4.2, the sequence of sets $(B_n)_{n=1}^\infty$ is a sequence of measurable sets. Hence by Theorem 1.4.16, we have

$$\mathsf{m}\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mathsf{m}(B_n). \tag{1.48}$$

It is easily seen from the construction of $(B_n)_{n=1}^\infty$ that

$$\text{for every } k \in \mathbb{Z}^+, \quad A_k = \bigcup_{n=1}^{k} B_n \quad \text{(disjoint union)}.$$

Hence by Theorem 1.4.16 (finite additivity of m), we have

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}(A_k) = \sum_{n=1}^{k} \mathsf{m}(B_n). \tag{1.49}$$

From (1.48) and (1.49), we get

$$\lim_{k\to\infty} \mathsf{m}(A_k) = \sum_{n=1}^{\infty} \mathsf{m}(B_n) = \mathsf{m}\Big(\bigcup_{n=1}^{\infty} B_n\Big)$$

which yields

$$\lim_{n\to\infty} \mathsf{m}(A_n) = \mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big)$$

since $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$ □

It is natural to guess that in Theorem 1.4.17, if in the assumption, *increasing* is replaced by *decreasing*, then in the conclusion, *union* should be replaced by *intersection*. Below, we will show that this is true if we make an additional assumption. First we give the following simple result.

Proposition 1.4.18 *Let A and B be measurable subsets of $\mathbb{R}$. Suppose $A \subseteq B$ and $\mathsf{m}(A) < \infty$. Then we have $\mathsf{m}(B \setminus A) = \mathsf{m}(B) - \mathsf{m}(A)$.*

Idea Use the additivity of the Lebesgue measure: B is the disjoint union of A and $B \setminus A$.

Proof Since $\mathsf{m}(A) < \infty$, to show that $\mathsf{m}(B \setminus A) = \mathsf{m}(B) - \mathsf{m}(A)$, it is sufficient (and also necessary) to show that

$$\mathsf{m}(B) = \mathsf{m}(A) + \mathsf{m}(B \setminus A).$$

For this, note that by the assumption $A \subseteq B$, we have

$$B = A \cup (B \setminus A) \quad \text{(disjoint union)}.$$

Since both sets A and $B \setminus A$ are measurable, by the additivity of the Lebesgue measure, we have

$$\mathsf{m}(B) = \mathsf{m}(A) + \mathsf{m}(B \setminus A).$$

□

Remark If $\mathsf{m}(A) = \infty$, then $\mathsf{m}(B) = \infty$ also. The right side of the equality in the conclusion of the above proposition is meaningless. If $\mathsf{m}(A) < \infty$ and $\mathsf{m}(B) = \infty$, then the equality becomes $\infty = \infty$.

Theorem 1.4.19 *Let $(A_n)_{n=1}^{\infty}$ be a decreasing sequence of measurable subsets of $\mathbb{R}$. Suppose there exists $n_0 \in \mathbb{Z}^+$ such that $\mathsf{m}(A_{n_0}) < \infty$. Then we have*

$$\mathsf{m}\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mathsf{m}(A_n).$$

Idea Apply Theorem 1.4.17 to the increasing sequence $(A_{n_0} \setminus A_n)_{n=1}^{\infty}$ of measurable sets.

Proof Denote $(B_n)_{n=1}^{\infty}$ to be the sequence of subsets of $\mathbb{R}$ given by

$$B_n = A_{n_0} \setminus A_n \quad (n \in \mathbb{Z}^+).$$

It is clear that $(B_n)_{n=1}^{\infty}$ is an increasing sequence of measurable sets. Hence by Theorem 1.4.17, we have

$$\mathsf{m}\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \lim_{n\to\infty} \mathsf{m}(B_n),$$

that is,

$$\mathsf{m}\Big(A_{n_0} \setminus \bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mathsf{m}(A_{n_0} \setminus A_n). \tag{1.50}$$

Note that

$$\mathsf{m}\Big(\bigcap_{n=1}^{\infty} A_n\Big) < \infty \qquad \text{and} \qquad \text{for every } n \geq n_0, \quad \mathsf{m}(A_n) < \infty.$$

Hence by (1.50) and Proposition 1.4.18, we get

$$\begin{aligned} \mathsf{m}(A_{n_0}) - \mathsf{m}\Big(\bigcap_{n=1}^{\infty} A_n\Big) &= \lim_{n\to\infty} \big(\mathsf{m}(A_{n_0}) - \mathsf{m}(A_n)\big) \\ &= \mathsf{m}(A_{n_0}) - \lim_{n\to\infty} \mathsf{m}(A_n). \end{aligned}$$

The required equality then follows since $\mathsf{m}(A_{n_0}) < \infty$. □

Remark If the sets A_n's are not assumed to be measurable and the Lebesgue measure is replaced by the Lebesgue outer measure, then Theorem 1.4.19 is not valid (see Exercise 1.3); however, Theorem 1.4.17 remains valid in this case (see exercise).

The following example illustrates that Theorem 1.4.19 does not hold if we do not assume $\mathsf{m}(A_n)$ to be finite for some n.

Example Denote $(A_n)_{n=1}^{\infty}$ to be the sequence of subsets of $\mathbb{R}$ given by

$$A_n = (n, \infty), \quad (n \in \mathbb{Z}^+).$$

It is clear that $(A_n)_{n=1}^{\infty}$ is a decreasing sequence of measurable subsets of $\mathbb{R}$. However, we have

$$\mathsf{m}\Big(\bigcap_{n=1}^{\infty} A_n\Big) < \lim_{n\to\infty} \mathsf{m}(A_n).$$

Indeed, we have $\lim_{n\to\infty} \mathsf{m}(A_n) = \infty$ and $\mathsf{m}\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \mathsf{m}(\emptyset) = 0$.

The next theorem means that every measurable set can be obtained by deleting a null set from a G_δ set or adding a null set to an F_σ set.

Theorem 1.4.20 *Let A be a subset of $\mathbb{R}$. Then the following statements are equivalent.*

(1) *A is measurable.*

(2) *For every $\epsilon > 0$, there exists an open subset G of $\mathbb{R}$ such that $G \supseteq A$ and that $\mathsf{m}^*(G \setminus A) < \epsilon$.*

(3) *For every $\epsilon > 0$, there exists a closed subset F of $\mathbb{R}$ such that $F \subseteq A$ and that $\mathsf{m}^*(A \setminus F) < \epsilon$.*

(4) *There exists a subset E of $\mathbb{R}$ which is a G_δ-set such that $E \supseteq A$ and that $\mathsf{m}^*(E \setminus A) = 0$.*

(5) *There exists a subset F of $\mathbb{R}$ which is an F_σ-set such that $E \subseteq A$ and that $\mathsf{m}^*(A \setminus E) = 0$.*

Idea For (1) $\Longrightarrow$ (2): if G is an open subset of $\mathbb{R}$ and $A \subseteq G$, then $\mathsf{m}(G \setminus A) = \mathsf{m}(G) - \mathsf{m}(A)$ provided that $\mathsf{m}(A) < \infty$; if $\mathsf{m}(A) = \infty$, consider $A = \bigcup_{k=1}^{\infty} A \cap [-k, k]$. For (2) $\Longrightarrow$ (4): taking $\epsilon = \frac{1}{n}$ (where $n \in \mathbb{Z}^+$), we get an open set G_n. For (4) $\Longrightarrow$ (1): G_δ-sets and null sets are measurable.

Proof Below we show that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (4) $\Longrightarrow$ (1) and (1) $\Longrightarrow$ (3) $\Longrightarrow$ (5) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2) We divide the proof into two steps. In Step 1, we prove the result for $\mathsf{m}(A) < \infty$. In Step 2, we prove the result in general.

(*Step 1*) Suppose A is a measurable subset of $\mathbb{R}$ and $\mathsf{m}(A) < \infty$. We want to show that (2) holds.

Let $\epsilon > 0$. By the definition of $\mathsf{m}^*(A)$, there exists a countable family $\{(a_n, b_n)\}_{n \in \Lambda}$ of open intervals such that $A \subseteq \bigcup_{n \in \Lambda} (a_n, b_n)$ and that

$$\sum_{n \in \Lambda} (b_n - a_n) < \mathsf{m}(A) + \epsilon. \tag{1.51}$$

Denote $G = \bigcup_{n \in \Lambda} (a_n, b_n)$ which is an open subset of $\mathbb{R}$ with $A \subseteq G$. We want to show that $\mathsf{m}(G \setminus A) < \epsilon$.

By the countable subadditivity of the Lebesgue measure, we have

$$\mathsf{m}(G) \leq \sum_{n \in \Lambda} (b_n - a_n). \tag{1.52}$$

Combining (1.51) and (1.52), we get

$$\mathsf{m}(G) < \mathsf{m}(A) + \epsilon. \tag{1.53}$$

Since $\mathsf{m}(A) < \infty$, it follows from Proposition 1.4.18 that

$$\mathsf{m}(G \setminus A) = \mathsf{m}(G) - \mathsf{m}(A)$$

which, together with (1.53), yields

$$\mathsf{m}(G \setminus A) < \epsilon.$$

(*Step 2*) Suppose A is a measurable subset of $\mathbb{R}$. We want to show that (2) holds.

Let $\epsilon > 0$. Denote $(A_k)_{k=1}^{\infty}$ to be the sequence of subsets of $\mathbb{R}$ given by

$$A_k = A \cap [-k, k], \quad (k \in \mathbb{Z}^+).$$

Note that $(A_k)_{k=1}^{\infty}$ is a sequence of measurable subsets of $\mathbb{R}$ with finite measures. Applying the result in Step 1 to each A_k (using the positive number $2^{-k}\epsilon$), we see that there exists a sequence $(G_k)_{k=1}^{\infty}$ of open subsets of $\mathbb{R}$, such that

$$\text{for every } k \in \mathbb{Z}^+, \quad A_k \subseteq G_k \tag{1.54}$$

and that

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}(G_k \setminus A_k) < 2^{-k}\epsilon. \tag{1.55}$$

Denote $G = \bigcup_{k=1}^{\infty} G_k$ which is an open subset of $\mathbb{R}$. Since $A = \bigcup_{k=1}^{\infty} A_k$, it follows from (1.54) that $A \subseteq G$. We want to show that $\mathsf{m}(G \setminus A) < \epsilon$.

Note that

$$G \setminus A = \Big(\bigcup_{k=1}^{\infty} G_k\Big) \setminus \Big(\bigcup_{k=1}^{\infty} A_k\Big) \subseteq \bigcup_{k=1}^{\infty} (G_k \setminus A_k).$$

Hence by the monotonicity and the countability subadditivity of the Lebesgue measure, we have

$$\begin{aligned} \mathsf{m}(G \setminus A) &\leq \mathsf{m}\Big(\bigcup_{k=1}^{\infty} (G_k \setminus A_k)\Big) \\ &\leq \sum_{k=1}^{\infty} \mathsf{m}(G_k \setminus A_k) \\ &< \sum_{k=1}^{\infty} 2^{-k}\epsilon \;=\; \epsilon \end{aligned}$$

where the last inequality follows from (1.55).

(2) $\Longrightarrow$ (4) Suppose (2) holds. Then by taking $\epsilon = n^{-1}$ $(n \in \mathbb{Z}^+)$, we see that there exists a sequence $(G_n)_{n=1}^{\infty}$ of open subsets of $\mathbb{R}$, such that

$$\text{for every } n \in \mathbb{Z}^+, \quad A \subseteq G_n$$

and that

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathsf{m}^*(G_n \setminus A) < n^{-1}. \tag{1.56}$$

Denote $E = \bigcap_{n=1}^{\infty} G_n$. Then E is a G_δ-set and $A \subseteq E$. We want to show that $\mathsf{m}^*(E \setminus A) = 0$. Note that

$$\text{for every } n \in \mathbb{Z}^+, \quad E \setminus A \subseteq G_n \setminus A.$$

Hence by the monotonicity of the Lebesgue outer measure and (1.56), we have

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathsf{m}^*(E \setminus A) \leq \mathsf{m}^*(G_n \setminus A) < n^{-1},$$

from which we obtain $\mathsf{m}^*(E \setminus A) = 0$ as required.

(4) $\Longrightarrow$ (1) Suppose (4) holds, that is, there exists a subset E of $\mathbb{R}$ which is a G_δ-set such that $E \supseteq A$ and that $\mathsf{m}^*(E \setminus A) = 0$. Since $A \subseteq E$, it follows that

$$A = E \setminus (E \setminus A).$$

To show that A is measurable, by Corollary 1.4.9, it suffices to show that both sets E and $E \setminus A$ are measurable.

- ⋄ Since E is the intersection of a countable family of open subsets of $\mathbb{R}$, it follows from Theorem 1.4.14 and Theorem 1.4.11 that the set E is measurable.
- ⋄ Since $\mathsf{m}^*(E \setminus A) = 0$, it follows from Theorem 1.4.5 that the set $E \setminus A$ is measurable.

(1) $\Longrightarrow$ (3) Suppose A is measurable. We want to show that (3) holds.

Let $\epsilon > 0$. Note that the set A^c is measurable. By what we obtain in (1) $\Longrightarrow$ (2), replacing A by A^c, we see that there exists an open subset G of $\mathbb{R}$ such that $A^c \subseteq G$ and that

$$\mathsf{m}^*(G \setminus A^c) < \epsilon. \tag{1.57}$$

Denote $F = \mathbb{R} \setminus G$. Then F is a closed subset of $\mathbb{R}$ and $F \subseteq A$. Moreover, we have

$$\mathsf{m}^*(A \setminus F) < \epsilon. \tag{1.58}$$

Indeed, (1.58) is just (1.57) since

$$A \setminus F = A \setminus G^c = A \cap G = G \setminus A^c.$$

(3) $\Longrightarrow$ (5) The proof is similar to that for (2) $\Longrightarrow$ (4). Instead of taking countable intersection, we take countable union.

(5) $\Longrightarrow$ (1) The proof is similar to that for (4) $\Longrightarrow$ (1).

□

To close this section, we prove a result that will be used later.

Proposition 1.4.21 *Let $(A_n)_{n=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}$. Suppose that $\sum_{n=1}^{\infty} \mathsf{m}(A_n) < \infty$. Then we have $\mathsf{m}\big(\limsup_{n\to\infty} A_n\big) = 0$.*

Idea For every $\epsilon > 0$, there exists n_0 such that $\sum_{n=n_0}^{\infty} \mathsf{m}(A_n) < \epsilon$.

Proof Recall that $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. We want to show that

$$\text{for every } \epsilon > 0, \quad \mathsf{m}\Big(\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k\Big) < \epsilon.$$

Let $\epsilon > 0$. Since $\sum_{n=1}^{\infty} \mathsf{m}(A_n) < \infty$, it follows that there exists $n_0 \in \mathbb{Z}^+$ such that

$$\sum_{n=n_0}^{\infty} \mathsf{m}(A_n) < \epsilon,$$

which, by the countable subadditivity of the Lebesgue measure, implies that

$$\mathsf{m}\Big(\bigcup_{k=n_0}^{\infty} A_k\Big) \leq \sum_{n=n_0}^{\infty} \mathsf{m}(A_n) < \epsilon. \tag{1.59}$$

Since $\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k \subseteq \bigcup_{k=n_0}^{\infty} A_k$, it follows from the monotonicity of the Lebesgue measure and (1.59) that

$$\mathsf{m}\Big(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k\Big) \leq \mathsf{m}\Big(\bigcup_{k=n_0}^{\infty} A_k\Big) < \epsilon.$$

□

Exercise 1.4

1. Let $(S_n)_{n=1}^{\infty}$ be a sequence of subsets of $\mathbb{R}$. Suppose there exists a disjoint sequence $(A_n)_{n=1}^{\infty}$ of measurable subsets of $\mathbb{R}$ such that for every $n \in \mathbb{Z}^+$, we have $S_n \subseteq A_n$. Show that $\mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} S_n\Big) = \sum_{n=1}^{\infty} \mathsf{m}^*(S_n)$.
2. Let A be a measurable subset of $\mathbb{R}$. Show that for every real number y, the set $A+y$ is measurable.
3. Let A and B be measurable subsets of $\mathbb{R}$. Show that $\mathsf{m}(A\cup B)+\mathsf{m}(A\cap B) = \mathsf{m}(A)+\mathsf{m}(B)$.
4. Let A and B be subsets of $\mathbb{R}$.
   - (a) Show that $\mathsf{m}^*(A\cup B)+\mathsf{m}^*(A\cap B) \leq \mathsf{m}^*(A)+\mathsf{m}^*(B)$.
   - (b) Suppose that both $\mathsf{m}^*(A)$ and $\mathsf{m}^*(B)$ are finite. Show that $|\mathsf{m}^*A - \mathsf{m}^*B| \leq \mathsf{m}^*(A\triangle B)$.
   - (c) Suppose that A and B are non-empty and $d(A,B)>0$. Show that $\mathsf{m}^*(A\cup B) = \mathsf{m}^*(A) + \mathsf{m}^*(B)$.
5. Let $(A_n)_{n=1}^{\infty}$ be an increasing sequence of subsets of $\mathbb{R}$. Show that $\lim_{n\to\infty} \mathsf{m}^*A_n) = \mathsf{m}^*\Big(\bigcup_{n=1}^{\infty} A_n\Big)$.
6. Let A be a measurable subset of $\mathbb{R}$ with $\mathsf{m}(A)<\infty$. Show that for every $\epsilon>0$, there exists an open subset G of $\mathbb{R}$, which is the union of a finite family of open intervals, such that $\mathsf{m}(A\triangle G)<\epsilon$.
7. (a) Give an example of an open subset G of $\mathbb{R}$ such that $\mathsf{m}(G) < \mathsf{m}(G^-) < \infty$, where G^- denotes the closure of G.
   - (b) Give an example of two open subsets G_1 and G_2 of $\mathbb{R}$ with $G_1 \subsetneq G_2$ such that $\mathsf{m}(G_1) = \mathsf{m}(G_2) < \infty$.
8. Show that for every subset S of $\mathbb{R}$, we have

$$\mathsf{m}^*(S) = \inf\{\ell(G) : G \text{ is a non-empty open subset of } \mathbb{R} \text{ and } S\subseteq G\}.$$

9. Let $\{I_\alpha\}_{\alpha\in\Lambda}$ be a family of non-degenerate intervals. Show that $\bigcup_{\alpha\in\Lambda} I_\alpha$ is measurable.
10. Let $(A_n)_{n=1}^{\infty}$ be a sequence of measurable subsets of $\mathbb{R}$. Show that
    - (a) $\mathsf{m}\Big(\liminf_{n\to\infty} A_n\Big) \leq \liminf_{n\to\infty} \mathsf{m}(A_n)$,
    - (b) $\mathsf{m}\Big(\limsup_{n\to\infty} A_n\Big) \geq \limsup_{n\to\infty} \mathsf{m}(A_n)$ if there exists $n_0\in\mathbb{Z}^+$ such that $\mathsf{m}\Big(\bigcup_{n=n_0}^{\infty} A_n\Big) < \infty$.

11. Let A be a measurable subset of $\mathbb{R}$ with $\mathsf{m}(A) > 0$.
    (a) Show that for every $c \in (0,1)$, there exists an open interval I such that $\mathsf{m}(A \cap I)) > c \cdot \ell(I)$.
    (b) Show that there exists $\delta > 0$ such that for every $x \in (-\delta, \delta)$, the set $(A + x) \cap A$ is non-empty.
12. Let A be a measurable subset of $\mathbb{R}$. Suppose that for every $x \in \mathbb{R} \setminus \{0\}$, we have $\mathsf{m}\big(A \cap (A + x)\big) = 0$. Show that $\mathsf{m}(A) = 0$.
13. Let A be a bounded subset of $\mathbb{R}$. Suppose there exists a bounded interval I with $A \subseteq I$ such that $\ell(I) = \mathsf{m}^*(A) + \mathsf{m}^*(I \setminus A)$. Show that A is measurable.
14. (a) Denote m_* to be the function from $\mathcal{B}(\mathbb{R})$ into $\mathbb{R}$ given by

$$\mathsf{m}_*(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \sup A - \inf A - \mathsf{m}^*\big([\inf A, \sup A] \setminus A\big) & \text{if } A \neq \emptyset, \end{cases}$$

    where $\mathcal{B}(\mathbb{R})$ denotes the set of all bounded subsets of $\mathbb{R}$.
    Show that if a and b are real numbers such that $A \subseteq [a,b]$, then $\mathsf{m}_*(A) = b - a - \mathsf{m}^*\big([a,b] \setminus A\big)$.
    (b) Show that a subset A of $\mathbb{R}$ is measurable if and only if for every $n \in \mathbb{Z}^+$, we have $\mathsf{m}^*\big(A \cap [-n,n]\big) = \mathsf{m}_*\big(A \cap [-n,n]\big)$.

1.5 An Uncountable Null Set

In this section, we give an example of an uncountable subset of $\mathbb{R}$ that has measure zero. First we generalize the concept of decimal expansions of numbers in $[0,1)$.

Notation Let p be an integer greater than 1. Denote by $\mathcal{S}_p$ the set of all sequences $(a_n)_{n=1}^\infty$ of integers such that $0 \leq a_n \leq p-1$ for every $n \in \mathbb{Z}^+$ and that $a_n < p-1$ for infinitely many $n \in \mathbb{Z}^+$.

Proposition 1.5.1 *Let p be an integer greater than 1. Then for every real number $x \in [0,1)$, there exists a unique $(a_n)_{n=1}^\infty \in \mathcal{S}_p$ such that*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

Idea For existence, divide $[0,1)$ into p subintervals (of equal lengths) in the form $[s,t)$ to get a_1. Repeat the process to get the remaining a_n's. For uniqueness, use $a_n < p-1$ for infinitely many n.

Proof Let $x \in [0,1)$.

(*Existence*) First we construct a sequence $(a_n)_{n=1}^\infty$ in $\mathbb{Z}_p$ by induction, where $\mathbb{Z}_p = \{n \in \mathbb{Z} : 0 \leq n \leq p-1\}$. Then we show that $x = \sum\limits_{n=1}^{\infty} \frac{a_n}{p^n}$ and that the sequence $(a_n)_{n=1}^\infty$ belongs to $\mathcal{S}_p$.

(1) Note that $[0,1)$ is the union of the disjoint family $\left\{\left[\frac{k}{p}, \frac{k+1}{p}\right)\right\}_{k\in\mathbb{Z}_p}$ of intervals. We define a_1 to be the unique integer in $\mathbb{Z}_p$ such that

$$x \in \left[\tfrac{a_1}{p}, \tfrac{a_1+1}{p}\right).$$

(2) Suppose $a_1, \ldots, a_n \in \mathbb{Z}_p$ have been defined such that

$$x \in \left[\sum_{i=1}^{n} \frac{a_i}{p^i}, \sum_{i=1}^{n} \frac{a_i}{p^i} + \frac{1}{p^n}\right).$$

We define a_{n+1} to be the unique integer in $\mathbb{Z}_p$ such that

$$x \in \left[\sum_{i=1}^{n+1} \frac{a_i}{p^i}, \sum_{i=1}^{n+1} \frac{a_i}{p^i} + \frac{1}{p^{n+1}}\right).$$

It is clear from the construction of the sequence $(a_n)_{n=1}^{\infty}$ that,

$$\text{for every } n \in \mathbb{Z}^+, \quad x \in \left[\sum_{i=1}^{n} \frac{a_i}{p^i}, \sum_{i=1}^{n} \frac{a_i}{p^i} + \frac{1}{p^n}\right). \tag{1.60}$$

Hence we have $x = \displaystyle\sum_{n=1}^{\infty} \frac{a_n}{p^n}$.

Next, we want to show that $(a_n)_{n=1}^{\infty} \in \mathcal{S}_p$. By construction, for every $n \in \mathbb{Z}^+$, we have $0 \le a_n \le p-1$. Now suppose $a_n < p-1$ for only finitely many $n \in \mathbb{Z}^+$, that is, the set $\{n \in \mathbb{Z}^+ : a_n < p-1\}$ is finite. We want to obtain a contradiction.

Denote n_0 to be the positive integer given by

$$n_0 = \begin{cases} 1 & \text{if } \{n \in \mathbb{Z}^+ : a_n < p-1\} = \emptyset. \\ \max\{n \in \mathbb{Z}^+ : a_n < p-1\} & \text{if } \{n \in \mathbb{Z}^+ : a_n < p-1\} \neq \emptyset. \end{cases}$$

It follows from the construction of n_0 that, for every positive integer $n > n_0$, we have $a_n = p-1$ (since $0 \le a_n \le p-1$), from which we obtain

$$\begin{aligned} x &= \sum_{i=1}^{n_0} \frac{a_i}{p^i} + \sum_{i=n_0+1}^{\infty} \frac{p-1}{p^i} \\ &= \sum_{i=1}^{n_0} \frac{a_i}{p^i} + \frac{1}{p^{n_0}} \end{aligned}$$

which contradicts (1.60).

(*Uniqueness*) Suppose $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ belong to $\mathcal{S}_p$ and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n} = \sum_{n=1}^{\infty} \frac{b_n}{p^n}. \tag{1.61}$$

We want to show that for every $n \in \mathbb{Z}^+$, we have $a_n = b_n$.
Suppose not, that is, the set $\{k \in \mathbb{Z}^+ : a_k \neq b_k\}$ is non-empty. Denote $n_0 = \min\{k \in \mathbb{Z}^+ : a_k \neq b_k\}$. By symmetry, we may assume that $a_{n_0} > b_{n_0}$, that is,

$$a_{n_0} \geq b_{n_0} + 1. \tag{1.62}$$

Since $(b_n)_{n=1}^{\infty} \in \mathcal{S}_p$, it follows that

$$\sum_{n=n_0+1}^{\infty} \frac{b_n}{p^n} < \sum_{n=n_0+1}^{\infty} \frac{p-1}{p^n}. \tag{1.63}$$

By the construction of n_0 and (1.63) and (1.62), we get

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_n}{p^n} - \sum_{n=1}^{\infty} \frac{b_n}{p^n} &= \frac{a_{n_0}}{p^{n_0}} + \sum_{n=n_0+1}^{\infty} \frac{a_n}{p^n} - \frac{b_{n_0}}{p^{n_0}} - \sum_{n=n_0+1}^{\infty} \frac{b_n}{p^n} \\ &\geq \frac{a_{n_0} - b_{n_0}}{p^{n_0}} - \sum_{n=n_0+1}^{\infty} \frac{b_n}{p^n} \\ &> \frac{a_{n_0} - b_{n_0}}{p^{n_0}} - \sum_{n=n_0+1}^{\infty} \frac{p-1}{p^n} \\ &= \frac{a_{n_0} - b_{n_0}}{p^{n_0}} - \frac{1}{p^{n_0}} \\ &\geq 0, \end{aligned}$$

which contradicts (1.61).

□

Terminology Let p be an integer greater than 1. For every $x \in [0, 1)$, the unique sequence $(a_n)_{n=1}^{\infty} \in \mathcal{S}_p$ such that $x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}$ is called the *canonical p-nary expansion* of x. Traditionally, we write

$$x = 0.a_1 a_2 a_3 \ldots \quad \text{(base } p\text{)}.$$

For $p = 2$, canonical 2-nary expansion is called canonical *binary* expansion and for $p = 3$, canonical 3-nary expansion is called canonical *ternary* expansion.

Remark Let p be a positive integer greater than 1. Then for every $(a_n)_{n=1}^{\infty} \in \mathcal{S}_p$, we have

$$0 \leq \sum_{n=1}^{\infty} \frac{a_n}{p^n} < 1.$$

Moreover, the canonical p-nary expansion of the number $\sum_{n=1}^{\infty} \frac{a_n}{p^n}$ is $(a_n)_{n=1}^{\infty}$.

In the example below, we will construct a subset of $[0, 1]$, called the *Cantor ternary set*, having the following properties:

- the measure of the Cantor ternary set is zero;
- the Cantor ternary set is uncountable.

The Cantor ternary set is obtained by removing from $[0,1]$ countably many open intervals such that the sum of the lengths of the open intervals equals 1. The intervals to be removed have the property that if x is an element belonging to any one of the intervals, then the canonical ternary expansion of x has at least one term equals 1 (hence if $x \in [0,1)$ and all the terms in the canonical ternary expansion of x are 0 or 2, then x belongs to the Cantor ternary set).

First, we described how to obtain the Cantor ternary set geometrically.

Denote $I = [0,1]$.

- Denote the open interval in the middle third of I by $G_{1,1}$, that is,
$$G_{1,1} = \left(\tfrac{1}{3}, \tfrac{2}{3}\right).$$
The complement of $G_{1,1}$ in I is the union of two disjoint closed intervals, denoted by $I_{1,1}$ and $I_{1,2}$ where
$$I_{1,1} = \left[0, \tfrac{1}{3}\right] \quad \text{and} \quad I_{1,2} = \left[\tfrac{2}{3}, 1\right].$$
- Denote the open interval in the middle third of $I_{1,1}$ by $G_{2,1}$ and that in the middle third of $I_{1,2}$ by $G_{2,2}$, that is,
$$G_{2,1} = \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \quad \text{and} \quad G_{2,2} = \left(\tfrac{7}{9}, \tfrac{8}{9}\right).$$
The complement of $G_{2,1}$ in $I_{1,1}$ is the union of two disjoint closed intervals, denoted by $I_{2,1}$ and $I_{2,2}$ and the complement of $G_{2,2}$ in $I_{1,2}$ is the union of two disjoint closed intervals, denoted by $I_{2,3}$ and $I_{2,4}$, where
$$I_{2,1} = \left[0, \tfrac{1}{9}\right], \quad I_{2,2} = \left[\tfrac{2}{9}, \tfrac{1}{3}\right], \quad I_{2,3} = \left[\tfrac{2}{3}, \tfrac{7}{9}\right], \quad \text{and} \quad I_{2,4} = \left[\tfrac{8}{9}, 1\right].$$
- Repeating the above process, we get a family of open intervals $\{G_{n,i}\}_{1 \le i \le 2^{n-1},\, n \in \mathbb{Z}^+}$.

The *Cantor ternary set*, is defined to be the complement in I of the union of the $G_{n,i}$'s, that is,
$$I \setminus \Big(\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{2^{n-1}} G_{n,i} \Big).$$

In the example below, we will write down the family of open intervals explicitly. Instead of using $1 \le i \le 2^{n-1}$, we consider i belonging to the set of all finite sequences $(\alpha_k)_{i=1}^{n-1}$ with index set $\{1, \ldots, n-1\}$ such that $\alpha_k = 0$ or 2 for every $k = 1, \ldots, n-1$. Readers are suggested to check that the family of open intervals $\{G_{n,i}\}_{1 \le i \le 2^{n-1},\, n \in \mathbb{Z}^+}$ that we describe geometrically is the same as the family $\{G_{n,i}\}_{i \in \Lambda_n, n \in \mathbb{Z}^+}$ constructed in the example below.

Example Denote $\Lambda_1 = \{1\}$. For each positive integer $n \geq 2$, denote Λ_n to be the set of all finite sequences $(\alpha_k)_{k=1}^{n-1}$ in $\{0, 2\}$ with index set $\{1, \ldots, n-1\}$. Denote $\{G_{n,i}\}_{i\in\Lambda_n, n\in\mathbb{Z}^+}$ to be the family of open intervals given by

$$\begin{aligned} G_{1,1} &= \left(\frac{1}{3}, \frac{2}{3}\right) \\ G_{n,i} &= \left(\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n}, \sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{2}{3^n}\right) \quad \text{where } i = (\alpha_k)_{k=1}^{n-1} \in \Lambda_n, \quad n \geq 2. \end{aligned}$$

The *Cantor ternary set* is defined to be the complement in $[0, 1]$ of the union of the family $\{G_{n,i}\}_{i\in\Lambda_n, n\in\mathbb{Z}^+}$, that is,

$$[0, 1] \setminus \bigcup_{n\in\mathbb{Z}^+} \bigcup_{i\in\Lambda_n} G_{n,i}.$$

Before considering the main results on the Cantor ternary set, we give some properties of the family $\{G_{n,i}\}_{i\in\Lambda_n, n\in\mathbb{Z}^+}$. The first two properties are obvious.

(P1) For every $n \in \mathbb{Z}^+$, the number of elements in Λ_n is 2^{n-1}.

(P2) For every $n \in \mathbb{Z}^+$ and for every $i \in \Lambda_n$, we have $\mathsf{m}(G_{n,i}) = 3^{-n}$.

(P3) The family $\{G_{n,i}\}_{i\in\Lambda_n, n\in\mathbb{Z}^+}$ is a disjoint family (of measurable subsets of $[0, 1]$).

Idea If I_1, I_2 belong to the family and $I_1 \neq I_2$, then either $b_1 < a_2$ or $b_2 < a_1$ where a_j and b_j are the left and right endpoints of I_j respectively.

Proof Suppose $(m, i) \neq (n, j)$ where $m, n \in \mathbb{Z}^+$, $i \in \Lambda_m$ and $j \in \Lambda_n$. We want to show that

$$G_{m,i} \cap G_{n,j} = \emptyset. \tag{1.64}$$

For this, we consider the following two cases (with subcases and subsubcases):

(*Case 1*) $m = 1$ or $n = 1$

To show that (1.64) holds, by symmetry, we may assume that $m = 1$. In this case, we have

$$G_{m,i} = \left(\frac{1}{3}, \frac{2}{3}\right).$$

Moreover, since Λ_1 has only one element and $(m, i) \neq (n, j)$, it follows that $n \geq 2$ and so

$$G_{n,j} = \left(\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n}, \sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{2}{3^n}\right)$$

where $j = (\alpha_k)_{k=1}^{n-1} \in \Lambda_n$.

(*Subcase 1.1*) $\alpha_1 = 0$

In this subcase, we have

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{2}{3^n} \quad < \quad \frac{1}{3}$$

and so $G_{m,i} \cap G_{n,j} = \emptyset$.

(*Subcase 1.2*) $\alpha_1 = 2$

In this subcase, we have

$$\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n} \quad > \quad \frac{2}{3}$$

and so $G_{m,i} \cap G_{n,j} = \emptyset$.

(*Case 2*) $m \geq 2$ and $n \geq 2$

In this case, we have

$$G_{m,i} \quad = \quad \left(\sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{1}{3^m}, \sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{2}{3^m}\right)$$

$$G_{n,j} \quad = \quad \left(\sum_{k=1}^{n-1} \frac{\beta_k}{3^k} + \frac{1}{3^n}, \sum_{k=1}^{n-1} \frac{\beta_k}{3^k} + \frac{2}{3^n}\right)$$

where $i = (\alpha_k)_{k=1}^{m-1} \in \Lambda_m$ and $j = (\beta_k)_{k=1}^{n-1} \in \Lambda_n$.

To show that (1.64) holds, by symmetry, we may assume that $m \leq n$.

(*Subcase 2.1*) $\{k \in \mathbb{Z} : 1 \leq k \leq m-1 \text{ and } \alpha_k \neq \beta_k\} = \emptyset$

In this subcase, we have

$$\text{for every } k = 1, \ldots, m-1, \quad \alpha_k = \beta_k.$$

Since $(m, i) \neq (n, j)$, it follows that $m < n$.

(*Subsubcase 2.1.1*) $\beta_m = 0$

In this subsubcase, we have

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{\beta_k}{3^k} + \frac{2}{3^n} \quad &= \quad \begin{cases} \displaystyle\sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{0}{3^m} + \frac{2}{3^n} & \text{if } n-1 = m, \\ \displaystyle\sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{0}{3^m} + \sum_{k=m+1}^{n-1} \frac{\beta_k}{3^k} + \frac{2}{3^n} & \text{if } n-1 > m \end{cases} \\ &< \quad \sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{1}{3^m} \end{aligned}$$

and so $G_{m,i} \cap G_{n,j} = \emptyset$.

(*Subsubcase 2.1.2*) $\beta_m = 2$

In this subsubcase, using similar argument as in Subsubcase 2.1.1, we see that

$$\sum_{k=1}^{n-1} \frac{\beta_k}{3^k} + \frac{1}{3^n} \quad > \quad \sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{2}{3^m}$$

and so $G_{m,i} \cap G_{n,j} = \emptyset$.

(*Subcase 2.2*) $\{k \in \mathbb{Z} : 1 \le k \le m-1 \text{ and } \alpha_k \ne \beta_k\} \ne \emptyset$

In this subcase, denote $k_0 = \min\{k \in \mathbb{Z} : 1 \le k \le m-1 \text{ and } \alpha_k \ne \beta_k\}$

(*Subsubcase 2.2.1*) $\alpha_{k_0} = 2$ and $\beta_{k_0} = 0$

In this subsubcase, it is straightforward to check that

$$\sum_{k=1}^{n-1} \frac{\beta_k}{3^k} + \frac{2}{3^n} \quad < \quad \sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{1}{3^m}$$

and so $G_{m,i} \cap G_{n,j} = \emptyset$.

(*Subsubcase 2.2.2*) $\alpha_{k_0} = 0$ and $\beta_{k_0} = 2$

In this subsubcase, it is straightforward to check that

$$\sum_{k=1}^{m-1} \frac{\alpha_k}{3^k} + \frac{2}{3^m} < \sum_{k=1}^{n-1} \frac{\beta_k}{3^k} + \frac{1}{3^n}$$

and so $G_{m,i} \cap G_{n,j} = \emptyset$.

□

(P4.1) Suppose $x \in G_{1,1}$ and $(b_k)_{k=1}^{\infty}$ is the canonical ternary expansion of x. Then we have $b_1 = 1$.

Proof Note that $G_{1,1} = \left(\frac{1}{3}, \frac{2}{3}\right)$. The result follows from the proof of (the existence part of) Proposition 1.5.1. Alternatively, we can use the method given in the proof of (P4.2) below.

□

(P4.2) Suppose $x \in G_{n,i}$ where $n \ge 2$ and $i = (\alpha_k)_{k=1}^{n-1} \in \Lambda_n$ and $(b_k)_{k=1}^{\infty}$ is the canonical ternary expansion of x. Then we have $b_n = 1$ and for every $k = 1, \dots, n-1$, we have $b_k = \alpha_k$.

Idea Use uniqueness of canonical ternary expansion.

Proof By (the uniqueness part of) Proposition 1.5.1, it suffices to show that there exists $(c_n)_{n=1}^{\infty} \in \mathcal{S}_3$ with $c_n = 1$ and for every $k = 1, \dots, n-1$, $c_k = \alpha_k$, such that $x = \sum_{k=1}^{\infty} \frac{c_k}{3^k}$. Denote y to be the real number given by

$$y = x - \left(\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n} \right). \tag{1.65}$$

Since $x \in G_{n,i} = \left(\sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n}, \sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{2}{3^n} \right)$, it follows that

$$0 < y < \frac{1}{3^n}.$$

Denote $(d_k)_{k=1}^{\infty}$ to be the canonical ternary expansion of the number $3^n y$. Then we have

$$3^n y = \sum_{k=1}^{\infty} \frac{d_k}{3^k}. \tag{1.66}$$

Denote $(c_k)_{k=1}^{\infty}$ be the sequence given by

$$\begin{aligned} c_k &= \alpha_k \quad (1 \leq k \leq n-1), \\ c_n &= 1, \\ c_k &= d_{k-n} \quad (k > n). \end{aligned}$$

We want to show that the sequence $(c_k)_{k=1}^{\infty}$ satisfies the following two conditions:

(1) $(c_k)_{k=1}^{\infty} \in \mathcal{S}_3$

(2) $x = \sum\limits_{k=1}^{\infty} \frac{c_k}{3^k}$

Condition (1) is obvious since $(d_k)_{k=1}^{\infty} \in \mathcal{S}_3$.

Condition (2) follows from the construction of $(c_k)_{k=1}^{\infty}$ and (1.66) and (1.65). Indeed, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{c_k}{3^k} &= \sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n} + \sum_{k=n+1}^{\infty} \frac{d_{k-n}}{3^k} \\ &= \sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n} + \sum_{k=1}^{\infty} \frac{d_k}{3^{k+n}} \\ &= \sum_{k=1}^{n-1} \frac{\alpha_k}{3^k} + \frac{1}{3^n} + y \\ &= x. \end{aligned}$$

□

Result 1 The Cantor ternary set is a null set.

Idea Sum of the lengths of the open intervals in the family $\{G_{n,i}\}_{i \in \Lambda_n, n \in \mathbb{Z}^+}$ is 1.

Proof We want to show that the measure of the Cantor ternary set is 0.

By (P3), using the countable additivity of the Lebesgue measure, and by (P2) and (P1), we have

$$\begin{aligned} \mathsf{m}\Big(\bigcup_{n \in \mathbb{Z}^+} \bigcup_{i \in \Lambda_n} G_{n,i} \Big) &= \sum_{n=1}^{\infty} \sum_{i \in \Lambda_n} \mathsf{m}(G_{n,i}) \\ &= \sum_{n=1}^{\infty} \sum_{i \in \Lambda_n} 3^{-n} \\ &= \sum_{n=1}^{\infty} 2^{n-1} \times 3^{-n} \\ &= 1. \end{aligned}$$

Hence by construction, the measure of the Cantor ternary set is

$$\mathsf{m}[0,1] - \mathsf{m}\Big(\bigcup_{n \in \mathbb{Z}^+} \bigcup_{i \in \Lambda_n} G_{n,i} \Big) = 0$$

□

Result 2 The Cantor ternary set is equipotent to $\mathbb{R}$ (hence the Cantor ternary set is uncountable).

Idea Construct an injective function from $[0, 1)$ into $\mathbb{R}$ such that the range of the function is a subset of the Cantor ternary set.

Proof Denote the Cantor ternary set by C_0. Since $C_0 \subseteq \mathbb{R}$, to show that C_0 is equipotent to $\mathbb{R}$, by the Bernstein Theorem, it suffices to show that there exists an injective function from $\mathbb{R}$ into C_0, or equivalently, that there exists an injective function from $[0, 1)$ into C_0, since $\mathbb{R}$ and $[0, 1)$ are equipotent.

Denote f to be the function from $[0, 1)$ into $\mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n}, \quad \text{where } (a_n)_{n=1}^{\infty} \text{ is the canonical binary expansion of } x.$$

We want to show that the following two conditions are satisfied (hence there exists an injective function from $[0, 1)$ into C_0).

(1) The range of f is a subset of C_0.

(2) The function f is injective

To prove (1) Let $x \in [0, 1)$. We want to show that $\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in C_0$ where $(a_n)_{n=1}^{\infty}$ is the canonical binary expansion of x.

Since $(a_n)_{n=1}^{\infty} \in \mathcal{S}_2$, it follows from definition that $a_n = 0$ for infinitely many $n \in \mathbb{Z}^+$. Hence we have

$$(2a_n)_{n=1}^{\infty} \in \mathcal{S}_3 \quad \text{and} \quad 0 \leq \sum_{n=1}^{\infty} \frac{2a_n}{3^n} < 1.$$

Therefore, the sequence $(2a_n)_{n=1}^{\infty}$ is the canonical ternary expansion of the number $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$.

Since none of the terms in $(2a_n)_{n=1}^{\infty}$ is 1, it follows from (P4.1) and (P4.2) that for every $n \in \mathbb{Z}^+$ and for every $i \in \Lambda_n$, the number $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ does not belong to $G_{n,i}$. Hence we have $\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in C_0$.

To prove (2) Suppose $x_1, x_2 \in [0, 1)$ and $x_1 \neq x_2$. We want to show that $\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \neq \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$, where $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are the canonical binary expansions of x_1 and x_2 respectively.

Since $x_1 \neq x_2$, that is,

$$\sum_{n=1}^{\infty} \frac{a_n}{2^n} \neq \sum_{n=1}^{\infty} \frac{b_n}{2^n},$$

it follows that the sequences $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are not equal (which means that there exists at least one $k \in \mathbb{Z}^+$ such that $a_k \neq b_k$).

Since $(2a_n)_{n=1}^{\infty}$ and $(2b_n)_{n=1}^{\infty}$ are not equal and $(2a_n)_{n=1}^{\infty}$ and $(2b_n)_{n=1}^{\infty}$ are the canonical ternary expansions of the numbers $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ and $\sum_{n=1}^{\infty} \frac{2b_n}{3^n}$ respectively, by (the uniqueness part of) Proposition 1.5.1, we have $\sum_{n=1}^{\infty} \frac{2a_n}{3^n} \neq \sum_{n=1}^{\infty} \frac{2b_n}{3^n}$. □

Exercise 1.5

1. Show that the set of all measurable subsets of $\mathbb{R}$ is equipotent to $\mathcal{P}(\mathbb{R})$.
2. Show that the Cantor ternary set is perfect.
3. Let r be a positive real number. Construct a perfect subset F of $\mathbb{R}$ such that $\mathsf{m}(F) = r$.
4. Give an example of a measurable subset A of $\mathbb{R}$ such that for every open interval (a, b), the measures of both $(a, b) \cap A$ and $(a, b) \setminus A$ are positive.

1.6 A Nonmeasurable Set

Since the Lebesgue outer measure is not additive, it follows that there exist non-measurable subsets of $\mathbb{R}$. In fact, we have the following

Result The set X constructed in Section 1.3 is not measurable.

Idea If X is measurable, then all the X_n's are measurable.

Proof Suppose X is measurable. Then the sequence of sets $(X_n)_{n=1}^{\infty}$ constructed in Section 1.3 is a sequence of measurable sets. This is because for every $r \in [0, 1)$,

$$X \overset{\circ}{+} r = \Big(\big(X \cap [0, 1-r)\big) + r\Big) \cup \Big(\big(X \cap [1-r, 1)\big) + r - 1\Big)$$

and both sets $\big(X \cap [0, 1-r)\big) + r$ and $\big(X \cap [1-r, 1)\big) + r - 1$ are measurable.

By the countable additivity of the Lebesgue measure, we have

$$\mathsf{m}\Big(\bigcup_{n=1}^{\infty} X_n\Big) = \sum_{n=1}^{\infty} \mathsf{m}(X_n),$$

which is a contradiction because

- $\mathsf{m}\Big(\bigcup_{n=1}^{\infty} X_n\Big) = \mathsf{m}[0, 1) = 1;$
- $\sum_{n=1}^{\infty} \mathsf{m}(X_n) = \sum_{n=1}^{\infty} \mathsf{m}(X) = \infty.$

□

Exercise 1.6

1. In this exercise, X is the non-measurable set constructed in Section 1.3.
   (*a*) Give an example of a subset S of $\mathbb{R}$ such that $\mathsf{m}^*(S) < \mathsf{m}^*(S \cap X) + \mathsf{m}^*(S \cap X^c)$.
   (*b*) Suppose A is a subset of X and A is measurable. Show that $\mathsf{m}(A) = 0$.
2. Let A be a subset of $\mathbb{R}$ with $\mathsf{m}^*(A) > 0$. Show that there exists a non-measurable set B such that $B \subseteq A$.

Chapter 2

Lebesgue Measurable Functions

2.1 Introduction

The aim of this chapter is to introduce the concept of *measurable functions* and to study their properties. Our goal is to obtain a collection of "nice" functions (which will be called *measurable functions*) such that if f is a bounded "nice" function defined on a bounded interval (or a measurable subset of $\mathbb{R}$ with finite measure), then the "integral" of f exists.

To consider integrals, the simplest way is to look at functions that take finitely many values only. Suppose f is a real-valued function defined on a (non-empty) bounded subset S of $\mathbb{R}$ such that the range of f is a finite set. Then there exist a disjoint finite family $\{S_i\}_{1\leq i\leq n}$ of subsets of S and a distinct finite family $\{c_i\}_{1\leq i\leq n}$ of real numbers such that

$$\text{for every } i = 1, \ldots, n, \quad \text{for every } x \in S_i, \quad f(x) = c_i.$$

It is natural to define the "integral" of f to be the value

$$\sum_{i=1}^{n} c_i \times \mathsf{m}(S_i), \tag{2.1}$$

provided that $S_1, \ldots, S_n$ are measurable.

From the above discussion, it seems reasonable that a real-valued function f with $\operatorname{dom}(f) \subseteq \mathbb{R}$ is considered to be "nice" if the following condition is satisfied:

(†) the inverse image of every singleton under f is a measurable subset of $\mathbb{R}$, that is, for every $y \in \mathbb{R}$, the set $\{x \in \operatorname{dom}(f) : f(x) = y\}$ is measurable.

However, there are many "weird" functions satisfying (†). In fact, if f is an injective function, then (†) is satisfied.

The Riemann integral over a closed interval $[a, b]$ of a continuous function $f : [a, b] \longrightarrow \mathbb{R}$ can be approximated by the sum

$$\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}),$$

provided that $\max\limits_{1 \le i \le n} (x_i - x_{i-1})$ is small, where $a = x_0 < x_1 < \cdots < x_n = b$ and $t_i \in (x_{i-1}, x_i)$ for $i = 1, \ldots, n$. The value $\sum\limits_{i=1}^{n} f(t_i)(x_i - x_{i-1})$ can be considered as the integral of the function $g : [a, b] \longrightarrow \mathbb{R}$ given by (for example)

$$g(x) = \begin{cases} f(t_i) & \text{if } x \in (x_{i-1}, x_i),\ i = 1, \ldots, n, \\ 0 & \text{otherwise.} \end{cases}$$

Such a function is called a step function.

Definition A real-valued function f is called a *step function* if the domain of f is a non-degenerate interval and there exists a subdivision $(x_i)_{i=0}^{n}$ of $\operatorname{dom}(f)$, such that for every $i = 1, \ldots, n$, the function f is constant on the open interval (x_{i-1}, x_i).

Replacing intervals by measurable subsets of $\mathbb{R}$, we have the following generalization of step functions.

Definition A real-valued function f is called a *simple function* if the domain of f is a non-empty subset of $\mathbb{R}$ and there exists a disjoint finite family $\{A_i\}_{i \in \Lambda}$ of subsets of $\operatorname{dom}(f)$ with $\bigcup\limits_{i \in \Lambda} A_i = \operatorname{dom}(f)$ such that for every $i \in \Lambda$, the set A_i is measurable and the function f is constant on A_i.

Remark The domain of every simple function is a measurable subset of $\mathbb{R}$.

Since the Riemann integral of a continuous function can be approximated by the integrals of step functions, we would like to extend the idea of “integral” so that the “integral” of a “nice” function can be approximated by “integrals” of simple functions, that is, sums in the form given in (2.1).

- For a continuous function $f : [a, b] \longrightarrow \mathbb{R}$, approximations of f by step functions are obtained by dividing its domain $[a, b]$ into finitely many subintervals. The oscillation of f on each of the subintervals is small if the lengths of the subintervals are small. This is guaranteed by the uniform continuity of f.
- For a bounded function f, to get small oscillations, instead of dividing its domain, we can divide its range into finitely many “small” (in the context of diameters) subsets. For simplicity, suppose f is a function with range contained in $[0, 1)$. Let n be a (large) positive integer. Denote $\{A_i\}_{1 \le i \le n}$ to be the finite family of subsets of $\operatorname{dom}(f)$ given by

$$A_i = \left\{x \in \operatorname{dom}(f) : \tfrac{i-1}{n} \le f(x) < \tfrac{i}{n}\right\} \qquad (i = 1, \ldots, n).$$

By construction, the family $\{A_i\}_{1\leq i\leq n}$ is disjoint and $\bigcup_{i=1}^{n} A_i = \mathrm{dom}\,(f)$. Moreover, for every $i = 1, \ldots, n$, the oscillation of f in A_i is at most $\frac{1}{n}$. So we can use the sums

$$\sum_{i=1}^{n} f(t_i) \times \mathsf{m}(A_i)$$

where $t_i \in A_i$ for $i = 1, \ldots, n$, to approximate the integral of f, provided all the A_i's are measurable.

From the above discussion, it seems reasonable that a function $f : \mathrm{dom}\,(f) \longrightarrow \mathbb{R}$ is considered to be "nice" if the following condition is satisfied:

(‡) the inverse image of every interval in the form $[a, b)$ under f is a measurable subset of $\mathbb{R}$, that is, for every pair a, b of real numbers with $a < b$, the set $\{x \in \mathrm{dom}\,(f) : a \leq f(x) < b\}$ is measurable.

Intervals in the form $[a, b)$ given in (‡) are quite artificial. In fact, they can be replaced by ones in the form (a, ∞). This is because the collection of all measurable subsets of $\mathbb{R}$ is closed under taking complement, countable union and countable intersection.

To close this section, we consider simple functions with "simple" range. The "simplest" simple functions are constant functions defined on (non-empty) measurable subsets of $\mathbb{R}$. The "next simplest" simple functions are those which takes two values only. Let f be a simple function with range equals to $\{0, 1\}$. Denote $A = \{x \in \mathrm{dom}\,(f) : f(x) = 1\}$. Then the function f is given by

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \mathrm{dom}\,(f) \setminus A. \end{cases}$$

In general, we have the following:

Definition Let A be a subset of a non-empty set X. The *characteristic function* of A on X, denoted by $\chi_{A,X}$, is the function from X into $\mathbb{R}$ given by

$$\chi_{A,X}(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in X \setminus A. \end{cases}$$

Alternatively, instead of $\chi_{A,X}$, we also write $\chi_A : X \longrightarrow \mathbb{R}$ to indicate that the function is defined on X. If the set X is understood, we simply say the *characteristic function* of A and write χ_A.

Remark

- If $A = X$ or $A = \emptyset$, then $\chi_{A,X}$ is a constant function.
- The domain of a characteristic function may not be a subset of $\mathbb{R}$.

- If X is a non-empty subset of $\mathbb{R}$, then $\chi_{A,X}$ is a simple function if and only if both sets X and A are measurable (exercise).
- Every simple function f can be expressed as a linear combination of characteristic functions $\chi_{A_i,\mathrm{dom}\,(f)}$ where each A_i is a measurable subset of $\mathrm{dom}\,(f)$. Indeed, we have

$$f = \sum_{i=1}^{n} a_i \cdot \chi_{A_i,\mathrm{dom}\,(f)},$$

  where $a_1, \ldots, a_n$ are the distinct values of the range of f and $A_i = \{x \in \mathrm{dom}\,(f) : f(x) = a_i\}$ for $i = 1, \ldots, n$.
- Given a simple function f, there are more than one ways to express f as linear combinations of characteristic functions.

  Example Denote $f : [0,1] \longrightarrow \mathbb{R}$ to be the Dirichlet function defined on $[0,1]$. Since both $[0,1] \cap \mathbb{Q}$ and $[0,1] \setminus \mathbb{Q}$ are measurable subsets of $\mathbb{R}$, it follows that f is a simple function. We can express f in the following ways (for example):

  ◇ $f = \chi_{[0,1]\cap\mathbb{Q}}$;

  ◇ $f = \chi_{[0,\frac{1}{2}]\cap\mathbb{Q}} + \chi_{(\frac{1}{2},1]\cap\mathbb{Q}}$;

  ◇ $f = \frac{1}{2} \cdot \chi_{[0,\frac{1}{2}]\cap\mathbb{Q}} + \frac{1}{2} \cdot \chi_{(\frac{1}{2},1]\cap\mathbb{Q}} + \frac{1}{2} \cdot \chi_{[0,1]\cap\mathbb{Q}}$;

  where the domains of the above characteristic functions are understood to be $[0,1]$.

Notation Let X be a non-empty set. We denote 0_X and 1_X to be the functions from X into $\mathbb{R}$ given by

$$0_X = \chi_{\emptyset,X} \qquad \text{and} \quad 1_X = \chi_{X,X}.$$

Exercise 2.1

1. Let f and g be step functions having the same domain. Show that the functions $f+g$, fg, $|f|$, $\max\{f,g\}$ and $\min\{f,g\}$ are steps functions.
2. Let f and g be simple functions having the same domain. Show that the functions $f+g$, fg, $|f|$, $\max\{f,g\}$ and $\min\{f,g\}$ are simple functions.
3. Let X be a non-empty subset of $\mathbb{R}$ and let A be a subset of X. Show that the characteristic function $\chi_{A,X}$ of A on X is a simple function if and only if both X and A are measurable.
4. Let A and B be subsets of a non-empty set X. Show that

   (a) $\chi_{A\cap B} = \chi_A \cdot \chi_B = \min\{\chi_A, \chi_B\}$;

   (b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B = \max\{\chi_A, \chi_B\}$;

   (c) $\chi_{X\setminus A} = 1 - \chi_A$,

   where the domains of the characteristic functions are understood to be X.

5. Let $(A_n)_{n=1}^{\infty}$ be a sequence of subsets of a non-empty set X. Show that
   (a) $\limsup\limits_{n\to\infty} \chi_{A_n} = \chi_{\limsup\limits_{n\to\infty} A_n}$;
   (b) $\liminf\limits_{n\to\infty} \chi_{A_n} = \chi_{\liminf\limits_{n\to\infty} A_n}$,
   where the domains of the characteristic functions are understood to be X.

2.2 Measurable Functions

Definition A real-valued function f is said to be *Lebesgue measurable*, or simply *measurable*, if the domain of f is a (non-empty) subset of $\mathbb{R}$ and for every real number a, the set

$$\{x \in \mathrm{dom}\,(f) : f(x) > a\}$$

is measurable.

Terminology For simplicity, "(Lebesgue) measurable real-valued functions" will be called "(Lebesgue) measurable functions".

Proposition 2.2.1 *The domain of every measurable function is a (non-empty) measurable subset of* $\mathbb{R}$.

Idea Apply definition. Use $\mathbb{R} = \bigcup\limits_{n=1}^{\infty} (-n, \infty)$.

Proof Let f be a measurable function. By definition, the sequence of sets

$$\big(\{x \in \mathrm{dom}\,(f) : f(x) > -n\}\big)_{n=1}^{\infty}$$

is a sequence of measurable sets. Since

$$\mathrm{dom}\,(f) = \bigcup_{n=1}^{\infty} \{x \in \mathrm{dom}\,(f) : f(x) > -n\},$$

it follows from Theorem 1.4.11 that the set $\mathrm{dom}\,(f)$ is measurable . □

Theorem 2.2.2 below shows that in the definition of measurable function, the inequality $>$ can be replaced by other types of inequalities. For convenience, we introduce the following

Notation Let f be a real-valued function and let a be a real number. We denote

- $[f > a] = \{x \in \mathrm{dom}\,(f) : f(x) > a\}$;
- $[f \geq a] = \{x \in \mathrm{dom}\,(f) : f(x) \geq a\}$;
- $[f < a] = \{x \in \mathrm{dom}\,(f) : f(x) < a\}$;
- $[f \leq a] = \{x \in \mathrm{dom}\,(f) : f(x) \leq a\}$;
- $[f = a] = \{x \in \mathrm{dom}\,(f) : f(x) = a\}$.

Theorem 2.2.2 *Let f a real-valued function defined on a (non-empty) subset of $\mathbb{R}$. Then the following statements are equivalent.*

(1) *For every $a \in \mathbb{R}$, the set $[f > a]$ is measurable.*

(2) *For every $a \in \mathbb{R}$, the set $[f \geq a]$ is measurable.*

(3) *For every $a \in \mathbb{R}$, the set $[f < a]$ is measurable.*

(4) *For every $a \in \mathbb{R}$, the set $[f \leq a]$ is measurable.*

Idea For (1) $\Longrightarrow$ (2), use $[f \geq a] = \bigcap_{n=1}^{\infty} [f > a - \frac{1}{n}]$. For (2) $\Longrightarrow$ (3), use $[f < a] = \operatorname{dom}(f) \setminus [f \geq a]$, noting that (2) implies that $\operatorname{dom}(f)$ is measurable.

Proof

(1) $\Longrightarrow$ (2) Note that for every $a \in \mathbb{R}$, we have

$$[f \geq a] = \bigcap_{n=1}^{\infty} [f > a - \tfrac{1}{n}].$$

The required implication follows from Corollary 1.4.12.

(2) $\Longrightarrow$ (3) First we note that (2) implies that $\operatorname{dom}(f)$ is measurable. This is because

$$\operatorname{dom}(f) = \bigcup_{n=1}^{\infty} [f \geq -n].$$

Also note that for every $a \in \mathbb{R}$, we have

$$[f < a] = \operatorname{dom}(f) \setminus [f \geq a].$$

The required implication follows from Proposition 1.4.2.

(3) $\Longrightarrow$ (4) The proof is similar to that for (1) $\Longrightarrow$ (2).

(4) $\Longrightarrow$ (1) The proof is similar to that for (2) $\Longrightarrow$ (3).

□

Corollary 2.2.3 *Let f be a measurable function. Then for every $a \in \mathbb{R}$, the set $[f = a]$ is measurable.*

Idea Use $[f = a] = [f \geq a] \cap [f \leq a]$.

Proof Note that

$$[f = a] = [f \geq a] \cap [f \leq a]$$

The required result follows from Theorem 2.2.2 and Corollary 1.4.12. □

The converse of Corollary 2.2.3 is not true in general (see exercise). However, it is true if the range of f is countable.

Proposition 2.2.4 *Let f be a real-valued function defined on a (non-empty) subset of $\mathbb{R}$. Suppose that the range of f is countable and that for every real number a, the set $[f = a]$ is measurable. Then the function f is measurable.*

Idea If $[f > a]$ is non-empty, then it can be written as the union of countably many sets in the form $[f = y_i]$.

Proof Let $a \in \mathbb{R}$. We want to show that the set $[f > a]$ is measurable. For this, we consider the following two cases:

(*Case 1*) $[f > a] = \emptyset$

In this case, the set $[f > a]$ is measurable by Corollary 1.4.3.

(*Case 2*) $[f > a] \neq \emptyset$

In this case, the set $\{y \in \text{range}\,(f) : y > a\}$, denoted by Z_a, is non-empty. Since $\big\{[f = y]\big\}_{y \in Z_a}$ is a countable family of measurable sets and

$$[f > a] = \bigcup_{y \in Z_a} [f = y],$$

it follows that the set $[f > a]$ is measurable.

□

Remark Simple functions are measurable.

Proposition 2.2.5 *Let X be a non-empty measurable subset of $\mathbb{R}$. Then every continuous function defined on X is measurable.*

Idea Apply definition. Use $[f > a] = f^{-1}\big((a, \infty)\big)$.

Proof Let f be a continuous function defined on X. To show that f is measurable, we apply definition. Let $a \in \mathbb{R}$. We want to show that the set $[f > a]$ is measurable.

Note that $[f > a] = f^{-1}\big((a, \infty)\big)$. Since f is continuous, by Proposition 0.5.1, there exists an open subset G of $\mathbb{R}$ such that

$$[f > a] = G \cap X.$$

Since both sets G and X are measurable, it follows that the set $[f > a]$ is measurable. □

Proposition 2.2.6 *Let X be a non-empty measurable subset of $\mathbb{R}$ with $\mathsf{m}(X) = 0$. Then every real-valued function defined on X is measurable.*

Idea Apply definition. Subsets of a null set are null sets.

Proof Let f be a real-valued function defined on X. To show that f is measurable, we apply definition. Let $a \in \mathbb{R}$. We want to show that the set $[f > a]$ is measurable.

Since X is a null set and $[f > a] \subseteq X$, it follows that $[f > a]$ is a null set. Hence by Theorem 1.4.5, the set $[f > a]$ is measurable. □

Proposition 2.2.7 *Let f be a measurable function and let A be a non-empty measurable subset of $\mathbb{R}$ with $A \subseteq \operatorname{dom}(f)$. Then the function $f|_A$ is measurable.*

Idea Apply definition. Use $[f|_A > a] = [f > a] \cap A$.

Proof To show that the function $f|_A$ is measurable, we apply definition. Let $a \in \mathbb{R}$. We want to show that the set $[f\big|_A > a]$ is measurable.

Since f is a measurable function, it follows that the set $[f > a]$ is measurable. Since the set A is measurable and

$$[f|_A > a] = [f > a] \cap A,$$

it follows from Corollary 1.4.12 that the set $[f\big|_A > a]$ is measurable. □

Below we will show that the sum and product of two measurable functions are measurable.

Proposition 2.2.8 *Let f and g be measurable functions having the same domain. Then the function $f+g$ is measurable.*

Idea Apply definition. For $a \in \mathbb{R}$, the set $[f+g > a]$ can be written as the union of a countable family of sets in the form $[f > r] \cap [g > a - r]$.

Proof Let $a \in \mathbb{R}$. We want to show that the set $[f+g > a]$ is measurable.

Note that the set $[f+g > a]$ can be written as follows:

$$[f+g > a] = \bigcup_{r \in \mathbb{Q}} \big([f > r] \cap [g > a - r]\big). \tag{2.2}$$

Indeed, if $(f+g)(x) > a$, then $f(x) > a - g(x)$. Since $\mathbb{Q}$ is order dense in $\mathbb{R}$, there exists $s \in \mathbb{Q}$ such that

$$f(x) > s \quad \text{and} \quad s > a - g(x),$$

that is, $x \in [f > s] \cap [g > a - s]$. Hence we have $[f+g > a] \subseteq \bigcup_{r \in \mathbb{Q}} \big([f > r] \cap [g > a - r]\big)$. The reverse inclusion is obvious.

Since f and g are measurable functions, it follows from Lemma 1.4.8 that the family

$$\big\{[f > r] \cap [g > a - r]\big\}_{r \in \mathbb{Q}}$$

is a family of measurable sets. Since $\mathbb{Q}$ is countable, it follows from Theorem 1.4.11 and (2.2) that the set $[f+g > a]$ is measurable. □

Before proving that the product of two measurable functions is measurable, we prove two special cases in the following two lemmas.

Lemma 2.2.9 *Let f be a measurable function. Then the function f^2 is measurable.*

Idea Apply definition. For $a \geq 0$, we have $[f^2 > a] = [f > \sqrt{a}] \cup [f < -\sqrt{a}]$.

Proof Let $a \in \mathbb{R}$. We want to show that the set $[f^2 > a]$ is measurable. For this, we consider the following two cases:

(*Case 1*) $a < 0$

In this case, we have $[f^2 > a] = \text{dom}\,(f)$ which is measurable by Proposition 2.2.1.

(*Case 2*) $a \geq 0$

In this case, we have

$$[f^2 > a] = [f > \sqrt{a}] \cup [f < -\sqrt{a}].$$

Since the function f is measurable, it follows that the sets $[f > \sqrt{a}]$ and $[f < -\sqrt{a}]$ are measurable and so by Theorem 1.4.11, the set $[f^2 > a]$ is measurable.

□

Lemma 2.2.10 *Let f be a measurable function and let c be a real number. Then the function cf is measurable.*

Idea Apply definition. Consider three cases: (i) $c = 0$; (ii) $c > 0$; (iii) $c < 0$.

Proof To prove that the function cf is measurable, we consider the following three cases:

(*Case 1*) $c = 0$,

In this case, the function cf is the function $0_{\text{dom}\,(f)}$. Note that $\text{dom}\,(f)$ is a measurable set. By Proposition 2.2.4, the function cf is measurable.

(*Case 2*) $c > 0$

In this case, for every $a \in \mathbb{R}$, we have

$$[cf > a] = \left[f > \tfrac{a}{c}\right]$$

which is a measurable set since f is a measurable function. Hence the function cf is measurable.

(*Case 3*) $c < 0$

In this case, for every $a \in \mathbb{R}$, we have

$$[cf > a] = \left[f < \tfrac{a}{c}\right]$$

which is a measurable set since f is a measurable function. Hence the function cf is measurable.

□

Corollary 2.2.11 *Let f and g be measurable functions with the same domain. Then the function $f-g$ is measurable.*

Idea Use $f-g=f+(-1)g$.

Proof Note that

$$f-g=f+(-1)g.$$

The required result follows from Lemma 2.2.10 and Proposition 2.2.8. □

Proposition 2.2.12 *Let f and g be measurable functions with the same domain. Then the function fg is measurable.*

Idea Use $fg=\frac{1}{2}\big((f+g)^2-f^2-g^2\big)$.

Proof Note that

$$fg=\frac{1}{2}\big((f+g)^2-f^2-g^2\big).$$

The required result follows from Proposition 2.2.8, Lemma 2.2.9, Corollary 2.2.11 and Lemma 2.2.10. □

Lemma 2.2.13 *Let f be a measurable function. Suppose that for every $x \in \operatorname{dom}(f)$, we have $f(x)\neq 0$. Then the function $1/f$ is measurable.*

Idea Apply definition. For the measurability of $[\frac{1}{f}>a]$, consider three cases: (i) $a=0$; (ii) $a>0$; (iii) $a<0$.

Proof Let $a\in\mathbb{R}$. We want to show that the set $[\frac{1}{f}>a]$ is measurable. For this, we consider the following three cases:

(*Case 1*) $a=0$

In this case, we have

$$[\tfrac{1}{f}>a]=[f>0]$$

which is a measurable set since f is a measurable function.

(*Case 2*) $a>0$

In this case, we have

$$[\tfrac{1}{f}>a]=[f<\tfrac{1}{a}]\cap[f>0]$$

which is a measurable set since f is a measurable function.

(*Case 3*) $a<0$

In this case, we have

$$[\tfrac{1}{f}>a]=[f<\tfrac{1}{a}]\cup[f>0]$$

which is a measurable set since f is a measurable function.

□

Corollary 2.2.14 *Let f and g be measurable functions having the same domain X. Suppose that for every $x \in X$, we have $g(x) \neq 0$. Then the function f/g is measurable.*

Proof Since $f/g = f \cdot \frac{1}{g}$, the required result follows from Lemma 2.2.13 and Proposition 2.2.12. □

Proposition 2.2.15 *Let $\{f_n\}_{n\in\Lambda}$ be a countable family of measurable functions having the same domain X. Suppose that for every $x \in X$, the set $\{f_n(x) : n \in \Lambda\}$ is bounded above. Then the function $\sup_{n\in\Lambda} f_n$ is measurable.*

Idea Apply definition. Use $\left[\sup_{n\in\Lambda} f_n > a\right] = \bigcup_{n\in\Lambda} [f_n > a]$.

Proof Let $a \in \mathbb{R}$. We want to show that the set $\left[\sup_{n\in\Lambda} f_n > a\right]$ is measurable.

It is easy to see that

$$\left[\sup_{n\in\Lambda} f_n > a\right] = \bigcup_{n\in\Lambda} [f_n > a]. \tag{2.3}$$

Since the family $\{f_n\}_{n\in\Lambda}$ is a family of measurable functions, it follows that the family

$$\big\{[f_n > a]\big\}_{n\in\Lambda}$$

is a family of measurable sets. Since Λ is countable, it follows from (2.3) that the set $\left[\sup_{n\in\Lambda} f_n > a\right]$ is measurable. □

Corollary 2.2.16 *Let $\{f_n\}_{n\in\Lambda}$ be a countable family of measurable functions with the same domain X. Suppose that for every $x \in X$, the set $\{f_n(x) : n \in \Lambda\}$ is bounded below. Then the function $\inf_{n\in\Lambda} f_n$ is measurable.*

Idea For every $x \in \mathbb{R}$, the set $\{(-f_n)(x) : n \in \Lambda\}$ is bounded above.

Proof Note that for every $x \in X$, the set

$$\{(-f_n)(x) : n \in \Lambda\}$$

is bounded above. Hence by Proposition 2.2.15, the function $\sup_{n\in\Lambda}(-f_n)$ is measurable. Since

$$\inf_{n\in\Lambda} f_n = -\sup_{n\in\Lambda}(-f_n)$$

it follows from Lemma 2.2.10 that the function $\inf_{n\in\Lambda} f_n$ is measurable. □

Definition Let f be a real-valued function. The *positive part* and *negative part* of f, denoted by f^+ and f^- respectively, are the functions from dom (f) into $\mathbb{R}$ given by

$$f^+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0, \\ 0 & \text{if } f(x) < 0, \end{cases} \qquad f^+(x) = \begin{cases} 0 & \text{if } f(x) > 0, \\ -f(x) & \text{if } -f(x) \leq 0. \end{cases}$$

Note The functions f^+ and f^- are non-negative. Moreover, we have

$$f = f^+ - f^- \qquad \text{and} \qquad |f| = f^+ + f^-.$$

Corollary 2.2.17 *Let f be a measurable function. Then the functions f^+, f^- and $|f|$ are measurable.*

Idea $f^+ = \max\{f, 0\}$ and $f^- \max\{-f, 0\}$.

Proof Note that

$$f^+ = \max\{f, 0_{\operatorname{dom}(f)}\} \quad \text{and} \quad f^- \max\{-f, 0_{\operatorname{dom}(f)}\}.$$

By Proposition 2.2.15, the functions f^+ and f^- are measurable. Hence by Proposition 2.2.8, the function $|f|$ is measurable. □

The next result means that if we alter the values $f(x)$ of a measurable function f for x belonging to a null set, then the resulted function is also measurable. First we introduce the concept of *equal almost everywhere.*

Definition Let f and g be real-valued functions having the same domain X where $X \subseteq \mathbb{R}$. Suppose that the set $\{x \in X : f(x) \neq g(x)\}$ is a null set. Then we say that f and g are *equal almost everywhere* and we write

$$f = g \text{ a.e.}$$

Notation Let f and g be real-valued functions having the same domain X. We denote

- $[f = g] = \{x \in X : f(x) = g(x)\}$;
- $[f \neq g] = \{x \in X : f(x) \neq g(x)\}$;
- $[f > g] = \{x \in X : f(x) > g(x)\}$;
- $[f < g] = \{x \in X : f(x) < g(x)\}$;
- $[f \geq g] = \{x \in X : f(x) \geq g(x)\}$;
- $[f \leq g] = \{x \in X : f(x) \leq g(x)\}$.

Theorem 2.2.18 *Let f be a measurable function. Suppose g is a real-valued function defined on* $\operatorname{dom}(f)$ *such that f and g are equal almost everywhere. Then the function g is measurable.*

Idea Apply definition. Use $[g > a] = \big([f > a] \cap [f = g]\big) \cup \big([g > a] \cap [f \neq g]\big)$.

Proof Let $a \in \mathbb{R}$. We want to show that the set $[g > a]$ is measurable.

It is straightforward to check that

$$[g > a] = \big([f > a] \cap [f = g]\big) \cup \big([g > a] \cap [f \neq g]\big)$$

To show that the set $[g > a]$ is measurable, it suffices to show that the follows three sets are measurable:

$$[f > a], \qquad [f = g], \qquad [g > a] \cap [f \neq g].$$

- Since f is a measurable function, it follows that $[f > a]$ is a measurable set.
- Since f and g are equal almost everywhere, it follows that the set $[f \neq g]$ is a null set and so it is a measurable set. Hence its complement $[f = g]$ is a measurable set.
- Since $[f \neq g]$ is a null set and $[g > a] \cap [f \neq g] \subseteq [f \neq g]$, it follows that $[g > a] \cap [f \neq g]$ is a null set and so it is a measurable set.

□

To close this section, we introduce a few notations that will be used later.

Notation

- Let f be a real-valued functions with $\operatorname{dom}(f) \subseteq \mathbb{R}$. Suppose that the set $[f \neq 0]$ is a null set. Then we write $f = 0$ a.e.
- Let f and g be real-valued functions having the same domain X. Suppose that for every $x \in X$, we have $f(x) \leq g(x)$. Then we write $f \leq g$.
- Let f and g be real-valued functions having the same domain X where $X \subseteq \mathbb{R}$.
  - ◇ Suppose that the set $[f > g]$ is a null set. Then we write $f \leq g$ a.e..
  - ◇ Suppose that the set $[f \geq g]$ is a null set. Then we write $f < g$ a.e..

Exercise 2.2

1. Show that every monotone function defined on a non-empty measurable subset of $\mathbb{R}$ is measurable.
2. Let f be a real-valued function defined on a non-empty subset of $\mathbb{R}$. Suppose there exist non-empty measurable subsets Y and Z of $\mathbb{R}$ such that $\operatorname{dom}(f) = Y \cup Z$ and that $f|_Y$ and $f|_Z$ are measurable. Show that f is measurable.
3. Let f be a real-valued function defined on a non-empty subset of $\mathbb{R}$. Suppose that for every $r \in \mathbb{Q}$, the set $[f > r]$ is measurable. Show that f is measurable.
4. Let f and g be measurable functions having the same domain. Show that the sets $[f > g]$ and $[f \geq g]$ are measurable.
5. Let f be a measurable function.
   - (a) Show that for every open subset G of $\mathbb{R}$, the set $f^{-1}(G)$ is measurable.
   - (b) Does the result in (a) still hold if "open set" is replaced by "closed set"?
   - (c) Does the result in (a) still hold if "open set" is replaced by "measurable set"?

6. Let f and g be real-valued functions defined on $\mathbb{R}$. Suppose f is continuous and g is measurable.
   (a) Show that the function $f \circ g$ is measurable.
   (b) Is the function $g \circ f$ measurable?
7. Give an example of a non-measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for every $a \in \mathbb{R}$, the set $[f = a]$ is measurable.

2.3 Convergence of Measurable Functions

In this section, we will consider the concepts of *pointwise convergence* and *almost everywhere convergence* of sequences of functions. First we will show that the limit of a sequence of measurable functions is measurable. In fact, we have more general results.

Proposition 2.3.1 *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of measurable functions having the same domain* X*. Suppose that for every* $x \in X$*, the set* $\{f_n(x) : n \in \mathbb{Z}^+\}$ *is bounded. Then the function* $\limsup_{n\to\infty} f_n$ *is measurable.*

Idea Use $\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{Z}^+} \left(\sup_{i\geq n} f_i\right)$.

Proof Note that

$$\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{Z}^+} \left(\sup_{i\geq n} f_i\right)$$

By Proposition 2.2.15, the family

$$\left\{\sup_{i\geq n} f_i\right\}_{n\in\mathbb{Z}^+}$$

is a family of measurable functions. Since $\mathbb{Z}^+$ is countable, it follows from Corollary 2.2.16 that the function $\limsup_{n\to\infty} f_n$ is measurable. □

Corollary 2.3.2 *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of measurable functions having the same domain* X*. Suppose that for every* $x \in X$*, the set* $\{f_n(x) : n \in \mathbb{Z}^+\}$ *is bounded. Then the function* $\liminf_{n\to\infty} f_n$ *is measurable.*

Proof The proof is similar to that of Corollary 2.2.16. Instead of using Proposition 2.2.15, we use Proposition 2.3.1. □

Corollary 2.3.3 *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of measurable functions having the same domain* X*. Suppose that for every* $x \in X$*, the limit* $\lim_{n\to\infty} f_n(x)$ *exists in* $\mathbb{R}$*. Then the function* $\lim_{n\to\infty} f_n$ *is measurable.*

Proof The existence of $\lim\limits_{n\to\infty} f_n$ implies that

$$\lim_{n\to\infty} f_n = \limsup_{n\to\infty} f_n.$$

Hence by Proposition 2.3.1, the function $\lim\limits_{n\to\infty} f_n$ is measurable. □

The above corollary can be generalized to almost everywhere limits. First we have the concept of *convergence almost everywhere.*

Definition Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions having the same domain X where $X \subseteq \mathbb{R}$ and let g be a real-valued function defined on X. Suppose that the set

$$\{x \in X : \lim_{n\to\infty} f_n(x) \neq g(x)\}$$

is a null set. Then we say that the sequence of functions $(f_n)_{n=1}^{\infty}$ *converges almost everywhere* to the function g and we write

$$f_n \longrightarrow g \quad \text{a.e.}$$

Notation Let $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers and let b be a real number. The notation $\lim\limits_{n\to\infty} a_n \neq b$ means that $\lim\limits_{n\to\infty} a_n$ does not exist in $\mathbb{R}$ or $\lim\limits_{n\to\infty} a_n$ exists in $\mathbb{R}$ but the limit is not equal to b.

For convenience, we introduce the following

Notation Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions having the same domain X and let g be a real-valued function defined on X. We denote

- $[f_n \to g] = \{x \in X : \lim\limits_{n\to\infty} f_n(x) = g(x)\}$;
- $[f_n \not\to g] = \{x \in X : \lim\limits_{n\to\infty} f_n(x) \neq g(x)\}$.

Theorem 2.3.4 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X. Suppose g is a real-valued function defined on X such that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to g. Then the function g is measurable.*

Idea Replace $(f_n)_{n=1}^{\infty}$ by a sequence $(g_n)_{n=1}^{\infty}$ of measurable functions such that for every $x \in X$, we have $\lim\limits_{n\to\infty} g_n(x) = g(x)$.

Proof Denote $(g_n)_{n=1}^{\infty}$ to be the sequence of functions from X into $\mathbb{R}$ given by

$$g_n(x) = \begin{cases} g(x) & \text{if } x \in [f_n \not\to g], \\ f_n(x) & \text{if } x \in [f_n \to g], \end{cases} \qquad (n \in \mathbb{Z}^+).$$

It is clear from the construction of $(g_n)_{n=1}^{\infty}$ that

$$\text{for every } x \in X, \quad \lim_{n\to\infty} g_n(x) = g(x) \tag{2.4}$$

and that

$$\text{for every } n \in \mathbb{Z}^+, \quad [f_n \neq g_n] \subseteq [f_n \not\to g].$$

Since the set $[f_n \not\to g]$ is a null set, it follows that for every $n \in \mathbb{Z}^+$, the set $[f_n \neq g_n]$ is a null set, that is,

$$\text{for every } n \in \mathbb{Z}^+, \quad f_n = g_n \ \text{ a.e..}$$

Hence by Theorem 2.2.18, the sequence $(g_n)_{n=1}^{\infty}$ is a sequence of measurable functions. In view of (2.4), by Corollary 2.3.3, the function g is measurable. □

It is well-known that pointwise convergence does not imply uniform convergence. For example, denote $(f_n)_{n=1}^{\infty}$ to be the sequence of functions from $[0, 1]$ into $\mathbb{R}$ given by

$$f_n(x) = x^n \quad (n \in \mathbb{Z}^+).$$

and denote $g : [0, 1] \longrightarrow \mathbb{R}$ to be the function given by

$$g = \begin{cases} 0 & \text{if } 0 \leq x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

It is clear that $(f_n)_{n=1}^{\infty}$ converges pointwise to g. However, the sequence of functions $(f_n)_{n=1}^{\infty}$ does not converge uniformly in $[0, 1]$ to g; for otherwise, the limit function g would be continuous.

For the above example, it is easy to see (for example, using Dini's Theorem) that by deleting a non-degenerate subinterval I with $1 \in I$, where the length of I can be smaller than any given positive number, the sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly in $[0, 1] \setminus I$ to the function g. This is true in general if intervals are replaced by measurable sets provided that the domain has finite measure (see the Egoroff Theorem below). Before proving the Egoroff Theorem, we give a result which describes the set of points of convergence.

Theorem 2.3.5 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions having the same domain X and g be a real-valued function defined on X. Then we have*

$$[f_n \to g] = \bigcap_{k=1}^{\infty} \liminf_{n\to\infty} \left[|f_n - f| < \tfrac{1}{k}\right].$$

Idea Use definition to show that $[f_n \to g]$ and $\bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \left[|f_i - g| < \frac{1}{k}\right]$ are subsets of each other.

Proof Recall that $\liminf_{n\to\infty} \left[|f_n - g| < \frac{1}{k}\right] = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \left[|f_i - g| < \frac{1}{k}\right]$. Denote $(A_k)_{k=1}^{\infty}$ to be the sequence of subsets of X given by

$$A_k = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} \left[|f_i - g| < \tfrac{1}{k}\right] \qquad (k \in \mathbb{Z}^+).$$

We want to show that $[f_n \to g] = \bigcap_{k=1}^{\infty} A_k$.

($\subseteq$) Let $x \in [f_n \to g]$, that is, $\lim_{n\to\infty} f_n(x) = g(x)$. We want to show that

$$\text{for every } k \in \mathbb{Z}^+, \quad x \in A_k.$$

Let $k \in \mathbb{Z}^+$. Since $\lim_{n\to\infty} f_n(x) = g(x)$, by the definition of limit, we see that there exists $n_0 \in \mathbb{Z}^+$ such that

$$\text{for every } i \geq n_0, \quad |f_i(x) - g(x)| < \tfrac{1}{k},$$

that is, $x \in \bigcap_{i=n_0}^{\infty} \left[|f_i - g| < \frac{1}{k}\right]$ which is a subset of A_k. Hence we have $x \in A_k$.

($\supseteq$) Let $x \in \bigcap_{k=1}^{\infty} A_k$, that is, for every $k \in \mathbb{Z}^+$, we have $x \in A_k$. We want to show that $x \in [f_n \to g]$, that is,

$$\lim_{n\to\infty} f_n(x) = g(x).$$

Let $\epsilon > 0$. Denote $k_0 = [\epsilon^{-1}] + 1$. Since $x \in A_{k_0}$, it follows from the construction of A_{k_0} that there exists $n_0 \in \mathbb{Z}^+$ such that

$$x \in \bigcap_{i=n_0}^{\infty} \left[|f_i - g| < \tfrac{1}{k_0}\right],$$

that is,

$$\text{for every } i \geq n_0, \quad |f_i(x) - g(x)| < \tfrac{1}{k_0},$$

which implies that (since $\frac{1}{k_0} < \epsilon$)

$$\text{for every } i \geq n_0, \quad |f_i(x) - g(x)| < \epsilon.$$

Therefore, we have $\lim_{n\to\infty} f_n(x) = g(x)$.

□

Egoroff Theorem *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let g be a measurable function defined on X. Suppose that $\mathsf{m}(X) < \infty$ and that the sequence of functions $(f_n)_{n=1}^{\infty}$ converges almost everywhere to the function g. Then for every $\epsilon > 0$, there exists a measurable set A with $A \subseteq X$ such that $\mathsf{m}(A) < \epsilon$ and that $(f_n)_{n=1}^{\infty}$ converges uniformly in $X \setminus A$ to g.*

Idea We want to get a measurable set Y with $Y \subseteq X$ such that the measure of $X \setminus Y$ is small and that $(f_n)_{n=1}^{\infty}$ converges uniformly in Y to g. The second requirement means that

(†) $\forall\, k \in \mathbb{Z}^+$, $\exists\, n \in \mathbb{Z}^+$ such that $\forall\, x \in Y$, $\forall\, i \geq n$, we have $|f_i(x) - g(x)| < \frac{1}{k}$.

It is straightforward to check that (†) is equivalent to the following:

(‡) $\forall\, k \in \mathbb{Z}^+$, $\exists\, n \in \mathbb{Z}^+$ such that $Y \subseteq \bigcap_{i=n}^{\infty} [|f_i - g| < \frac{1}{k}]$.

In view of (‡), we can take Y to be

$$\bigcap_{k=1}^{\infty}\bigcap_{i=n_k}^{\infty}[|f_i-g|<\tfrac{1}{k}]$$

where $n_1, n_2, \ldots$ are chosen such that

$$X \setminus Y = \bigcup_{k=1}^{\infty}\Big(X \setminus \bigcap_{i=n_k}^{\infty}[|f_i-g|<\tfrac{1}{k}]\Big)$$

has small measure. This can be achieved because by Theorem 2.3.5, we have

$$\bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}[|f_i-g|<\tfrac{1}{k}] = [f_n \to g],$$

and so we can take $(n_k)_{k=1}^{\infty}$ such that for each $k \in \mathbb{Z}^+$, the measure of $\bigcap\limits_{i=n_k}^{\infty}[|f_i-g|<\tfrac{1}{k}]$ is close to $\mathsf{m}(X)$ since $\mathsf{m}(X) = \mathsf{m}[f_n \to g]$.

Proof Denote $N = [f_n \not\to g]$ which is a null set by assumption. Denote $(A_k)_{k=1}^{\infty}$ to be the sequence of subsets of X given by

$$A_k = \liminf_{n\to\infty}\big[|f_n-g|<\tfrac{1}{k}\big] \qquad (k \in \mathbb{Z}^+),$$

that is,

$$A_k = \bigcup_{n=1}^{\infty}\bigcap_{i=n}^{\infty}\big[|f_i-g|<\tfrac{1}{k}\big], \qquad (k \in \mathbb{Z}^+).$$

By Theorem 2.3.5, we have

$$X \setminus N = \bigcap_{k=1}^{\infty} A_k$$

which implies that

$$\text{for every } k \in \mathbb{Z}^+, \quad X \setminus N \subseteq A_k. \tag{2.5}$$

Since $\mathsf{m}(X \setminus N) = \mathsf{m}(X)$ and for every $k \in \mathbb{Z}^+$, we have $A_k \subseteq X$, it follows from (2.5) that

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}(A_k) = \mathsf{m}(X).$$

Hence by Theorem 1.4.17 together with the construction of $(A_k)_{k=1}^{\infty}$, we have

$$\text{for every } k \in \mathbb{Z}^+, \quad \lim_{n\to\infty}\mathsf{m}\Big(\bigcap_{i=n}^{\infty}\big[|f_i-g|<\tfrac{1}{k}\big]\Big) = \mathsf{m}(X). \tag{2.6}$$

Let $\epsilon > 0$. Applying definition to each of the limits in (2.6), we see that there exists a sequence of positive integers $(n_k)_{k=1}^{\infty}$ such that

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}\Big(\bigcap_{i=n_k}^{\infty}\big[|f_i-g|<\tfrac{1}{k}\big]\Big) > \mathsf{m}(X) - \frac{\epsilon}{2^k}. \tag{2.7}$$

Denote A to be the set given by

$$A = X \setminus \bigcap_{k=1}^{\infty} \bigcap_{i=n_k}^{\infty} \left[|f_i - g| < \tfrac{1}{k}\right].$$

It is clear that A is a measurable set with $A \subseteq X$. We want to show that $\mathsf{m}(A) < \epsilon$ and that $(f_n)_{n=1}^{\infty}$ converges uniformly in $X \setminus A$ to g.

- Note that

$$A = \bigcup_{k=1}^{\infty} \Big(X \setminus \bigcap_{i=n_k}^{\infty} \left[|f_i - g| < \tfrac{1}{k}\right] \Big).$$

  By the countable subadditivity of the Lebesgue measure and Proposition 1.4.18 (using the assumption that $\mathsf{m}(X) < \infty$) and by (2.7), we have

$$\begin{aligned} \mathsf{m}(A) &\leq \sum_{k=1}^{\infty} \mathsf{m}\Big(X \setminus \bigcap_{i=n_k}^{\infty} \left[|f_i - g| < \tfrac{1}{k}\right] \Big) \\ &= \sum_{k=1}^{\infty} \Big(\mathsf{m}(X) - \mathsf{m}\Big(\bigcap_{i=n_k}^{\infty} \left[|f_i - g| < \tfrac{1}{k}\right] \Big) \Big) \\ &< \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon. \end{aligned}$$

- Let $\epsilon' > 0$. Denote $k_0 = [\epsilon'^{-1}] + 1$. Since $X \setminus A \subseteq \bigcap_{i=n_{k_0}}^{\infty} \left[|f_i - g| < \tfrac{1}{k_0}\right]$, it follows that

$$\text{for every } i \geq n_{k_0}, \quad \text{for every } x \in X \setminus A, \quad |f_i(x) - g(x)| < \tfrac{1}{k_0} < \epsilon'.$$

  Hence the sequence of functions $(f_n)_{n=1}^{\infty}$ converges uniformly in $X \setminus A$ to the function g. □

The following example illustrates that the Egoroff Theorem does not hold if the assumption that $\mathsf{m}(X) < \infty$ is removed.

Example Denote $(f_n)_{n=1}^{\infty}$ to be the sequence of real-valued functions defined on $\mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1 & \text{if } |x| \geq n, \\ 0 & \text{if } |x| < n, \end{cases} \qquad (n \in \mathbb{Z}^+).$$

- It is clear that $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions converging pointwise to the function $0_{\mathbb{R}}$.
- However, there does not exist any measurable subset A of $\mathbb{R}$ such that $\mathsf{m}(A)$ is finite and that $(f_n)_{n=1}^{\infty}$ converges uniformly in $\mathbb{R} \setminus A$ to the function $0_{\mathbb{R}}$. This is because if Y is an unbounded subset of $\mathbb{R}$, then $(f_n)_{n=1}^{\infty}$ does not converge uniformly in Y to $0_{\mathbb{R}}$.

Exercise 2.3

1. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued functions having the same domain X and let g be a real-valued function defined on X. Show that $[f_n \not\to g] = \bigcup_{k\to\infty} \limsup_{n\to\infty} \left[|f_n - g| \geq \frac{1}{k}\right]$.
2. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X. Show that the sets $\{x \in X : \lim_{n\to\infty} f_n(x) \text{ exists in } \mathbb{R}\}$ and $\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}$ are measurable.
3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X. Let f and g be real-valued functions defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to both f and g. Show that $f = g$ a.e..
4. Let $f : (a, b) \longrightarrow \mathbb{R}$ be a differentiable function. Show that the derived function f' of f is measurable.

2.4 Approximations for Measurable Functions

Given a measurable function f, we want to "*approximate*" it by a "*familiar*" function g, where "familiar function" can be continuous function, simple function etc. There are three interpretations of the term "*approximation*": uniform approximation, almost everywhere approximation and "nearly everywhere" approximation as described below in (1), (2) and (3) respectively.

(1) For every $\epsilon > 0$, there is a "familiar" function g defined on $\text{dom}(f)$ such that for every $x \in \text{dom}(f)$, we have $|f(x) - g(x)| < \epsilon$.

(2) There is a "familiar" function g defined on $\text{dom}(f)$ such that $f = g$ a.e., that is, $\mathsf{m}[f \neq g] = 0$.

(3) For every $\epsilon > 0$, there is a "familiar" function g defined on $\text{dom}(f)$ such that $\mathsf{m}[f \neq g] < \epsilon$.

Note

- If f is a discontinuous function, then (1) does not hold if "familiar function" means "continuous function". However, if "familiar function" means "simple function" and f is a bounded measurable function, then (1) holds (see Proposition 2.4.1).
- In general, (2) does not hold if "familiar function" means "continuous function".

  Example Denote f to be the function from $[0, 1]$ into $\mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \bigcup_{n=1}^{\infty} \left(\frac{1}{2n+1}, \frac{1}{2n}\right), \\ 0 & \text{otherwise.} \end{cases}$$

  It is clear that f is measurable. However, there does not exist any continuous function g from $[0, 1]$ into $\mathbb{R}$ such that $\mathsf{m}[f \neq g] = 0$.

  Proof Let $g : [0, 1] \longrightarrow \mathbb{R}$ be a measurable function such that $\mathsf{m}[f \neq g] = 0$. We want to show that g is not continuous at 0.

For each $k \in \mathbb{Z}^+$, since $\mathsf{m}[f \neq g] = 0$, it follows that

$$\mathsf{m}\Big(\big(\tfrac{1}{k+1}, \tfrac{1}{k}\big) \cap [f = g]\Big) = \tfrac{1}{k} - \tfrac{1}{k+1} > 0,$$

and so there exists $x_k \in \big(\tfrac{1}{k+1}, \tfrac{1}{k}\big) \cap [f = g]$. It is clear that

◇ both sequences $(x_{2n})_{n=1}^{\infty}$ and $(x_{2n+1})_{n=1}^{\infty}$ converge to 0;

◇ for every $n \in \mathbb{Z}^+$, we have $g(x_{2n}) = 1$ and $g(x_{2n+1}) = 0$.

Hence g is not continuous at 0. □

Remark The above function f is not continuous at 0 and we cannot modify the values of f in a null set such that the resulted function is continuous at 0. In the exercise, readers are asked to find a (everywhere discontinuous) measurable function $f : [0, 1] \longrightarrow \mathbb{R}$ such that if we modify the values of f in a null set, the resulted function is not continuous at any point in $[0, 1]$.

- If "familiar function" means "continuous function", Statement (3) means that we can modify the values of a measurable function f in a set with measure smaller than any given (small) positive number so that the resulted function is continuous. This result is true and is called the Lusin Theorem. The idea is to approximate bounded measurable functions by simple functions in the sense of (1), and then approximate simple functions by continuous functions in the sense of (3).

The following proposition means that bounded measurable functions can be approximated uniformly by simple functions. We give the result in a form that will be used to prove Proposition 2.4.2.

Proposition 2.4.1 *Let f be a bounded measurable function. Then for every $\epsilon > 0$, there exists a simple function g defined on* dom (f) *such that*

$$\text{for every } x \in \operatorname{dom}(f), \quad f(x) - \epsilon \leq g(x) \leq f(x).$$

Moreover, if in addition, f is non-negative, then g can be taken to be non-negative also.

Idea The range of f is a subset of a bounded interval. Divide the bounded interval into finitely many subintervals with lengths smaller than ϵ.

Proof By assumption, the range of f is a bounded (non-empty) subset of $\mathbb{R}$. Denote c and d to be the infimum and supremum of the range of f respectively.

Let $\epsilon > 0$. Denote $N = [(d - c)\epsilon^{-1}] + 1$. Denote $\{A_k\}_{k=0}^{N}$ to be the finite family of subsets of dom (f) given by

$$A_k = \{x \in \operatorname{dom}(f) : c + \tfrac{k}{N}(d - c) \leq f(x) < c + \tfrac{k+1}{N}(d - c)\}, \quad (k \in \{0, 1, \ldots, N).$$

Note that the family $\{A_k\}_{k=0}^{N}$ is disjoint and that $\bigcup\limits_{k=0}^{N} A_k = \operatorname{dom}(f)$. Denote g to be the function from dom (f) into $\mathbb{R}$ given by

$$g(x) = c + \tfrac{k}{N}(d - c) \quad \text{if } x \in A_k,\ k \in \{0, 1, \ldots, N\}.$$

Since f is measurable, it follows that for every $k = 0, 1, \ldots, N$, the set A_k is measurable. Hence the function g is a simple function. It is clear from the construction of g that

$$\text{for every } x \in \operatorname{dom}(f), \quad f(x) - \tfrac{1}{N}(d-c) < g(x) \leq f(x)$$

Since $\epsilon > \frac{1}{N}(d-c))$, it follows that

$$\text{for every } x \in \operatorname{dom}(f), \quad f(x) - \epsilon < g(x) \leq f(x).$$

Moreover, if f is non-negative, then $c \geq 0$ and so g is non-negative. □

Proposition 2.4.2 *Let f be a non-negative measurable function Then there exists an increasing sequence $(\varphi_n)_{n=1}^{\infty}$ of non-negative simple functions defined on* $\operatorname{dom}(f)$ *such that*

$$\textit{for every } x \in \operatorname{dom}(f), \quad \lim_{n\to\infty} \varphi_n(x) = f(x).$$

Idea There exists an increasing sequence $(f_n)_{n=1}^{\infty}$ of non-negative bounded measurable functions such that for every $x \in \operatorname{dom}(f)$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$. Apply Proposition 2.4.1 to each f_n with suitably chosen $\epsilon(n) > 0$.

Proof Denote $(f_n)_{n=1}^{\infty}$ to be the sequence of functions from $\operatorname{dom}(f)$ into $\mathbb{R}$ given by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n, \\ n & \text{otherwise,} \end{cases} \qquad (n \in \mathbb{Z}^+).$$

It is clear that $(f_n)_{n=1}^{\infty}$ is a sequence of non-negative bounded measurable functions. Applying Proposition 2.4.1 to each f_n (taking $\epsilon = \frac{1}{n}$), we see that there exists a sequence $(g_n)_{n=1}^{\infty}$ of non-negative simple functions defined on $\operatorname{dom}(f)$ such that

$$\text{for every } n \in \mathbb{Z}^+, \quad \text{for every } x \in \operatorname{dom}(f), \quad f_n(x) - \tfrac{1}{n} \leq g_n(x) \leq f_n(x).$$

which implies that

$$\text{for every } n \in \mathbb{Z}^+, \quad \text{for every } x \in \operatorname{dom}(f), \quad f_n(x) - \tfrac{1}{n} \leq g_n(x) \leq f(x). \tag{2.8}$$

Denote $(\varphi_n)_{n=1}^{\infty}$ to be the sequence of functions given by

$$\varphi_n = \max_{1\leq i\leq n} g_i \qquad (n \in \mathbb{Z}^+).$$

It is clear that $(\varphi_n)_{n=1}^{\infty}$ is an increasing sequence of non-negative simple functions defined on $\operatorname{dom}(f)$. We want to show that

$$\text{for every } x \in \operatorname{dom}(f), \quad \lim_{n\to\infty} \varphi_n(x) = f(x).$$

Let $x \in \operatorname{dom}(f)$. Note that

$$\text{for every } n \geq [f(x)] + 1, \quad f_n(x) = f(x).$$

Hence by (2.8) together with the construction of $(\varphi_n)_{n=1}^{\infty}$, we have

$$\text{for every } n \geq [f(x)] + 1, \quad f(x) - \tfrac{1}{n} \leq \varphi_n(x) \leq f(x).$$

From this we see that $\lim_{n\to\infty} \varphi_n(x) = f(x)$. □

Theorem 2.4.3 *Let f be a measurable function. Then there exists a sequence $(\varphi_n)_{n=1}^{\infty}$ of simple functions defined on* dom (f) *such that*

$$\textit{for every } x \in \operatorname{dom}(f), \quad \lim_{n\to\infty} \varphi_n(x) = f(x).$$

Idea Use $f = f^+ - f^-$.

Proof Note that the functions f^+ and f^- are non-negative measurable functions defined on dom (f). Applying Proposition 2.4.2 to the two functions f^+ and f^-, we see that there exist two sequences of (non-negative) simple functions $(g_n)_{n=1}^{\infty}$ and $(h_n)_{n=1}^{\infty}$ defined on dom (f) such that

$$\text{for every } x \in \operatorname{dom}(f), \quad \lim_{n\to\infty} g_n(x) = f^+(x) \text{ and } \lim_{n\to\infty} h_n(x) = f^-(x). \tag{2.9}$$

Denote $(\varphi_n)_{n=1}^{\infty}$ to be the sequence of functions given by

$$\varphi_n = g_n - h_n \quad (n \in \mathbb{Z}^+).$$

It is clear that $(\varphi_n)_{n=1}^{\infty}$ is a sequence of simple functions defined on dom (f). Moreover, by (2.9), we have

$$\text{for every } x \in \operatorname{dom}(f), \quad \lim_{n\to\infty} \varphi_n(x) = f^+(x) - f^-(x) = f(x).$$

□

In the above theorem, simple functions can be replaced by continuous functions if convergence pointwise is relaxed to convergence almost everywhere (see Theorem 2.4.6). First we prove the following two results.

Theorem 2.4.4 *Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences of measurable functions having the same domain X and let f be a measurable function having domain X. Suppose $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f and $\sum_{n=1}^{\infty} \mathsf{m}[f_n \neq g_n] < \infty$. Then $(g_n)_{n=1}^{\infty}$ converges almost everywhere to f.*

Idea The set $\limsup_{n\to\infty}[f_n \neq g_n] \cup [f_n \not\to f]$ is a null set. For x belonging to its complement in X, $\lim_{n\to\infty} g_n(x) = f(x)$.

Proof Denote $A = \limsup_{n\to\infty}[f_n \neq g_n]$, that is,

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} [f_k \neq g_k]$$

and denote $B = A \cup [f_n \not\to f]$. We want to show that $\mathsf{m}(B) = 0$ and that for every $x \in X \setminus B$, we have $\lim_{n\to\infty} g_n(x) = f(x)$.

- By assumption, the set $[f_n \not\to f]$ is a null set. By Proposition 1.4.21, we have $\mathsf{m}(A) = 0$. Hence by the subadditivity of the Lebesgue measure, we have $\mathsf{m}(B) = 0$.
- Let $x \in X \setminus B$.

  Since $x \notin [f_n \not\to f]$, it follows that

$$\lim_{k\to\infty} f_k(x) = f(x). \tag{2.10}$$

  Since $x \notin A$, it follows that there exists $n_0 \in \mathbb{Z}^+$ such that $x \notin \bigcup_{k=n_0}^{\infty} [f_k \neq g_k]$, that is,

$$\text{for every } k \geq n_0, \quad f_k(x) = g_k(x). \tag{2.11}$$

  Hence by (2.10) and (2.11), we get $\lim\limits_{k\to\infty} g_k(x) = f(x)$.

□

The following lemma is a special case of the Lusin Theorem.

Lemma 2.4.5 *Let f be a simple function. Then for every $\epsilon > 0$, there exists a continuous function g defined on* $\operatorname{dom}(f)$ *such that* $\mathsf{m}[f \neq g] < \epsilon$.

Idea There exist measurable sets $A_1, \ldots, A_n$ with $\bigcup_{i=1}^{n} A_i = \operatorname{dom}(f)$ such that f is constant (hence continuous) on each A_i. Replace each A_i by a closed set F_i with $F_i \subseteq A_i$ such that $\mathsf{m}(A_i \setminus F_i)$ is small.

Proof Since f is a simple function, it follows that there exist a finite family $\{c_i\}_{i=1}^n$ of real numbers and a disjoint finite family $\{A_i\}_{i=1}^n$ of measurable subsets of $\mathbb{R}$ with $\operatorname{dom}(f) = \bigcup_{i=1}^{n} A_i$ such that

$$\text{for every } i \in \{1, \ldots, n\}, \quad \text{for every } x \in A_i, \quad f(x) = c_i.$$

Let $\epsilon > 0$. Apply Theorem 1.4.20 to each A_i (using $\frac{\epsilon}{n} > 0$), we see that there exists a finite family $\{F_i\}_{i=1}^n$ of closed subsets of $\mathbb{R}$ such that

$$\text{for every } i \in \{1, \ldots, n\}, \quad F_i \subseteq A_i, \text{ and } \mathsf{m}(A_i \setminus F_i) < \frac{\epsilon}{n}. \tag{2.12}$$

Note that for every $i \in \{1, \ldots, n\}$, the restriction $f\big|_{F_i}$ of f on F_i is constant (hence continuous). Moreover, since $\{F_i\}_{i=1}^n$ is a disjoint finite family of closed subsets of $\mathbb{R}$, it follows from Proposition 0.5.2 that the function $f\big|_{\bigcup_{i=1}^{n} F_i}$ is continuous. Hence by Proposition 0.5.4, there exists a continuous function $g : \operatorname{dom}(f) \longrightarrow \mathbb{R}$ such that

$$\text{for every } x \in \bigcup_{i=1}^{n} F_i, \quad g(x) = f(x).$$

We want to show that

$$\mathsf{m}[f \neq g] < \epsilon.$$

For this, note that

$$[f \neq g] \subseteq \operatorname{dom}(f) \setminus \bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} (A_i \setminus F_i).$$

Hence by the monotonicity and the finite additivity of the Lebesgue measure and using (2.12), we have

$$\begin{aligned} \mathsf{m}[f \neq g] &\leq \mathsf{m}\Big(\bigcup_{i=1}^{n} (A_i \setminus F_i)\Big) \\ &= \sum_{i=1}^{n} \mathsf{m}(A_i \setminus F_i) \\ &< \sum_{i=1}^{n} \frac{\epsilon}{n} = \epsilon. \end{aligned}$$

□

Theorem 2.4.6 *Let f be a measurable function. Then there exists a sequence $(g_n)_{n=1}^{\infty}$ of continuous functions defined on* $\operatorname{dom}(f)$ *such that $(g_n)_{n=1}^{\infty}$ converges almost everywhere to f.*

Idea There exists a sequence of simple functions $(\varphi_n)_{n=1}^{\infty}$ such that for every $x \in \operatorname{dom}(f)$, $\lim_{n\to\infty} \varphi_n(x) = f(x)$. Replace each φ_n by a continuous function g_n such that $\mathsf{m}[\varphi_n \neq g_n]$ is small.

Proof By Theorem 2.4.3, there exists a sequence of simple functions $(\varphi_n)_{n=1}^{\infty}$ defined on $\operatorname{dom}(f)$ such that

$$\text{for every } x \in \operatorname{dom}(f), \quad \lim_{n\to\infty} \varphi_n(x) = f(x).$$

Applying Lemma 2.4.5 to each φ_n (taking $\epsilon = 2^{-n}$), we see that there exists a sequence $(g_n)_{n=1}^{\infty}$ of continuous functions defined on $\operatorname{dom}(f)$ such that

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathsf{m}[\varphi_n \neq g_n] < 2^{-n}. \tag{2.13}$$

Since $\sum_{n=1}^{\infty} \mathsf{m}[\varphi_n \neq g_n] < \infty$, it follows from Theorem 2.4.4 that $(g_n)_{n=1}^{\infty}$ converges almost everywhere to f. □

We are going to prove the Lusin Theorem. There are two equivalent versions.

Lusin Theorem (Version 1) *Let f be a measurable function. Then for every $\epsilon > 0$, there exists a measurable subset A of $\mathbb{R}$ with $A \subseteq \operatorname{dom}(f)$ and $\mathsf{m}(A) < \epsilon$ such that $f|_{\operatorname{dom}(f)\setminus A}$ is continuous.*

Idea For the case where $\operatorname{dom}(f)$ has finite measure, apply the Egoroff Theorem. For the general case, there exist subsets $E_1, E_2, \ldots$ of $\operatorname{dom}(f)$ such that $\mathsf{m}\big(\operatorname{dom}(f) \setminus \bigcup_{i=1}^{\infty} E_i\big)$ is small and that for each i, $\mathsf{m}(E_i) < \infty$ and $d(F_i, \bigcup_{k=1, k\neq i}^{\infty} F_k) > 0$.

Proof We divide the proof into two steps. In Step 1, we prove the result for the case where dom (f) has finite measure. In Step 2, we prove the result in general.

(*Step 1*) Suppose f is a measurable function with $\mathsf{m}\big(\mathrm{dom}\,(f)\big) < \infty$.

Let $\epsilon > 0$. By Theorem 2.4.6, there exists a sequence $(g_n)_{n=1}^{\infty}$ of continuous functions defined on dom (f) such that $(g_n)_{n=1}^{\infty}$ converges almost everywhere to f. By the Egoroff Theorem, there exists a measurable set A with $A \subseteq \mathrm{dom}\,(f)$ such that $\mathsf{m}(A) < \epsilon$ and that $(g_n)_{n=1}^{\infty}$ converges uniformly in $\mathrm{dom}\,(f) \setminus A$ to f. It is clear that $f\big|_{\mathrm{dom}\,(f)\setminus A}$ is continuous.

(*Step 2*) Let $\epsilon > 0$. Let $\{\epsilon_n\}_{n\in\mathbb{Z}}$ be a family of numbers in $(0, 1)$ such that

$$\sum_{n\in\mathbb{Z}} \epsilon_n < \frac{\epsilon}{2}. \tag{2.14}$$

Denote $\{E_n\}_{n\in\mathbb{Z}}$ to be the family of subsets of dom (f) given by

$$E_n = [n, n+1-\epsilon_n] \cap \mathrm{dom}\,(f) \qquad (n \in \mathbb{Z}).$$

Note that for every $n \in \mathbb{Z}$, the function $f\big|_{E_n}$ is measurable and $\mathsf{m}\big(\mathrm{dom}\,(f\big|_{E_n})\big) = \mathsf{m}(E_n) < \infty$. Applying Step 1 to each $f\big|_{E_n}$ (taking $\epsilon = \epsilon_n$), we see that there exists a family $\{A_n\}_{n\in\mathbb{Z}}$ of measurable subsets of $\mathbb{R}$ such that

$$\text{for every } n \in \mathbb{Z}, \qquad A_n \subseteq E_n \ \text{ and } \ \mathsf{m}(A_n) < \epsilon_n \ \text{ and } \ f\big|_{E_n\setminus A_n} \text{ is continuous.} \tag{2.15}$$

Denote A to be the subset of dom (f) given by

$$A = \Big(\bigcup_{n\in\mathbb{Z}} A_n\Big) \cup \Big(\mathrm{dom}\,(f) \setminus \bigcup_{n\in\mathbb{Z}} E_n\Big).$$

We want to show that $\mathsf{m}(A) < \epsilon$ and that $f|_{\mathrm{dom}\,(f)\setminus A}$ is continuous.

- Note that

$$\mathrm{dom}\,(f) \setminus \bigcup_{n\in\mathbb{Z}} E_n \ \subseteq \ \mathbb{R} \setminus \bigcup_{n\in\mathbb{Z}} [n, n+1-\epsilon_n] \ = \ \bigcup_{n\in\mathbb{Z}} (n+1-\epsilon_n, n+1).$$

  Hence by the monotonicity and the countable subadditivity of the Lebesgue measure, together with (2.15) and (2.14), we have

$$\begin{aligned} \mathsf{m}(A) &\leq \sum_{n\in\mathbb{Z}} \mathsf{m}(A_n) + \sum_{n\in\mathbb{Z}} \mathsf{m}(n+1-\epsilon_n, n+1) \\ &< \sum_{n\in\mathbb{Z}} \epsilon_n + \sum_{n\in\mathbb{Z}} \epsilon_n \\ &< \epsilon. \end{aligned}$$

- Note that
  - ◇ $\mathrm{dom}\,(f) \setminus A = \bigcup_{n\in\mathbb{Z}} (E_n \setminus A_n)$;

- $\diamond$ for every $n \in \mathbb{Z}$, the function $f\big|_{E_n \setminus A_n}$ is continuous;
- $\diamond$ for every $n \in \mathbb{Z}$, we have $d\Big(E_n \setminus A_n, \bigcup_{k \in \mathbb{Z}\setminus\{n\}} E_k \setminus A_k\Big) > 0$.

Hence by Proposition 0.5.3, the function $f\big|_{\mathrm{dom}\,(f)\setminus A}$ is continuous.

□

Lusin Theorem (Version 2) *Let f be a measurable function. Then for every $\epsilon > 0$, there exists a continuous function g defined on* $\mathrm{dom}\,(f)$ *such that* $\mathsf{m}[f \neq g] < \epsilon$.

Idea Apply Version 1 to get a measurable set A with $\mathsf{m}(A)$ small such that the restriction of f on $\mathrm{dom}\,(f) \setminus A$ is continuous. Replace $\mathrm{dom}\,(f) \setminus A$ by a closed subset F of $\mathbb{R}$ such that the measure of $(\mathrm{dom}\,(f) \setminus A) \setminus F$ is small.

Proof Let $\epsilon > 0$. By Version 1 (using the positive number $\frac{\epsilon}{2}$), we see that there exists a measurable subset A of $\mathbb{R}$ with $A \subseteq \mathrm{dom}\,(f)$ such that

$$\mathsf{m}(A) < \frac{\epsilon}{2} \tag{2.16}$$

and that $f\big|_{\mathrm{dom}\,(f)\setminus A}$ is continuous. Applying Theorem 1.4.20 to the measurable set $\mathrm{dom}\,(f) \setminus A$ (using the positive number $\frac{\epsilon}{2}$), we see that there exists a closed subset F of $\mathbb{R}$ with $F \subseteq \mathrm{dom}\,(f) \setminus A$ such that

$$\mathsf{m}\big((\mathrm{dom}\,(f) \setminus A) \setminus F\big) < \frac{\epsilon}{2}. \tag{2.17}$$

Since the function $f\big|_F$ is the restriction of the continuous function $f\big|_{\mathrm{dom}\,(f)\setminus A}$ on F, it follows that $f\big|_F$ is continuous. Hence by Proposition 0.5.4, there exists a continuous function $g : \mathrm{dom}\,(f) \longrightarrow \mathbb{R}$ such that

$$\text{for every } x \in F, \quad g(x) = f(x).$$

We want to show that

$$\mathsf{m}[f \neq g] < \epsilon.$$

For this, note that

$$[f \neq g] \subseteq \mathrm{dom}\,(f) \setminus F = A \cup \big((\mathrm{dom}\,(f) \setminus A) \setminus F\big).$$

Hence by the monotonicity and the additivity of the Lebesgue measure and (2.16) and (2.17), we have

$$\begin{aligned} \mathsf{m}[f \neq g] &\leq \mathsf{m}\Big(A \cup \big((\mathrm{dom}\,(f) \setminus A) \setminus F\big)\Big) \\ &= \mathsf{m}(A) + \mathsf{m}\big((\mathrm{dom}\,(f) \setminus A) \setminus F\big) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon. \end{aligned}$$

□

Exercise 2.4

1. Give an example of a measurable function $f : [0,1] \longrightarrow \mathbb{R}$ such that if $g : [0,1] \longrightarrow \mathbb{R}$ is a function with $\mathsf{m}[f \neq g] = 0$, then g is not continuous at any point in $[0,1]$.
2. Let f be a bounded measurable function. Suppose c and d are real numbers such that range$\,(f) \subseteq [c,d]$. Show that for every $\epsilon > 0$, there exists a continuous function $g : \text{dom}\,(f) \longrightarrow \mathbb{R}$ such that $\mathsf{m}[f \neq g] < \epsilon$ and that range$\,(g) \subseteq [c,d]$.
3. Let $f : [a,b] \longrightarrow \mathbb{R}$ be a measurable function. Show that
   (a) for every $\epsilon > 0$, there exists a polynomial function p on $[a,b]$ such that $\mathsf{m}[|f - p| \geq \epsilon] < \epsilon$,
   (b) there exists a sequence $(p_n)_{n=1}^{\infty}$ of polynomial functions on $[a,b]$ such that $p_n \longrightarrow f$ a.e..
4. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a measurable function. Suppose that for every pair of real numbers x and y, we have $f(x+y) = f(x) + f(y)$. Show that f is linear, that is, there exists a constant $c \in \mathbb{R}$ such that for every $x \in \mathbb{R}$, we have $f(x) = cx$.
   (b) Show that there exists a non-linear function $g : \mathbb{R} \longrightarrow \mathbb{R}$ with the property that for every pair of real numbers x and y, we have $g(x+y) = g(x) + g(y)$.

2.5 Convergence in Measure

In this section, we consider another concept of convergence called *convergence in measure.* If the underlying domain has measure equal to 1, it is known as *convergence in probability.*

Definition Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f be a measurable function defined on X. We say that $(f_n)_{n=1}^{\infty}$ *converges in measure to* f, denoted by $f_n \overset{m}{\Longrightarrow} f$, if for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \mathsf{m}[|f_n - f| \geq \epsilon] = 0.$$

Below we list some results on convergence in measure. These results (except the third one) are similar to that for convergence of sequences in $\mathbb{R}$.

(1) If $f_n \overset{m}{\Longrightarrow} f$, then for every subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$, we have $f_{n_k} \overset{m}{\Longrightarrow} f$.

(2) If $f_n \overset{m}{\Longrightarrow} f$ and $g_n \overset{m}{\Longrightarrow} g$, then we have $f_n + g_n \overset{m}{\Longrightarrow} f + g$,

(3) If $f_n \overset{m}{\Longrightarrow} f$ and $g_n \overset{m}{\Longrightarrow} g$ and the measure of the common domain is finite, then we have $f_n \cdot g_n \overset{m}{\Longrightarrow} f \cdot g$.

The proofs of the above results are left as exercises.

Next, we give an example to illustrate that convergence in measure does not imply convergence almost everywhere and an example to illustrate that convergence almost everywhere/pointwise does not imply convergence in measure.

Example Denote $(f_n)_{n=1}^{\infty}$ to be the sequence of functions from $\mathbb{R}$ into $\mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1 & \text{if } x > n, \\ 0 & \text{if } x \leq n, \end{cases} \qquad (n \in \mathbb{Z}^+).$$

It is clear that $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions. Moreover, we have the following:

- The sequence of functions $(f_n)_{n=1}^{\infty}$ converges pointwise to the function $0_{\mathbb{R}}$;
- The sequence of functions $(f_n)_{n=1}^{\infty}$ does not converge in measure to the function $0_{\mathbb{R}}$. Indeed, if $0 < \epsilon < 1$, then for every $n \in \mathbb{Z}^+$, we have $[|f_n - 0| \geq \epsilon] = (n, \infty)$ and so

$$\lim_{n\to\infty} \mathsf{m}[|f_n - 0| \geq \epsilon] = \infty.$$

Before writing down the sequence of functions in the next example, we describe how to obtain the functions geometrically.

- Denote $I_1 = [0, 1)$.
- Divide the interval $[0, 1)$ into two subintervals:

$$\big[0, \tfrac{1}{2}\big) \quad \text{and} \quad \big[\tfrac{1}{2}, 1\big),$$

  denoted by I_2 and I_3 respectively.
- Divide the interval $[0, 1)$ into three subintervals:

$$\big[0, \tfrac{1}{3}\big), \quad \big[\tfrac{1}{3}, \tfrac{2}{3}\big) \quad \text{and} \quad \big[\tfrac{2}{3}, 1\big),$$

  denoted by I_4, I_5 and I_6 respectively.

Repeating the above process, we get a sequence of intervals $(I_n)_{n=1}^{\infty}$ satisfying the following conditions:

(1) $\mathsf{m}(I_n) \to 0$ as $n \to \infty$

(2) For every $x \in [0, 1)$, there exist infinitely many $i \in \mathbb{Z}^+$ such that $x \in I_i$ and there exist infinitely many $j \in \mathbb{Z}^+$ such that $x \notin I_j$.

The sequence of functions that we give in the following example is the sequence of characteristic functions $(\chi_{I_n})_{n=1}^{\infty}$ with domain $[0, 1)$.

Example Denote $(f_n)_{n=1}^{\infty}$ to be the sequence of functions from $[0, 1)$ into $\mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1 & \text{if } x \in \big[\frac{j-1}{k+1}, \frac{j}{k+1}\big), \quad \text{where } k = \max\{i \in \mathbb{Z} : 0 + 1 + 2 + \cdots + i < n\} \\ & \qquad\qquad\qquad\quad\ \text{and } j = n - (0 + 1 + 2 + \cdots + k), \\ 0 & \text{otherwise}, \end{cases} \qquad (n \in \mathbb{Z}^+).$$

It is clear that $(f_n)_{n=1}^{\infty}$ is a sequence of measurable functions. Moreover, we have the following:

- The sequence of functions $(f_n)_{n=1}^{\infty}$ converges in measure to the function $0_{[0,1)}$.

  Indeed, we have

$$\mathsf{m}[|f_n - 0| \geq \epsilon] = \begin{cases} \frac{1}{k+1} & \text{if } 0 < \epsilon \leq 1, \quad \text{where } k = \max\{i \in \mathbb{Z} : 0 + 1 + 2 + \cdots + i < n\} \\ \emptyset & \text{if } \epsilon > 1, \end{cases} \qquad (n \in \mathbb{Z}^+).$$

- For every $x \in [0,1)$, the sequence of real numbers $\big(f_n(x)\big)_{n=1}^{\infty}$ diverges.

  Indeed, for every $x \in [0,1)$, there exist infinitely many $i \in \mathbb{Z}^+$ such that $f_i(x) = 1$ and there exist infinitely many $j \in \mathbb{Z}^+$ such that $f_j(x) = 0$.

Although convergence almost everywhere does not imply convergence in measure in general, we do have the implication if the underlying domain has finite measure. Before proving the result (Theorem 2.5.2), we give a condition that is equivalent to convergence in measure.

Proposition 2.5.1 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f be a measurable function defined on X. Then $(f_n)_{n=1}^{\infty}$ converges in measure to f if and only if the following condition is satisfied:*

(†) *For every $\epsilon > 0$, there exists $n_0 \in \mathbb{Z}^+$ such that for every $n \geq n_0$, we have $\mathsf{m}[|f_n - f| \geq \epsilon] < \epsilon$.*

Idea Apply definition of limit.

Proof

($\Longrightarrow$) Suppose $(f_n)_{n=1}^{\infty}$ converges in measure to f.

Let $\epsilon > 0$. By definition, we have $\lim_{n\to\infty} \mathsf{m}[|f_n - f| \geq \epsilon] = 0$. Hence there exists $n_0 \in \mathbb{Z}^+$ such that

$$\text{for every } n \geq n_0, \quad \mathsf{m}[|f_n - f| \geq \epsilon] < \epsilon.$$

($\Longleftarrow$) Suppose (†) holds.

Let $\epsilon > 0$. We want to show that $\lim_{n\to\infty} \mathsf{m}[|f_n - f| \geq \epsilon] = 0$, or equivalently,

$$\forall\, \epsilon' > 0,\ \exists\, n_0 \in \mathbb{Z}^+ \text{ such that for every } n \geq n_0,\ \mathsf{m}[|f_n - f|] < \epsilon'.$$

Let $\epsilon' > 0$. Denote $\epsilon'' = \min\{\epsilon, \epsilon'\}$. By assumption, there exists $n_0 \in \mathbb{Z}^+$ such that

$$\text{for every } n \geq n_0, \quad \mathsf{m}[|f_n - f| \geq \epsilon''] < \epsilon''. \tag{2.18}$$

Since $\epsilon'' \leq \epsilon$, it follows that

$$\text{for every } n \in \mathbb{Z}^+, \quad [|f_n - f| \geq \epsilon''] \supseteq [|f_n - f| \geq \epsilon].$$

Hence by (2.18) and the condition that $\epsilon'' \leq \epsilon'$, we obtain

$$\text{for every } n \geq n_0, \quad \mathsf{m}[|f_n - f| \geq \epsilon] < \epsilon'.$$ □

Theorem 2.5.2 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f be a measurable function defined on X. Suppose $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f and $\mathsf{m}(X) < \infty$. Then $(f_n)_{n=1}^{\infty}$ converges in measure to f.*

Idea Apply the Egoroff Theorem.

Proof We want to show that condition (†) in Proposition 2.5.1 holds.

Let $\epsilon > 0$. By the Egoroff Theorem, there exists a measurable set A with $A \subseteq X$ and $\mathsf{m}(A) < \epsilon$ such that $(f_n)_{n=1}^{\infty}$ converges uniformly in $X \setminus A$ to f. Hence there exists $n_0 \in \mathbb{Z}^+$ such that

$$\text{for every } n \geq n_0, \quad \text{for every } x \in X \setminus A, \quad |f_n(x) - f(x)| < \epsilon.$$

which implies that

$$\text{for every } n \geq n_0, \quad [|f_n - f| \geq \epsilon] \subseteq A.$$

Therefore, by the monotonicity of the Lebesgue measure and the condition on A, we obtain

$$\text{for every } n \geq n_0, \quad \mathsf{m}[|f_n - f| \geq \epsilon] < \epsilon.$$

□

Although convergence in measure does not imply convergence almost everywhere, we do have a partial implication.

Theorem 2.5.3 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f be a measurable function defined on X. Suppose $(f_n)_{n=1}^{\infty}$ converges in measure to f. Then there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that $(f_{n_k})_{k=1}^{\infty}$ converges almost everywhere to f.*

Idea We want to get a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ and a subset A of X with $\mathsf{m}(A) = 0$ such that for every $x \in X \setminus A$, $\lim\limits_{k\to\infty} f_{n_k}(x) = f(x)$. If we have a sequence $(A_k)_{k=1}^{\infty}$ of subsets of X such that $\sum\limits_{k=1}^{\infty} \mathsf{m}(A_k) < \infty$, then the set $\limsup\limits_{k\to\infty} A_k$ has measure zero. We can take $A_k = [|f_{n_k} - f| \geq \epsilon_k]$ where $\epsilon_k > 0$ is small and n_k is to be chosen so that $\mathsf{m}(A_k)$ is small.

Proof Using Condition (†) in Proposition 2.5.1 (taking $\epsilon = 2^{-k}$ for $k \in \mathbb{Z}^+$) we see that there exists a sequence $(m_k)_{k=1}^{\infty}$ of positive integers such that

$$\text{for every } n \geq m_k, \quad \mathsf{m}[|f_n - f| \geq 2^{-k}] < 2^{-k}. \tag{2.19}$$

Denote $(n_k)_{k=1}^{\infty}$ to be the sequence of positive integers defined inductively by

$$\begin{aligned} n_1 &= m_1, \\ n_k &= \max\{n_{k-1}, m_k\} + 1 \quad (k \geq 2). \end{aligned}$$

Note that $(n_k)_{k=1}^{\infty}$ is a strictly increasing sequence of positive integers. Hence the sequence of functions $(f_{n_k})_{k=1}^{\infty}$ is a subsequence of $(f_n)_{n=1}^{\infty}$. Denote A to be the subset of X given by

$$A = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} [|f_{n_k} - f| \geq 2^{-k}].$$

We want to show that $\mathsf{m}(A) = 0$ and that for every $x \in X \setminus A$, we have $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

- Denote $(A_k)_{k=1}^{\infty}$ to be the sequence of measurable subsets of $\mathbb{R}$ given by

$$A_k = [|f_{n_k} - f| \geq 2^{-k}] \qquad (k \in \mathbb{Z}^+).$$

By (2.19) together with the construction of $(n_k)_{k=1}^{\infty}$, we get

$$\text{for every } k \in \mathbb{Z}^+, \quad \mathsf{m}(A_k) < 2^{-k},$$

which implies that $\sum_{k=1}^{\infty} \mathsf{m}(A_k) < \infty$. Note that $A = \limsup_{k\to\infty} A_k$. Hence by Proposition 1.4.21, we get $\mathsf{m}(A) = 0$.

- Let $x \in X \setminus A$, that is,

$$x \in \bigcup_{j=1}^{\infty} \left(X \setminus \bigcup_{k=j}^{\infty} [|f_{n_k} - f| \geq 2^{-k}] \right)$$

Then there exists $j_0 \in \mathbb{Z}^+$ such that

$$x \notin \bigcup_{k=j_0}^{\infty} [|f_{n_k} - f| \geq 2^{-k}],$$

that is,

$$\text{for every } k \geq j_0, \quad |f_{n_k}(x) - f(x)| < 2^{-k}.$$

From this, we see that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$.

□

To close this section, we give the following result which means that the limit of convergence in measure is unique almost everywhere.

Proposition 2.5.4 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f and g be measurable functions defined on X. Suppose $(f_n)_{n=1}^{\infty}$ converges in measure to both f and g. Then f and g are equal almost everywhere.*

Idea If $(h_j)_{j=1}^{\infty}$ converges almost everywhere to both f and g, then $f = g$ a.e..

Proof Since $(f_n)_{n=1}^{\infty}$ converges in measure to f, by Theorem 2.5.3, there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that $(f_{n_k})_{k=1}^{\infty}$ converges almost everywhere to f. Note that the sequence $(f_{n_k})_{k=1}^{\infty}$ converges in measure to g. Hence by Theorem 2.5.3 again, there exists a subsequence $(f_{n_{k_j}})_{j=1}^{\infty}$ of $(f_{n_k})_{k=1}^{\infty}$ such that $(f_{n_{k_j}})_{j=1}^{\infty}$ converges almost everywhere to g. Since the sequence $(f_{n_{k_j}})_{j=1}^{\infty}$ converges almost everywhere to both f and g, it follows that f and g are equal almost everywhere.

□

Exercise 2.5

1. Let $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ be sequences of measurable functions having the same domain X and let f and g be measurable functions defined on X. Suppose $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ converges in measure to f and g respectively. Show that
   (a) every subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ converges in measure to f;
   (b) for every $c \in \mathbb{R}$, the sequence $(cf_n)_{n=1}^{\infty}$ converges in measure to cf;
   (c) the sequence $(f_n + g_n)_{n=1}^{\infty}$ converges in measure to $f + g$;
   (d) if, in addition, $\mathsf{m}(X) < \infty$, then the sequence $(f_n \cdot g_n)_{n=1}^{\infty}$ converges in measure to $f \cdot g$.

   Give an example to show that (d) is false if $\mathsf{m}(X)$ is not assumed to be finite.
2. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f and g be measurable functions defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges in measure to f and that for every $n \in \mathbb{Z}^+$, we have $f_n \leq g$ a.e.. Show that $f \leq g$ a.e..

2.6 Measurable Extended-valued Functions

In this section, we describe briefly the concept and properties measurable extended real-valued functions. Many results in the last few sections on measurable (real-valued) functions are also valid for measurable extended real-valued functions. Some results require slight modification. In some results, some restrictions can be removed. We will state the results only. The proofs are similar to that for real-valued functions and are left as exercises.

Terminology An *extended real-valued function* is a function whose codomain is $\mathbb{R}^*$.

Note Every real-valued function can be considered as an extended real-valued function in a natural way:

- If $f : X \longrightarrow \mathbb{R}$ is a real-valued function, then the function $f_e : X \longrightarrow \mathbb{R}^*$ given by

$$f_e(x) = f(x) \quad \text{for } x \in X$$

  is an extended real-valued function.

Conversely, every extended real-valued function whose range is a subset of $\mathbb{R}$ can be considered as a real-valued function:

- If $f : X \longrightarrow \mathbb{R}^*$ is an extended real-valued function with $\text{range}(f) \subseteq \mathbb{R}$, then the function $f_r : X \longrightarrow \mathbb{R}$ given by

$$f_r(x) = f(x) \quad \text{for } x \in X$$

  is a real-valued function.

Definition An extended real-valued function is said to be *Lebesgue measurable*, or simply *measurable*, if the domain of f is a (non-empty) measurable subset of $\mathbb{R}$ and for every real number a, the set

$$\{x \in \operatorname{dom}(f) : f(x) > a\}$$

is measurable.

Remark If f is a real-valued function satisfying the condition that for every $a \in \mathbb{R}$, the set $\{x \in \operatorname{dom}(f) : f(x) > a\}$ is measurable, then the domain of f is measurable (Proposition 2.2.1). However, this is not true if *real-valued function* is replaced by *extended real-valued function.*

Below we discuss the extended real-valued version of some definitions and results in Section 2.2.

- The notations $[f > a]$, $[f \leq a]$ etc. can also be defined for extended real-valued functions f and extended real numbers a.
- Theorem 2.2.2 remains valid if "*real-valued function defined on a (non-empty) subset of* $\mathbb{R}$" is replaced by "*extended real-valued function defined on a (non-empty) measurable subset of* $\mathbb{R}$".
- Corollary 2.2.3 remains valid if "*measurable function*" is replaced by "*measurable extended real-valued function*".
- Proposition 2.2.4 remains valid if "*real-valued function*" is replaced by "*extended real-valued function*" and "*for every real number a*" is replaced by "*for every extended real number a*".
- For Proposition 2.2.5, we omit the discussion of the result for extended real-valued functions. This is because we haven't defined a metric on $\mathbb{R}^*$; we can't consider continuity of extended real-valued function.
- Proposition 2.2.6 is also valid if "*real-valued function*" is replaced by "*extended real-valued function*".
- Proposition 2.2.7 remains valid if "*measurable function*" is replaced by "*measurable extended real-valued function*".
- Proposition 2.2.8 remains valid if "*measurable functions*" is replaced by "*measurable extended real-valued functions*" with the additional assumption that the function $f + g$ is defined. Similar modifications apply to other operations of functions.
- Proposition 2.2.15 and Corollary 2.2.16 remain if "*measurable functions*" is replaced by "*measurable extended real-valued functions*". Moreover, for extended real-valued functions, there is no need to assume that for every $x \in X$, the set $\{f_n(x) : n \in \Lambda\}$ is bounded above or below.
- Corollary 2.2.17 remains valid if "*measurable function*" is replaced by "*measurable extended real-valued function*". For the function $|f|$, we can define it as $f^+ + f^-$ or define $|\infty| = \infty$ and $|-\infty| = \infty$.
- The notations $[f = g]$, $[f < g]$ etc., and the concept of *equal almost everywhere* can also be defined for extended real-valued functions.

- Theorem 2.2.18 remains valid if "*measurable function*" is replaced by "*measurable extended real-valued function*" and "*real-valued function*" is replaced by "*extended real-valued function*".
- The notations $f = 0$ a.e., $f \le g$ and $f \le g$ a.e. can also be defined for extended real-valued functions.

In Section 2.3, we consider convergence of measurable functions. For convergence of measurable extended real-valued functions, we have to consider convergence of sequences of extended real numbers. If we consider a sequence $(x_n)_{n=1}^{\infty}$ of extended real numbers to be convergent if $\lim_{n\to\infty} x_n$ exists in $\mathbb{R}^*$ (and say that $(x_n)_{n=1}^{\infty}$ converges to $\lim_{n\to\infty} x_n$), then most of the results in Section 2.3 remain valid for extended real-valued functions. For the Egoroff Theorem, we have to introduce an additional concept called *finite almost everywhere*.

- Proposition 2.3.1 and Corollary 2.3.2 are valid if "*measurable functions*" is replaced by "*measurable extended real-valued functions*". Moreover, for extended real-valued functions, there is no need to assume that for every $x \in X$, the set $\{f_n(x) : n \in \Lambda\}$ is bounded.
- Corollary 2.3.3 remains valid if "*measurable functions*" is replaced by "*measurable extended real-valued functions*" and "*limit in* $\mathbb{R}$" is replaced by "*limit in* $\mathbb{R}^*$".
- The notations $[f_n \to g]$ and $[f_n \not\to g]$ and the concept of *convergence almost everywhere* can be defined for extended real-valued functions, where limit means limit in $\mathbb{R}^*$.
- Theorem 2.3.4 remains valid if "*measurable functions*" is replaced by "*measurable extended real-valued functions*" and "*real-valued function*" is replaced by "*extended real-valued function*".
- For Theorem 2.3.5, the version for extended real-valued functions is quite different since the concepts of $\lim_{n\to\infty} x_n = a$ (where $a \in \mathbb{R}$) and $\lim_{n\to\infty} x_n = \infty$ are defined differently: the ϵ-N definition for the first limit involves small positive ϵ whereas the r-N definition for the second limit involves large positive r.

To discuss the extended real-valued version of the Egoroff Theorem, we need the following

Definition Let f be an extended real-valued function defined on a (non-empty) measurable subset of $\mathbb{R}$. We say that f is *finite almost everywhere*, denoted by $|f| < \infty$ a.e., if the set $\{x \in \operatorname{dom}(f) : f(x) = \pm\infty\}$ is a null set.

Notation $a = \pm\infty$ means $a = \infty$ or $a = -\infty$.

Egoroff Theorem *Let $(f_n)_{n=1}^{\infty}$ be a sequence of finite almost everywhere measurable extended real-valued functions having the same domain X and let g be a finite almost everywhere measurable extended real-valued function defined on X. Suppose that $\mathsf{m}(X) < \infty$ and that the sequence of functions $(f_n)_{n=1}^{\infty}$ converges almost everywhere to the function g. Then for every $\epsilon > 0$, there exists a measurable set A with $A \subseteq X$ such that $\mathsf{m}(A) < \epsilon$ and that $(f_n)_{n=1}^{\infty}$ converges uniformly in $X \setminus A$ to g.*

In Section 2.4, we consider approximations for measurable functions. Most of the results remain valid for extended real-valued measurable functions. For the Lusin Theorem, we have to consider functions that are finite almost everywhere.

- Proposition 2.4.2, Theorem 2.4.3, Theorem 2.4.4 and Theorem 2.4.6 remain valid if "*measurable function*" is replaced by "*measurable extended real-valued function*" and limit means *limit in* $\mathbb{R}^*$ and convergence means *convergence in* $\mathbb{R}^*$.
- To make the Lusin Theorem (both versions) valid for measurable extended real-valued function, it is necessary and sufficient to impose the condition that the function is finite almost everywhere. This is because we only define the concept of continuity for real-valued functions.

Lusin Theorem (Version 1) *Let f be a measurable extended real-valued function that is finite almost everywhere. Then for every $\epsilon > 0$, there exists a measurable subset A of $\mathbb{R}$ with $A \subseteq \operatorname{dom}(f)$ and $\mathsf{m}(A) < \epsilon$ such that $f|_{\operatorname{dom}(f)\setminus A}$ is continuous.*

Lusin Theorem (Version 2) *Let f be a measurable extended real-valued function that is finite almost everywhere. Then for every $\epsilon > 0$, there exists a continuous function g defined on $\operatorname{dom}(f)$ such that $\mathsf{m}[f \neq g] < \epsilon$.*

The concept of *convergence in measure* discussed in Section 2.5 can also be defined for measurable extended real-valued functions that are finite almost everywhere with special treatment for the notation $[|g - f| \geq \epsilon]$. Readers are suggested to formulate the concept and consider whether the results in Section 2.5 remain valid (with modifications) or not.

Remark Measurable extended real-valued functions that are not finite almost everywhere are not interesting. This is because

- ◇ the Egoroff Theorem and the Lusin Theorem can't be applied to these functions;
- ◇ they are not integrable (see Proposition 3.6.1).

If f is a finite almost everywhere measurable extended real-valued function, then (see Proposition 3.6.2) there exists a real-valued function g defined on $\operatorname{dom}(f)$ such that the set $\{x \in \operatorname{dom}(f) : f(x) \neq g(x)\}$ is a null set. In Functional Analysis, when we consider L^p-spaces, functions that are equal almost everywhere are considered to be the same function (the meaning can be made precise by considering equivalence relation and quotient set). For this reason, it is sufficient to consider measurable real-valued functions only.

Chapter 3

Lebesgue Integration

3.1 Introduction

The aim of this chapter is to introduce the concept of *Lebesgue integrals* for *Lebesgue integrable functions* and to study their properties. The concept of Lebesgue integral is a generalization of that of Riemann integral in the sense that if f is a Riemann integrable function defined on a closed and bounded interval, then it is a Lebesgue integrable function and the Riemann integral of f and the Lebesgue integral of f are the same.

Recall that when we define Riemann integrability of a function f over a closed and bounded interval, we use upper and lower Darboux sums. For this we have to assume that the function f is bounded. To define Lebesgue integral, we can use a similar approach: we can first consider bounded measurable functions defined on sets with finite measures and then consider arbitrary measurable functions defined on arbitrary measurable sets. However, the approach we adopt here is different.

- In section 2, we will introduce a notation $\mathrm{I}(\varphi)$ where φ is a non-negative simple function. The value $\mathrm{I}(\varphi)$ can be considered to be the integral of φ. The idea is similar to that for the integrals of step functions:

  If φ is a non-negative simple function, then it can be written as a linear combination

  $$\sum_{i=1}^{n} c_i \cdot \chi_{A_i}$$

  of characteristic functions defined on $\mathrm{dom}\,(\varphi)$. We denote by $\mathrm{I}(\varphi)$ to be the value

  $$\sum_{i=1}^{n} c_i \times \mathsf{m}(A_i). \tag{3.1}$$

  This value can be considered as the "area" of the subset of $\mathbb{R}^2$ that lies below the graph of φ and above the x-axis. However there are two questions about the uniqueness of the value in (3.1).

(i) In the sum, we may have $c_i = 0$ and $\mathsf{m}(A_i) = \infty$. To make the value well-defined, we take the sum over all i's such that $c_i > 0$ for the case where φ is not identically 0.

(ii) There are more than one ways to express φ as linear combinations of characteristic functions. To make the value $\mathrm{I}(\varphi)$ well-defined, we take the canonical representation (to be defined in the next section) of φ. Later, we will see that the value $\mathrm{I}(\varphi)$ is unchanged if we take other linear combinations.

- In section 3, we will introduce a notation $\mathrm{L}(f)$ where f is a non-negative measurable function (the domain can have infinite measure). Usually, the notation $\mathrm{L}(f)$ is written as $\int f$ and is called the *Lebesgue integral* of f. The idea is similar to that for lower integral (Riemann):

  If f is a non-negative measurable function, then we define $\mathrm{L}(f)$ to be the supremum of the set of all values $\mathrm{I}(\varphi)$ where φ runs through the collection of all non-negative simple functions that are dominated by f.

- In section 4, we will introduce the concept of *Lebesgue integrable functions.* If f is a measurable function, then we can write

$$f = g - h$$

where g and h are non-negative measurable functions. It is natural to define the integral of f to be the difference of $\mathrm{L}(g)$ and $\mathrm{L}(h)$. However, there are more than one ways to write f as a difference of two non-negative measurable functions. Moreover, we may have $\infty - \infty$. To make the integral of f well-defined, we

(i) use the positive part and negative part of f and

(ii) impose the condition that at least one of the two extended real numbers $\mathrm{L}(f^+)$ and $\mathrm{L}(f^-)$ is not ∞.

We define a measurable function f to be *integrable* if the integral of f is a real number.

3.2 Integration of Non-negative Simple Functions

Recall that a real-valued function φ defined on a (non-empty measurable) subset of $\mathbb{R}$ is called a simple function if there exist a distinct family $\{c_i\}_{i=1}^n$ of real numbers and a disjoint family $\{A_i\}_{i=1}^n$ of measurable subsets of $\mathbb{R}$ with $\mathrm{dom}\,(\varphi) = \bigcup\limits_{i=1}^{n} A_i$ such that $\varphi = \sum\limits_{i=1}^{n} c_i \cdot \chi_{A_i, \mathrm{dom}\,(\varphi)}$.

There are many ways to express a non-negative simple function φ as a linear combination of characteristic functions $\sum\limits_{i=1}^{k} a_i \cdot \chi_{B_i, \mathrm{dom}\,(\varphi)}$ We take a natural one and call it the *canonical representation.* In order to avoid $0 \times \infty$ in the sum $\sum\limits_{i=1}^{k} a_i \times \mathsf{m}(B_i)$, we omit the terms where $a_i = 0$. For this, we have to assume that $\mathrm{range}\,(\varphi) \setminus \{0\} \neq \emptyset$, that is, φ is not identically 0.

Definition Let f be a non-negative real-valued function such that the set range(f) is finite and that the set range$(f) \setminus \{0\}$ is non-empty.

◇ Denote n to be the number of elements of range$(f) \setminus \{0\}$.

◇ Denote $\{c_i\}_{i=0}^n$ to be the family of real numbers defined (inductively) as follows:

$$\begin{aligned} c_0 &= 0, \\ c_i &= \min\big(\operatorname{range}(f) \setminus \{c_0, \ldots, c_{i-1}\}\big), \qquad (i = 1, \ldots, n). \end{aligned}$$

◇ Denote $\{A_i\}_{i=0}^n$ to be the family of subsets of dom(f) given by

$$A_i = \{x \in \operatorname{dom}(f) : f(x) = c_i\}, \qquad (i = 0, \ldots, n).$$

The ordered $(2n+2)$-tuple $(c_0, c_1, \ldots, c_n, A_0, A_1, \ldots, A_n)$ is called the *canonical representation* of f.

Remark

◇ The family $\{A_i\}_{i=0}^n$ is a disjoint family of subsets of dom(f) with $\bigcup\limits_{i=0}^{n} A_i = \operatorname{dom}(f)$. The set A_0 may be empty. For every $i = 1, \ldots, n$, the set A_i is non-empty.

◇ If $0 \in \operatorname{range}(f)$, then we have $A_0 \neq \emptyset$ and $\operatorname{range}(f) = \{c_0, c_1, \ldots, c_n\}$.
If $0 \notin \operatorname{range}(f)$, then we have $A_0 = \emptyset$ and $\operatorname{range}(f) = \{c_1, \ldots, c_n\}$.

◇ $f = \sum\limits_{i=0}^{n} c_i \cdot \chi_{A_i, \operatorname{dom}(f)} = \sum\limits_{i=1}^{n} c_i \cdot \chi_{A_i, \operatorname{dom}(f)}$

Example The canonical representation of the Dirichlet function defined on $[0,1]$ is $\big(0,\ 1,\ [0,1] \setminus \mathbb{Q},\ \mathbb{Q}\big)$.

Definition Let φ be a non-negative simple function.

- If φ is identically 0, we denote $\mathrm{I}(\varphi) = 0$.
- If φ is not identically 0, we denote $\mathrm{I}(\varphi)$ to be the non-negative extended real-number given by

$$\mathrm{I}(\varphi) = \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i),$$

where $(0, c_1, \ldots, c_n, A_0, A_1, \ldots, A_n)$ is the canonical representation of φ.

Remark If $\varphi = \sum\limits_{i=1}^{k} a_i \cdot \chi_{B_i, \operatorname{dom}(\varphi)}$, where $\{a_i\}_{i=1}^k$ is a family of positive real numbers and $\{B_i\}_{i=1}^k$ is a family of measurable sets that are subsets of dom(f), then we have $\mathrm{I}(\varphi) = \sum\limits_{i=1}^{k} a_i \times \mathsf{m}(B_i)$. We will prove the result for the case where the family $\{B_i\}_{i=1}^k$ is disjoint and $\bigcup\limits_{i=1}^{k} B_i = \operatorname{dom}(\varphi)$ in Lemma 3.2.4 (in fact, it is a special case of the result in the lemma), and we will prove the result in general in Corollary 3.2.7.

Remark If φ is a step function defined on a closed and bounded interval $[a, b]$, then we have $\mathrm{I}(\varphi) = \int_a^b \varphi(x)\,\mathrm{d}x$, the Riemann integral of φ over $[a, b]$. The proof of this result is left as an exercise.

Example Denote the Dirichlet function on $[0, 1]$ by φ. Since the canonical representation of φ is

$$\big(0,\ 1,\ [0,1] \setminus \mathbb{Q},\ \mathbb{Q}\big),$$

it follows from definition that

$$\mathrm{I}(\varphi) = 1 \times \mathsf{m}(\mathbb{Q}) = 0.$$

More generally, we have the following

Proposition 3.2.1 *Let X be a non-empty measurable subset of $\mathbb{R}$ and let A be a measurable subset of X. Then we have $\mathrm{I}(\chi_{A,X}) = \mathsf{m}(A)$.*

Idea Apply definition. Consider the two cases: (i) $A = \emptyset$ and (ii) $A \neq \emptyset$.

Proof To prove the result, we consider the following two cases (for simplicity, $\chi_{A,X}$ is written as χ_A):

(*Case 1*) $A = \emptyset$

In this case, the function χ_A is identically 0 and so we have $\mathrm{I}(\chi_A) = 0$.

(*Case 2*) $A \neq \emptyset$

In this case, the function χ_A is not identically 0. Since the canonical representation of χ_A is $(0, 1, X \setminus A, A)$, it follows from definition that

$$\mathrm{I}(\chi_A) = 1 \times \mathsf{m}(A) = \mathsf{m}(A).$$

□

Lemma 3.2.2 *Let φ be a non-negative simple function. Suppose $\varphi = 0$ a.e.. Then we have $\mathrm{I}(\varphi) = 0$.*

Idea For the case where φ is not identically 0, consider the canonical representation of φ. For $i \geq 1$, the set A_i is a null set.

Proof To prove that $\mathrm{I}(\varphi) = 0$, we consider the following two cases:

(*Case 1*) φ is identically 0

In this case, by definition, we have $\mathrm{I}(\varphi) = 0$.

(*Case 2*) φ is not identically 0

In this case, we have

$$\mathrm{I}(\varphi) = \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i),$$

where $(0, c_1, \ldots, c_n, A_0, A_1, \ldots, A_n)$ is the canonical representation of φ.

Note that for every $i = 1, \ldots, n$, the set A_i is a subset of $[\varphi \neq 0]$ which is a null set. Hence we have

$$\text{for every } i = 1, \ldots, n, \quad \mathsf{m}(A_i) = 0,$$

from which we obtain

$$\sum_{i=1}^{n} c_i \times \mathsf{m}(A_i) = 0,$$

that is, $\mathrm{I}(\varphi) = 0$.

□

Below, we will show that the operator I preserves multiplication by positive constants, preserves addition and preserves order. These results will be used in the next section.

Lemma 3.2.3 *Let* φ *be a non-negative simple function and let* k *be a positive real number. Then we have* $\mathrm{I}(k\varphi) = k \cdot \mathrm{I}(\varphi)$.

Idea Consider the canonical representation of φ for the case where φ is not identically 0.

Proof To prove that $\mathrm{I}(k\varphi) = k \cdot \mathrm{I}(\varphi)$, we consider the following two cases:

(*Case 1*) φ is identically 0

In this case, $k\varphi$ is also identically 0. Hence we have $\mathrm{I}(k\varphi) = 0 = k \cdot \mathrm{I}(\varphi)$.

(*Case 2*) φ is not identically 0 In this case, we have

$$\mathrm{I}(\varphi) = \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i),$$

where $(0, c_1, \ldots, c_n, A_0, A_1, \ldots, A_n)$ is the canonical representation of φ. It is easy to see that the canonical representation of $k\varphi$ is

$$(0, kc_1, \ldots, kc_n, A_0, A_1, \ldots, A_n).$$

Hence by definition, we have

$$\begin{aligned} \mathrm{I}(k\varphi) &= \sum_{i=1}^{n} kc_i \times \mathsf{m}(A_i) \\ &= k \cdot \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i) \\ &= k \cdot \mathrm{I}(\varphi). \end{aligned}$$

□

Unlike Lemma 3.2.3, to prove that the operator I preserves addition, that is, $\mathrm{I}(\varphi + \quad) = \mathrm{I}(\varphi) + \mathrm{I}(\quad)$ where φ and are non-negative simple functions having the same domain, we cannot use definition. This is because we cannot add canonical representations together. To prove the result, we need the following

Lemma 3.2.4 *Let φ be a non-negative simple function that is not identically* 0*. Suppose that* $\varphi = \sum_{j=1}^{p} a_j \cdot \chi_{B_j, \mathrm{dom}\,(\varphi)}$ *where* $\{a_j\}_{j=1}^p$ *is a family of non-negative real numbers and* $\{B_j\}_{j=1}^p$ *is a disjoint family of measurable sets with* $\bigcup_{j=1}^{p} B_j = \mathrm{dom}\,(\varphi)$*. Then we have*

$$\mathrm{I}(\varphi) = \sum_{j \in \Lambda_+} a_j \times \mathsf{m}(B_j),$$

where $\Lambda_+ = \{j \in \{1, \ldots, p\} : a_j > 0\}$*.*

Remark Since φ is not identically 0, it follows that $\Lambda_+ \neq \emptyset$. Note that the family of numbers $\{a_j\}_{j=1}^p$ may not be distinct and that some of the sets $B_1, \ldots, B_p$ may be empty.

Idea Discard the B_j's that are empty and group together the B_j's such that the values of the a_j's are the same.

Proof Denote $(0, c_1, \ldots, c_n, A_0, A_1, \ldots, A_n)$ to be the canonical representation of φ. Since $\{B_j\}_{j=1}^p$ is a disjoint family of measurable sets with $\bigcup_{j=1}^{p} B_j = \mathrm{dom}\,(\varphi)$ and $\varphi = \sum_{j=1}^{p} a_j \cdot \chi_{B_j, \mathrm{dom}\,(\varphi)}$, it follows that

$$\text{range of } \varphi \subseteq \{a_1, \ldots, a_p\},$$

which implies that

$$\{c_1, \ldots, c_n\} \subseteq \{a_1, \ldots, a_p\}.$$

Denote $\{\Lambda_i\}_{i=1}^n$ to be the family of subsets of Λ_+ given by

$$\Lambda_i = \{j \in \Lambda_+ : a_j = c_i\}, \qquad (1 \leq i \leq n). \tag{3.2}$$

It is clear that

(†) the family of sets $\{\Lambda_i\}_{i=1}^n$ is disjoint.

It is also clear that if $j \in \Lambda_+ \setminus \bigcup_{i=1}^{n} \Lambda_i$, then a_j does not belong to the range of f and so we have

$$\text{for every } j \in \Lambda_+ \setminus \bigcup_{i=1}^{n} \Lambda_i, \qquad B_j = \emptyset$$

which yields

$$\text{for every } j \in \Lambda_+ \setminus \bigcup_{i=1}^{n} \Lambda_i, \qquad \mathsf{m}(B_j) = 0. \tag{3.3}$$

Moreover, since $\{B_j\}_{j=1}^p$ is a disjoint family of measurable sets with $\bigcup_{j=1}^{p} B_j = \mathrm{dom}\,(\varphi)$ and

$$\text{for every } i = 1, \ldots, n, \qquad A_i = [\varphi = c_i],$$

it follows from the construction in (3.2) that for every $i = 1, \ldots, n$, the family $\{B_j\}_{j \in \Lambda_i}$ is a disjoint family of subsets of A_i with $\bigcup\limits_{j \in \Lambda_i} B_j = A_i$ and so we have

$$\text{for every } i = 1, \ldots, n, \qquad \mathsf{m}(A_i) = \sum_{j \in \Lambda_i} \mathsf{m}(B_j). \tag{3.4}$$

Hence by (†), (3.3), (3.2) and (3.4), we have

$$\begin{aligned}
\sum_{j \in \Lambda_+} a_j \times \mathsf{m}(B_j) &= \sum_{j \in \bigcup\limits_{i=1}^{n} \Lambda_i} a_j \times \mathsf{m}(B_j) + \sum_{j \in \Lambda \setminus \bigcup\limits_{j=1}^{n} \Lambda_j} a_j \times \mathsf{m}(B_j) \\
&= \sum_{i=1}^{n} \sum_{j \in \Lambda_i} a_j \times \mathsf{m}(B_j) + \sum_{j \in \Lambda \setminus \bigcup\limits_{j=1}^{n} \Lambda_j} a_j \times 0 \\
&= \sum_{i=1}^{n} c_i \times \left(\sum_{j \in \Lambda_i} \mathsf{m}(B_j) \right) \\
&= \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i) \;=\; \mathrm{I}(\varphi)
\end{aligned}$$

□

Lemma 3.2.5 *Let φ and   be non-negative simple functions defined on the same domain. Then we have* $\mathrm{I}(\varphi + \quad) = \mathrm{I}(\varphi) + \mathrm{I}(\quad)$.

Idea For the case where φ and   are not identically 0, express both of them in the form $\sum\limits_{\alpha \in \Lambda} b_\alpha \chi_{S_\alpha}$ where $\{S_\alpha\}_{\alpha \in \Lambda}$ is a disjoint family of measurable sets whose union is the common domain of φ and   .

Proof To prove that $\mathrm{I}(\varphi + \quad) = \mathrm{I}(\varphi) + \mathrm{I}(\quad)$, we consider the following two cases:

(*Case 1*) φ or   is identically 0

By symmetry, it suffices to consider the case where φ is identically 0. In this case, we have

$$\mathrm{I}(\varphi + \quad) = \mathrm{I}(\quad) = 0 + \mathrm{I}(\quad) = \mathrm{I}(\varphi) + \mathrm{I}(\quad).$$

(*Case 2*) φ and   are not identically 0

In this case, we have

$$\mathrm{I}(\varphi) = \sum_{i=1}^{p} c_i \times \mathsf{m}(A_i) \qquad \text{and} \qquad \mathrm{I}(\quad) = \sum_{j=1}^{q} d_j \times \mathsf{m}(B_j)$$

where $(c_0, c_1, \ldots, c_p, A_0, A_1, \ldots, A_p)$ and $(d_0, d_1, \ldots, d_q, B_0, B_1, \ldots, B_q)$ are the canonical representations of φ and   respectively.

Denote $\Lambda = \{0, 1, \ldots, p\} \times \{0, 1, \ldots, q\}$ and denote $\{S_{(i,j)}\}_{(i,j)\in\Lambda}$ to be the family of subsets of $\mathbb{R}$ given by

$$S_{(i,j)} = A_i \cap B_j, \qquad (i, j) \in \Lambda.$$

It is clear that

(†) $\{S_{(i,j)}\}_{(i,j)\in\Lambda}$ is a disjoint family of measurable sets with $\bigcup\limits_{(i,j)\in\Lambda} S_{(i,j)} = X$, where X is the common domain of φ and .

Below, the domains of the characteristic functions are understood to be X.

Note that for every $i = 0, \ldots, p$, the family $\{S_{(i,j)}\}_{j=0}^{q}$ is a disjoint family of measurable sets with $\bigcup\limits_{j=0}^{q} S_{(i,j)} = A_i$. Hence we have

$$\text{for every } i = 0, \ldots, p, \qquad \chi_{A_i} = \sum_{j=0}^{q} \chi_{S_{(i,j)}}; \tag{3.5}$$

$$\text{for every } i = 0, \ldots, p, \qquad \mathsf{m}(A_i) = \sum_{j=0}^{q} \mathsf{m}(S_{(i,j)}). \tag{3.6}$$

Similarly, we have

$$\text{for every } j = 0, \ldots, q, \qquad \chi_{B_j} = \sum_{i=0}^{p} \chi_{S_{(i,j)}}; \tag{3.7}$$

$$\text{for every } j = 0, \ldots, q, \qquad \mathsf{m}(B_j) = \sum_{i=0}^{p} \mathsf{m}(S_{(i,j)}). \tag{3.8}$$

Since $\varphi = \sum\limits_{i=0}^{p} c_i \cdot \chi_{A_i}$ and $\quad = \sum\limits_{j=0}^{q} d_j \cdot \chi_{B_j}$, it follows from (3.5) and (3.7) that

$$\varphi = \sum_{(i,j)\in\Lambda} c_i \cdot \chi_{S_{(i,j)}} \qquad \text{and} \qquad \quad = \sum_{(i,j)\in\Lambda} d_i \cdot \chi_{S_{(i,j)}}$$

which yields

$$\varphi + \quad = \sum_{(i,j)\in\Lambda} (c_i + d_j)\chi_{S_{(i,j)}}.$$

In view of (†), by Lemma 3.2.4, we have

$$\mathrm{I}(\varphi + \quad) = \sum_{(i,j)\in\Lambda_+} (c_i + d_j)\mathsf{m}(S_{(i,j)}),$$

where $\Lambda_+ = \Lambda \setminus \{(0, 0)\}$. Grouping terms and rearranging, we get

$$\begin{aligned}
\mathrm{I}(\varphi + \quad) &= \sum_{i=1}^{p}\sum_{j=1}^{q}(c_i + d_j)\cdot \mathsf{m}(S_{(i,j)}) + \sum_{i=1}^{p}(c_i + 0)\cdot \mathsf{m}(S_{(i,0)}) + \sum_{j=1}^{q}(0 + d_j)\cdot \mathsf{m}(S_{(0,j)}) \\
&= \left(\sum_{i=1}^{p}\sum_{j=1}^{q} c_i \cdot \mathsf{m}(S_{(i,j)}) + \sum_{i=1}^{p} c_i \cdot \mathsf{m}(S_{(i,0)})\right) + \left(\sum_{j=1}^{q}\sum_{i=1}^{p} d_j \cdot \mathsf{m}(S_{(i,j)}) + \sum_{j=1}^{q} d_j \cdot \mathsf{m}(S_{(0,j)})\right) \\
&= \sum_{i=1}^{p}\sum_{j=0}^{q} c_i \cdot \mathsf{m}(S_{(i,j)}) + \sum_{j=1}^{q}\sum_{i=0}^{p} d_j \cdot \mathsf{m}(S_{(i,j)}) \\
&= \sum_{i=1}^{p} c_i \times \mathsf{m}(A_i) + \sum_{j=1}^{q} d_j \times \mathsf{m}(B_j) \\
&= \mathrm{I}(\varphi) + \mathrm{I}(\quad),
\end{aligned}$$

where the second last equality follows from (3.6) and (3.8).

□

Corollary 3.2.6 *Let φ and   be non-negative simple functions with the same domain. Suppose that $\varphi \leq \quad$. Then we have $\mathrm{I}(\varphi) \leq \mathrm{I}(\quad)$.*

Idea Use $\quad = (\quad - \varphi) + \varphi$.

Proof Note that $\quad - \varphi$ is a non-negative simple function and that

$$\quad = (\quad - \varphi) + \varphi.$$

It follows from Lemma 3.2.5 that

$$\mathrm{I}(\quad) = \mathrm{I}(\quad - \varphi) + \mathrm{I}(\varphi). \tag{3.9}$$

Since $\mathrm{I}(\quad - \varphi)$ is a non-negative extended real number, it follows from (3.9) that $\mathrm{I}(\quad) \geq \mathrm{I}(\varphi)$. □

Corollary 3.2.7 *Let φ be a non-negative simple function. Suppose that $\varphi = \sum\limits_{j=1}^{p} a_j \cdot \chi_{B_j, \mathrm{dom}\,(\varphi)}$ where $\{a_j\}_{j=1}^{p}$ is a family of positive real numbers and $\{B_j\}_{j=1}^{p}$ is a family of measurable sets that are subsets of* dom (φ). *Then we have*

$$\mathrm{I}(\varphi) = \sum_{j=1}^{p} a_j \times \mathsf{m}(B_j).$$

Idea The operator I preserves addition. Use induction.

Proof In the proof below, the domains of the characteristic functions are understood to be dom (φ).

Denote $\{\varphi_j\}_{j=1}^{p}$ to be the finite family of non-negative simple functions given by

$$\varphi_j = a_j \cdot \chi_{B_j} \qquad (j = 1, \ldots, p).$$

By Lemma 3.2.3 and Proposition 3.2.1, we have

$$\text{for every } j = 1, \ldots, p, \qquad \mathrm{I}(\varphi_j) = a_j \times \mathrm{I}(\chi_{B_j}) = a_j \times \mathsf{m}(B_j). \tag{3.10}$$

Since $\varphi = \sum_{j=1}^{p} \varphi_j$, it follows from Lemma 3.2.5 (and induction) that

$$\mathrm{I}(\varphi) = \sum_{j=1}^{p} \mathrm{I}(\varphi_j). \tag{3.11}$$

The required equality then follows from (3.10) and (3.11). □

To close this section, we prove one more result that will be used in the next section.

Lemma 3.2.8 *Let φ be a non-negative simple function. Suppose $(S_k)_{k=1}^{\infty}$ is an increasing sequence of measurable subsets of $\mathbb{R}$ such that $\bigcup_{k=1}^{\infty} S_k = \mathrm{dom}\,(f)$. Then we have*

$$\lim_{k\to\infty} \mathrm{I}\big(\varphi \cdot \chi_{S_k, \mathrm{dom}\,(\varphi)}\big) = \mathrm{I}(\varphi).$$

Idea Express $\varphi \cdot \chi_{S_k}$ in the form $\sum_{i=1}^{n} a_i \cdot \chi_{B_{k,i}}$ and consider $\lim_{k\to\infty} \mathsf{m}(B_{k,i})$.

Proof To prove the result, we consider the following two cases:

(*Case 1*) φ is identically 0

In this case, for every $k \in \mathbb{Z}^+$, the function $\varphi \cdot \chi_{S_k, \mathrm{dom}\,(\varphi)}$ is identically 0 and so we have

$$\lim_{k\to\infty} \mathrm{I}\big(\varphi \cdot \chi_{S_k, \mathrm{dom}\,(\varphi)}\big) = 0 = \mathrm{I}(\varphi).$$

(*Case 2*) φ is not identically 0

In this case, we have $\mathrm{I}(\varphi) = \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i)$ where $(0, c_1, \ldots, c_n, A_0, A_1, \ldots, A_n)$ is the canonical representation of φ.

In the proof below, the domains of the characteristic functions are understood to be $\mathrm{dom}\,(\varphi)$.

Since $\varphi = \sum_{i=1}^{n} c_i \cdot \chi_{A_i}$ and

$$\text{for every } i = 1, \ldots, n, \quad \text{for every } k \in \mathbb{Z}^+, \qquad \chi_{A_i} \cdot \chi_{S_k} = \chi_{A_i \cap S_k},$$

it follows that

$$\text{for every } k \in \mathbb{Z}^+, \qquad \varphi \cdot \chi_{S_k} = \sum_{i=1}^{n} c_i \cdot \chi_{A_i \cap S_k}.$$

Hence by Corollary 3.2.7, we have

$$\text{for every } k \in \mathbb{Z}^+, \qquad \mathrm{I}(\varphi \cdot \chi_{S_k}) = \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i \cap S_k). \tag{3.12}$$

Since $(S_k)_{k=1}^{\infty}$ is an increasing sequence of measurable sets and $\bigcup_{k=1}^{\infty} S_k = \mathrm{dom}\,(f)$, it follows that for every $i = 1, \ldots, n$, the sequence of sets $(A_i \cap S_k)_{k=1}^{\infty}$ is increasing and $\bigcup_{k=1}^{\infty} (A_i \cap S_k) = A_i$. Hence by Theorem 1.4.17, we have

$$\text{for every } i =, 1 \ldots, n, \qquad \lim_{k\to\infty} \mathsf{m}(A_i \cap S_k) = \mathsf{m}(A_i). \tag{3.13}$$

Therefore, by (3.12) and (3.13), we get

$$\begin{aligned} \lim_{k\to\infty} \mathrm{I}(\varphi \cdot \chi_{S_k}) &= \lim_{k\to\infty} \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i \cap S_k) \\ &= \sum_{i=1}^{n} c_i \times \lim_{k\to\infty} \mathsf{m}(A_i \cap S_k) \\ &= \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i) \\ &= \mathrm{I}(\varphi) \end{aligned}$$

□

Remark In the above lemma, the sequence of functions $(\varphi \cdot \chi_{S_k})_{k=1}^{\infty}$ is increasing and for every $x \in \mathrm{dom}\,(\varphi)$, we have $\lim_{k\to\infty} \varphi \cdot \chi_{S_k}(x) = \varphi(x)$. The result in the lemma is a special case of the Dominated Convergence Theorem that will be discussed in the next section.

Notation Let φ be a non-negative simple function and let A be a measurable set with $A \subseteq \mathrm{dom}\,(\varphi)$. We denote $\mathrm{I}_A(\varphi)$ to be the non-negative extended real number given by

$$\mathrm{I}_A(\varphi) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \mathrm{I}\big(\varphi|_A\big) & \text{if } A \neq \emptyset. \end{cases}$$

The following result can be used to give an alternative definition for $\mathrm{I}_A(\varphi)$.

Lemma 3.2.9 *Let φ be a non-negative simple function and let A be a measurable set with $A \subseteq \mathrm{dom}\,(\varphi)$. Then we have*

$$\mathrm{I}_A(\varphi) = \mathrm{I}(\varphi \cdot \chi_{A,\mathrm{dom}\,(\varphi)}).$$

Idea For the case where $A \neq \emptyset$ and $\varphi\big|_A$ is not identically 0, consider the canonical representation of φ.

Proof To prove the result, we consider the following two cases:

(*Case 1*) $A = \emptyset$

In this case, we have $\mathrm{I}_A(\varphi) = 0$. Note that the function $\varphi \cdot \chi_{A,\mathrm{dom}\,(\varphi)}$ is identically 0. Hence by definition, we have $\mathrm{I}(\varphi \cdot \chi_{A,\mathrm{dom}\,(\varphi)}) = 0$.

(*Case 2*) $A \neq \emptyset$

In this case, to prove the result, we consider the following two subcases:

(*Subcase 2a*) $\varphi\big|_A$ is identically 0

In this subcase, the function $\varphi \cdot \chi_{A,\mathrm{dom}\,(\varphi)}$ is also identically 0. Hence by definition, we have

$$\mathrm{I}_A(\varphi) = \mathrm{I}\big(\varphi|_A\big) = 0 \qquad \text{and} \qquad \mathrm{I}(\varphi \cdot \chi_{A,\mathrm{dom}\,(\varphi)}) = 0.$$

(*Subcase 2b*) $\varphi\big|_A$ is not identically 0

In this subcase, the function φ is not identically 0. Denote $(0, c_1, \ldots, c_n, A_0, A_1, \ldots, A_n)$ to be the canonical representation of φ. Then we have

$$\varphi = \sum_{i=1}^{n} c_i \cdot \chi_{A_i,\mathrm{dom}\,(\varphi)}.$$

Note that

$$\varphi \cdot \chi_{A,\mathrm{dom}\,(\varphi)} = \sum_{i=1}^{n} c_i \cdot \chi_{A_i \cap A,\mathrm{dom}\,(\varphi)} \qquad \text{and} \qquad \varphi\big|_A = \sum_{i=1}^{n} c_i \cdot \chi_{A_i \cap A, A}.$$

Hence by Corollary 3.2.7, we get

$$\mathrm{I}_A(\varphi) = \mathrm{I}\big(\varphi|_A\big) = \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i \cap A) = \mathrm{I}(\varphi \cdot \chi_{A,\mathrm{dom}\,(\varphi)}).$$

□

Exercise 3.2

1. Let φ be a step function defined on a closed and bounded interval $[a, b]$. Show that $\mathrm{I}(\varphi) = \int_a^b \varphi(x)\,\mathrm{d}x$.
2. Let φ be a non-negative simple function. Suppose $\varphi = \sum\limits_{j=1}^{p} a_j \cdot \chi_{B_j,\mathrm{dom}\,(\varphi)}$ where $\{a_j\}_{j=1}^{p}$ is a family of real numbers and $\{B_j\}_{j=1}^{p}$ is a family of measurable sets that are subsets of $\mathrm{dom}\,(\varphi)$ such that for every $j = 1, \ldots, p$, we have $\mathsf{m}(B_j) < \infty$. Show that $\mathrm{I}(\varphi) = \sum\limits_{j=1}^{p} a_j \times \mathsf{m}(B_j)$.
3. Let φ be non-negative simple function and let $\{A_n\}_{n\in\Lambda}$ be a disjoint countable family of measurable sets that are subsets of $\mathrm{dom}\,(\varphi)$. Show that $\mathrm{I}\big(\varphi \cdot \chi_{\bigcup\limits_{n\in\Lambda} A_n}\big) = \sum\limits_{n\in\Lambda} \mathrm{I}\big(\varphi \cdot \chi_{A_n}\big)$, where the domains of the characteristic functions are understood to be $\mathrm{dom}\,(\varphi)$.

3.3 Integration of Non-negative Measurable Functions

In this section, we will introduce the notation $\mathrm{L}(f)$ where f is a non-negative measurable. Usually, this notation is denoted by $\int f$ and is called the Lebesgue integral of f. However, we reserve the notation $\int f$ for measurable functions satisfying some conditions (see next section).

Definition Let f be a non-negative measurable function. We denote $\mathrm{L}(f)$ to be the extended real-number given by

$$\mathrm{L}(f) = \sup\Big\{\mathrm{I}(\varphi) : \varphi \text{ is a non-negative simple function defined on } \mathrm{dom}\,(f) \text{ and } \varphi \leq f\Big\}$$

The following proposition means that the operator L is an extension of the operator I.

Proposition 3.3.1 *Let f be a non-negative simple function. Then we have*

$$\mathrm{L}(f) = \mathrm{I}(f).$$

Idea The operator I preserves order.

Proof

- By taking $\varphi = f$ in the definition of $\mathrm{L}(f)$, we get

$$\mathrm{L}(f) \geq \mathrm{I}(f).$$

- By Corollary 3.2.6, we have

$$\text{for every non-negative simple functions } \varphi \text{ defined on } \mathrm{dom}\,(f) \text{ with } \varphi \leq f, \quad \mathrm{I}(\varphi) \leq \mathrm{I}(f).$$

  Hence by definition, we get

$$\mathrm{L}(f) \leq \mathrm{I}(f).$$

□

Lemma 3.3.2 *Let f be a non-negative measurable function. Suppose $f = 0$ a.e.. Then we have*

$$\mathrm{L}(f) = 0.$$

Idea If φ is a non-negative simple function and $\varphi \leq f$, then $\varphi = 0$ a.e..

Proof To prove the required equality, it is sufficient (and also necessary) to show that

$$\text{for every non-negative simple functions } \varphi \text{ defined on } \mathrm{dom}\,(f) \text{ with } \varphi \leq f, \quad \mathrm{I}(\varphi) = 0.$$

Let φ be a non-negative simple function defined on $\mathrm{dom}\,(f)$ with $\varphi \leq f$. Since $[f \neq 0]$ is a null set and

$$[\varphi \neq 0] \subseteq [f \neq 0],$$

it follows that $[\varphi \neq 0]$ is a null set, that is, $\varphi = 0$ a.e.. Hence by Lemma 3.2.2, we have $\mathrm{I}(\varphi) = 0$. □

Lemma 3.3.3 *Let f and g be non-negative measurable functions having the same domain. Suppose that $f \leq g$. Then we have*

$$\mathrm{L}(f) \leq \mathrm{L}(g).$$

Idea Use definition.

Proof To prove the required inequality, by the definition of $\mathrm{L}(f)$, it is sufficient (and also necessary) to show that

$$\text{for every non-negative simple function } \varphi \text{ defined on } X \text{ with } \varphi \leq f, \quad \mathrm{I}(\varphi) \leq \mathrm{L}(g),$$

where X is the common domain of f and g.

Let φ be a non-negative simple function defined on X with $\varphi \leq f$. Since $f \leq g$, it follows that

$$\varphi \leq g.$$

Hence by the definition of $\mathrm{L}(g)$, we get

$$\mathrm{I}(\varphi) \leq \mathrm{L}(g).$$

□

The following result is the converse of Lemma 3.3.2.

Proposition 3.3.4 *Let f be a non-negative measurable function. Suppose that $\mathrm{L}(f) = 0$. Then we have*

$$f = 0 \ \text{ a.e..}$$

Idea Use $[f > 0] = \bigcup\limits_{n=1}^{\infty} [f \geq \frac{1}{n}]$.

Proof We want to show that $[f \neq 0]$ is a null set. For this, it is sufficient (and also necessary) to show that $[f > 0]$ is a null set. This is because f is non-negative.

Denote $(A_n)_{n=1}^{\infty}$ to be the sequence of measurable sets given by

$$A_n = [f \geq \tfrac{1}{n}], \quad (n \in \mathbb{Z}^+).$$

It is clear from the construction of $(A_n)_{n=1}^{\infty}$ that

$$\text{for every } n \in \mathbb{Z}^+, \qquad \tfrac{1}{n} \cdot \chi_{A_n} \leq f \cdot \chi_{A_n} \leq f,$$

where χ_{A_n} means $\chi_{A_n, \mathrm{dom}\,(f)}$. Hence by Lemma 3.3.3 we have

$$\text{for every } n \in \mathbb{Z}^+, \qquad \mathrm{L}\big(\tfrac{1}{n} \cdot \chi_{A_n}\big) \leq \mathrm{L}(f). \tag{3.14}$$

By Corollary 3.2.7, we have

$$\text{for every } n \in \mathbb{Z}^+, \qquad \mathrm{L}\big(\tfrac{1}{n} \cdot \chi_{A_n}\big) = \frac{1}{n} \times \mathsf{m}(A_n). \tag{3.15}$$

Since $\mathrm{L}(f) = 0$, it follows from (3.14) and (3.15) that

$$\text{for every } n \in \mathbb{Z}^+, \qquad \mathsf{m}(A_n) = 0.$$

Since $[f > 0] = \bigcup\limits_{n=1}^{\infty} A_n$, it follows that $[f > 0]$ is a null set. □

Proposition 3.3.5 *Let f be a non-negative measurable function and let k be a positive real number. Then we have*

$$\mathrm{L}(kf) = k \cdot \mathrm{L}(f).$$

Idea To show that $\mathrm{L}(kf) \geq k \cdot \mathrm{L}(f)$, apply definition for $\mathrm{L}(f)$ and use the fact that the operator I preserves multiplication by positive constants.

Proof We divide the proof into two steps. In Step 1, we prove the inequality $\geq$. In Step 2, we prove the inequality $\leq$.

(*Step 1*) Let f be a non-negative measurable function and let k be a positive real number. We want to show that

$$\mathrm{L}(kf) \geq k \cdot \mathrm{L}(f).$$

For this, by the definition of $\mathrm{L}(f)$, it is sufficient (and also necessary) to show that

$$\text{for every non-negative simple function } \varphi \text{ defined on } \mathrm{dom}\,(f) \text{ with } \varphi \leq f, \quad \mathrm{L}(kf) \geq k \cdot \mathrm{I}(\varphi).$$

Let φ be a non-negative simple function defined on $\mathrm{dom}\,(f)$ with $\varphi \leq f$. Then the function $k\varphi$ is a non-negative simple function defined on $\mathrm{dom}\,(kf)$ and $k\varphi \leq kf$. Hence by the definition of $\mathrm{L}(kf)$, we have

$$\mathrm{I}(k\varphi) \leq \mathrm{L}(kf). \tag{3.16}$$

By Lemma 3.2.3, we have

$$\mathrm{I}(k\varphi) = k \cdot \mathrm{I}(\varphi). \tag{3.17}$$

Combining (3.16) and (3.17), we get

$$\mathrm{L}(kf) \geq k \cdot \mathrm{I}(\varphi).$$

(*Step 2*) Let f be a non-negative measurable function and let k be a positive real number. We want to show that

$$\mathrm{L}(kf) \leq k \cdot \mathrm{L}(f).$$

Note that the function kf is non-negative and measurable and the number k^{-1} is non-negative. Hence by Step 1, we get

$$\mathrm{L}\big(k^{-1}(kf)\big) \geq k^{-1} \cdot \mathrm{L}\big((kf)\big),$$

that is,

$$k \cdot \mathrm{L}(f) \geq \mathrm{L}(kf).$$

□

Since the operator I preserves addition, it is natural to guess that the operator L preserves addition also, that is, if f and g are non-negative measurable functions (having the same domain), then we have $\mathrm{L}(f+g) = \mathrm{L}(f) + \mathrm{L}(g)$. We can use the idea in Step 1 for the proof of Proposition 3.3.5

to show that $\mathrm{L}(f)+\mathrm{L}(g) \leq \mathrm{L}(f+g)$. However, for the reverse inequality, we cannot apply the idea in Step 2. Below, we will prove a result called the Monotone Convergence Theorem and obtain the result that $\mathrm{L}(f)+\mathrm{L}(g)=\mathrm{L}(f+g)$ as a corollary.

Monotone Convergence Theorem *Let $(f_n)_{n=1}^{\infty}$ be an increasing sequence of non-negative measurable functions having the same domain X. Suppose that f is a non-negative real-valued function defined on X such that for every $x \in X$, we have $\lim_{n\to\infty} f_n(x) = f(x)$. Then we have*

$$\lim_{n\to\infty} \mathrm{L}(f_n) = \mathrm{L}(f),$$

where limit means limit in $\mathbb{R}^*$.

Idea The difficult part is to show that $\lim_{n\to\infty} \mathrm{L}(f_n) \geq \mathrm{L}(f)$. For this, we replace $\mathrm{L}(f)$ by $\mathrm{I}(\varphi)$ where φ is a non-negative simple function with $\varphi \leq f$ and then replace $\mathrm{I}(\varphi)$ by $c \cdot \mathrm{I}(\varphi)$ where c is a positive real number less that 1. With such c and φ, we construct $A_n = \{x \in X : f_n(x) \geq c\varphi(x)\}$. The conditions on c and φ together with the assumptions on $(f_n)_{n=1}^{\infty}$ and f imply that $(A_n)_{n=1}^{\infty}$ is increasing and $\bigcup_{n=1}^{\infty} A_n = \mathrm{dom}\,(f)$ and so we can apply Lemma 3.2.8.

Proof Note that by Corollary 2.3.3, the non-negative function f is measurable. By Lemma 3.3.3, we see that the sequence of extended real numbers $\big(\mathrm{L}(f_n)\big)_{n=1}^{\infty}$ is increasing and that

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{L}(f_n) \leq \mathrm{L}(f).$$

Hence $\lim_{n\to\infty} \mathrm{L}(f_n)$ exists in $\mathbb{R}^*$ and we have

$$\lim_{n\to\infty} \mathrm{L}(f_n) \leq \mathrm{L}(f).$$

To prove the reverse inequality, it is sufficient (and also necessary) to prove that

$$\text{for every non-negative simple function } \varphi \text{ defined on } X \text{ with } \varphi \leq f, \quad \lim_{n\to\infty} \mathrm{L}(f_n) \geq \mathrm{I}(\varphi).$$

or equivalently that,

- for every non-negative simple function φ defined on X with $\varphi \leq f$ and for every $c \in (0,1)$, we have $\lim_{n\to\infty} \mathrm{L}(f_n) \geq \mathrm{I}(\varphi)$.

Let φ be a non-negative simple function defined on X with $\varphi \leq f$ and let c be a real number with $0 < c < 1$. Denote $(A_n)_{n=1}^{\infty}$ to be the sequence of subsets of X given by

$$A_n = \{x \in X : f_n(x) \geq c\varphi(x)\}, \qquad (n \in \mathbb{Z}^+).$$

Since the sequence of functions $(f_n)_{n=1}^{\infty}$ is increasing, it follows that the sequence of measurable sets $(A_n)_{n=1}^{\infty}$ is increasing. Moreover, since for every $x \in X$, we have $\lim_{n\to\infty} f_n(x) = f(x)$ and $c\varphi(x) < f(x)$, it follows that

$$\bigcup_{n=1}^{\infty} A_n = X.$$

Hence by Lemma 3.2.8, we have

$$\mathrm{I}(\varphi) = \lim_{n\to\infty} \mathrm{I}\big(\varphi \cdot \chi_{A_n}\big), \tag{3.18}$$

where χ_{A_n} means $\chi_{A_n,X}$.

Note that for every $n \in \mathbb{Z}^+$, the function $c \cdot \varphi \cdot \chi_{A_n}$ is a non-negative simple function. Moreover, by the construction of $(A_n)_{n=1}^{\infty}$, we see that

$$\text{for every } n \in \mathbb{Z}^+, \quad c \cdot \varphi \cdot \chi_{A_n} \le f_n \cdot \chi_{A_n},$$

which implies that

$$\text{for every } n \in \mathbb{Z}^+, \quad c \cdot \varphi \cdot \chi_{A_n} \le f_n.$$

Hence, by the definition of $\mathrm{L}(f_n)$, we get

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{L}(f_n) \ge \mathrm{I}(c \cdot \varphi \cdot \chi_{A_n}),$$

which, by Lemma 3.2.3, implies that

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{L}(f_n) \ge c \cdot \mathrm{I}(\varphi \cdot \chi_{A_n}). \tag{3.19}$$

Therefore, by (3.18) and (3.19), we get

$$\begin{aligned} c \cdot \mathrm{I}(\varphi) &= c \cdot \lim_{n\to\infty} \mathrm{I}(\varphi \cdot \chi_{A_n}) \\ &= \lim_{n\to\infty} c \cdot \mathrm{I}(\varphi \cdot \chi_{A_n}) \\ &\le \lim_{n\to\infty} \mathrm{L}(f_n) \end{aligned}$$

□

Remark In the above prove, we cannot omit the number $c < 1$ and simply take $A_n = [f_n \ge \varphi]$. This is because $\bigcup_{n=1}^{\infty} [f_n \ge \varphi]$ may not be equal to X since there may exist $x_0 \in X$ such that $\varphi(x_0) = f(x_0)$ and that for every $n \in \mathbb{Z}^+$, $f_n(x_0) < f(x_0)$.

The following result is another version of the Monotone Convergence Theorem (it may better be called the *Monotone Divergence Theorem*).

Proposition 3.3.6 *Let $(f_n)_{n=1}^{\infty}$ be an increasing sequence of non-negative measurable functions having the same domain X. Suppose that the measure of the set $\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}$ is positive. Then we have*

$$\lim_{n\to\infty} \mathrm{L}(f_n) = \infty.$$

Remark The set $\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}$ is measurable (see Exercise 2.3).

Idea The idea is similar to that for the Monotone Convergence Theorem. Instead of applying Lemma 3.2.8, we use Theorem 1.4.17.

Proof It is clear that $\lim_{n\to\infty} \mathrm{L}(f_n)$ exists in $\mathbb{R}^*$. To show that $\lim_{n\to\infty} \mathrm{L}(f_n) = \infty$, it is sufficient (and also necessary) to show that

$$\text{for every real number } r, \quad \lim_{n\to\infty} \mathrm{L}(f_n) \geq r.$$

Let r be a real number. Denote c to be the positive real number given by

$$c = \begin{cases} 1 & \text{if } \mathsf{m}\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\} = \infty, \\ \dfrac{r}{\mathsf{m}\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}} & \text{if } \mathsf{m}\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\} < \infty. \end{cases}$$

We want to show that

$$\lim_{n\to\infty} \mathrm{L}(f_n) \geq c \cdot \mathsf{m}\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}$$

and hence we have $\lim_{n\to\infty} \mathrm{L}(f_n) \geq r$ as required.

Denote $(A_n)_{n=1}^{\infty}$ to be the sequence of subsets of X given by

$$A_n = \{x \in X : f_n(x) \geq c\}, \qquad (n \in \mathbb{Z}^+).$$

Since the sequence of functions $(f_n)_{n=1}^{\infty}$ is increasing, it follows that the sequence of measurable sets $(A_n)_{n=1}^{\infty}$ is increasing. Hence by Theorem 1.4.17, we have

$$\lim_{n\to\infty} \mathsf{m}(A_n) = \mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big). \tag{3.20}$$

Moreover, since

$$\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\} \subseteq \bigcup_{n=1}^{\infty} A_n,$$

it follows from the monotonicity of the Lebesgue measure that

$$\mathsf{m}\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\} \leq \mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big). \tag{3.21}$$

Note that for every $n \in \mathbb{Z}^+$, the function $c \cdot \chi_{A_n}$ is a non-negative simple function, where χ_{A_n} means $\chi_{A_n,X}$. Moreover, by the construction of $(A_n)_{n=1}^{\infty}$, we see that

$$\text{for every } n \in \mathbb{Z}^+, \quad c \cdot \chi_{A_n} \leq f_n \cdot \chi_{A_n},$$

which implies that

$$\text{for every } n \in \mathbb{Z}^+, \quad c \cdot \chi_{A_n} \leq f_n.$$

Hence by the definition of $\mathrm{L}(f_n)$, we get

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{I}(c \cdot \chi_{A_n}) \leq \mathrm{L}(f_n). \tag{3.22}$$

By Corollary 3.2.7, we have

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{I}(c \cdot \chi_{A_n}) = c \cdot \mathsf{m}(A_n). \tag{3.23}$$

Combining (3.22), (3.23), (3.20) and (3.21) and using the condition that $c > 0$, we get

$$\begin{aligned} \lim_{n\to\infty} \mathrm{L}(f_n) &\geq \lim_{n\to\infty} \mathrm{I}(c \cdot \chi_{A_n}) \\ &= \lim_{n\to\infty} c \cdot \mathsf{m}(A_n) \\ &= c \cdot \lim_{n\to\infty} \mathsf{m}(A_n) \\ &= c \cdot \mathsf{m}\Big(\bigcup_{n=1}^{\infty} A_n\Big) \\ &\geq c \cdot \mathsf{m}\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}. \end{aligned}$$

□

Remark If $\mathsf{m}\big(\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}\big) = 0$, then there exists a real-valued function f defined on X such that $(f_n)_{n=1}^{\infty}$ converges to f almost everywhere. In this case, we also have $\lim_{n\to\infty} \mathrm{L}(f_n) = \mathrm{L}(f)$. This result is the Monotone Convergence Theorem in a more general form. It will be proved in Theorem 3.3.12.

The following result means that the operator L is additive. It is a simple consequence of the Monotone Convergence Theorem.

Theorem 3.3.7 *Let f and g be non-negative measurable functions having the same domain. Then we have*

$$\mathrm{L}(f + g) = \mathrm{L}(f) + \mathrm{L}(g).$$

Idea For each of the two functions f and g, there exists an increasing sequence of non-negative simple functions converging pointwise to the function. Apply the Monotone Convergence Theorem and use the fact that the operator I is additive.

Proof By Proposition 2.4.2, there exist increasing sequences of non-negative simple functions $(\varphi_n)_{n=1}^{\infty}$ and $(\quad_n)_{n=1}^{\infty}$ such that

$$\text{for every } x \in X, \qquad \lim_{n\to\infty} \varphi_n(x) = f(x) \quad \text{and} \quad \lim_{n\to\infty} \quad_n(x) = g(x).$$

By the Monotone Convergence Theorem, we have

$$\lim_{n\to\infty} \mathrm{L}(\varphi_n) = \mathrm{L}(f) \quad \text{and} \quad \lim_{n\to\infty} \mathrm{L}(\quad_n) = \mathrm{L}(g). \tag{3.24}$$

Since $(\varphi_n + \quad_n)_{n=1}^{\infty}$ is an increasing sequence of non-negative simple functions and

$$\text{for every } x \in X, \quad \lim_{n\to\infty} (\varphi_n + \quad_n)(x) = (f + g)(x),$$

again by the Monotone Convergence Theorem, we have

$$\lim_{n\to\infty} \mathrm{L}(\varphi_n + \quad {}_n) = \mathrm{L}(f+g). \tag{3.25}$$

Combining (3.24) and (3.25), we get

$$\begin{aligned}
\mathrm{L}(f) + \mathrm{L}(g) &= \lim_{n\to\infty} \mathrm{L}(\varphi_n) + \lim_{n\to\infty} \mathrm{L}(\quad {}_n) \\
&= \lim_{n\to\infty} \mathrm{I}(\varphi_n) + \lim_{n\to\infty} \mathrm{I}(\quad {}_n) \\
&= \lim_{n\to\infty} \mathrm{I}(\varphi_n + \quad {}_n) \\
&= \lim_{n\to\infty} \mathrm{L}(\varphi_n + \quad {}_n) \\
&= \mathrm{L}(f+g)
\end{aligned}$$

where the second and fourth equality follows from Proposition 3.3.1 and the third equality follows from Lemma 3.2.5. □

Corollary 3.3.8 *Let f be a measurable function. Then we have*

$$\mathrm{L}\big(|f|\big) = \mathrm{L}(f^+) + \mathrm{L}(f^-).$$

Idea Use $|f| = f^+ + f^-$.

Proof Note that the functions $|f|$, f^+ and f^- are non-negative measurable functions (having the same domain) and that

$$|f| = f^+ + f^-.$$

Hence by Theorem 3.3.7, we get

$$\mathrm{L}\big(|f|\big) = \mathrm{L}(f^+) + \mathrm{L}(f^-).$$

□

Corollary 3.3.9 *Let f and g be measurable functions having the same domain. Then we have*

$$\mathrm{L}\big(|f+g|\big) \le \mathrm{L}\big(|f|\big) + \mathrm{L}\big(|g|\big).$$

Idea Use $|f+g| \le |f| + |g|$.

Proof Note that the functions $|f+g|$, $|f|$ and $|g|$ are non-negative measurable functions and that

$$|f+g| \le |f| + |g|.$$

Hence by Lemma 3.3.3 and Theorem 3.3.7, we have

$$\begin{aligned}
\mathrm{L}\big(|f+g|\big) &\le \mathrm{L}\big(|f| + |g|\big) \\
&= \mathrm{L}\big(|f|\big) + \mathrm{L}\big(|g|\big).
\end{aligned}$$

□

The following result is a generalization of Lemma 3.3.2.

Corollary 3.3.10 *Let f and g be non-negative measurable functions having the same domain. Suppose that f and g are equal almost everywhere. Then we have*

$$\mathrm{L}(f) = \mathrm{L}(g).$$

Idea The operator L is additive. Write $f = f_1 + f_2$ and $g = g_1 + g_2$ in a way so that $f_1 = g_1$ a.e. and $f_2 = g_2 = 0$ a.e..

Proof Denote X to be the common domain of f and g. Denote A and B to be the subsets of X given by

$$A = [f = g] \qquad \text{and} \qquad B = [f \neq g].$$

Since $A \cap B = \emptyset$ and $A \cup B = X$, it follows that

$$\text{for every } x \in X, \quad \chi_A(x) + \chi_B(x) = 1,$$

where the domains of the characteristic functions are understood to be X. Hence we have

$$\begin{aligned} f &= f \cdot \chi_A + f \cdot \chi_B \\ g &= g \cdot \chi_A + g \cdot \chi_B. \end{aligned}$$

Therefore, by Theorem 3.3.7, we get

$$\mathrm{L}(f) = \mathrm{L}(f \cdot \chi_A) + \mathrm{L}(f \cdot \chi_B) \tag{3.26}$$

$$\mathrm{L}(g) = \mathrm{L}(g \cdot \chi_A) + \mathrm{L}(g \cdot \chi_B). \tag{3.27}$$

It is clear from the construction of A that $f \cdot \chi_A = g \cdot \chi_A$ and so we have

$$\mathrm{L}(f \cdot \chi_A) = \mathrm{L}(g \cdot \chi_A). \tag{3.28}$$

By assumption, the set B is a null set. Since

$$[f \cdot \chi_B \neq 0] \subseteq B \quad \text{and} \quad [g \cdot \chi_B \neq 0] \subseteq B,$$

it follows that both sets $[f \cdot \chi_B \neq 0]$ and $[g \cdot \chi_B \neq 0]$ are null sets, that is,

$$f \cdot \chi_B = 0 \ \text{ a.e.} \qquad \text{and} \qquad g \cdot \chi_B = 0 \ \text{ a.e.}$$

Hence by Lemma 3.3.2, we have

$$\mathrm{L}(f \cdot \chi_B) = 0 \qquad \text{and} \qquad \mathrm{L}(g \cdot \chi_B) = 0. \tag{3.29}$$

The required result follows from (3.26), (3.27), (3.28) and (3.29). □

The following result is a generalization of Lemma 3.3.3.

Corollary 3.3.11 *Let f and g be non-negative measurable functions having the same domain. Suppose that $f \leq g$ a.e. Then we have*

$$\mathrm{L}(f) \leq \mathrm{L}(g).$$

Idea There exists a non-negative measurable function h such that $f \leq h$ and that $g = h$ a.e..

Proof Denote the common domain of f and g by X and denote

$$A = \{x \in X : f(x) > g(x)\}.$$

Denote h to be the function from X into $\mathbb{R}$ given by

$$h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in X \setminus A. \end{cases}$$

By construction, we have

$$\text{for every } x \in X, \quad f(x) \leq h(x).$$

Hence by Lemma 3.3.3, we have

$$\mathrm{L}(f) \leq \mathrm{L}(h). \tag{3.30}$$

By assumption, the set A is a null set. Since

$$[g \neq h] \subseteq A,$$

it follows that the set $[g \neq h]$ is a null set, that is,

$$g = h \quad \text{a.e.}$$

Hence by Corollary 3.3.10, we have

$$\mathrm{L}(g) = \mathrm{L}(h). \tag{3.31}$$

The required inequality follows from (3.30) and (3.31). □

The following result is the Monotone Convergence Theorem in a more general form.

Theorem 3.3.12 *Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative measurable functions having the same domain X such that for every $n \in \mathbb{Z}^+$, we have $f_n \leq f_{n+1}$ a.e.. Suppose that f is a non-negative real-valued function defined on X such that $(f_n)_{n=1}^{\infty}$ converges to f almost everywhere. Then we have*

$$\lim_{n\to\infty} \mathrm{L}(f_n) = \mathrm{L}(f).$$

Idea There exists an increasing sequence of non-negative measurable functions $(g_n)_{n=1}^{\infty}$ such that for every $x \in X$, $\lim\limits_{n\to\infty} g_n(x) = f(x)$ and that for every $n \in \mathbb{Z}^+$, $f_n = g_n$ a.e..

Proof Denote $A_0 = [f_n \not\to f]$ and denote $(A_n)_{n=1}^{\infty}$ to be the sequence of subsets of X given by

$$A_n = [f_n > f_{n+1}] \qquad (n \in \mathbb{Z}^+).$$

By assumption, we have

$$\text{for every } n \in \mathbb{Z}^+ \cup \{0\}, \quad \mathsf{m}(A_n) = 0. \tag{3.32}$$

Denote $A = \bigcup_{n=0}^{\infty} A_n$. By (3.32) together with the countable subadditivity of the Lebesgue measure, we see that A is a null set. Moreover, it follows from the construction of A that

$$\text{for every } x \in X \setminus A, \quad \text{for every } n \in \mathbb{Z}^+, \qquad f_n(x) \leq f_{n+1}(x), \tag{3.33}$$

$$\text{for every } x \in \setminus A, \qquad \lim_{n\to\infty} f_n(x) = f(x). \tag{3.34}$$

Denote $(g_n)_{n=1}^{\infty}$ to be the sequence of functions from X into $\mathbb{R}$ given by

$$g_n(x) = \begin{cases} f(x) & \text{if } x \in A, \\ f_n(x) & \text{if } x \in X \setminus A, \end{cases} \qquad (n \in \mathbb{Z}^+).$$

Since A is a null set, it follows from the construction of $(g_n)_{n=1}^{\infty}$ that

$$\text{for every } n \in \mathbb{Z}^+, \quad f_n = g_n \ \text{ a.e..}$$

Hence by Corollary 3.3.10, we get

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{L}(f_n) = \mathrm{L}(g_n). \tag{3.35}$$

By (3.33) and (3.34), together with the construction of $(g_n)_{n=1}^{\infty}$, we see that the sequence of measurable functions $(g_n)_{n=1}^{\infty}$ is increasing and that

$$\text{for every } x \in X, \quad \lim_{n\to\infty} g_n(x) = f(x).$$

Hence by the Monotone Convergence Theorem, we get

$$\lim_{n\to\infty} \mathrm{L}(g_n) = \mathrm{L}(f). \tag{3.36}$$

The required equality then follows from (3.35) and (3.36). □

Fatou's Lemma *Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative measurable functions having the same domain X.*

(1) *Suppose that for every $x \in X$, we have $\lim_{n\to\infty} f_n(x) \neq \infty$. Then we have*

$$\mathrm{L}\Big(\liminf_{n\to\infty} f_n\Big) \leq \liminf_{n\to\infty} \mathrm{L}(f_n).$$

(2) *Suppose that the measure of the set $\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}$ is positive. Then we have*

$$\lim_{n\to\infty} \mathrm{L}(f_n) = \infty.$$

Remark If the set $\{x \in X : \lim_{n\to\infty} f_n(x) = \infty\}$ is a null set, then there exists a sequence $(h_n)_{n=1}^{\infty}$ of non-negative measurable functions defined on X such that for every $n \in \mathbb{Z}^+$, the functions f_n and h_n are equal almost everywhere and that for every $x \in X$, we have $\lim_{n\to\infty} h_n(x) \neq \infty$.

Idea Apply the Monotone Convergence Theorem to $\big(\inf_{k\geq n} f_k\big)_{n=1}^{\infty}$ which is an increasing sequence of non-negative measurable functions.

Proof Denote $(g_n)_{n=1}^{\infty}$ to be the sequence of real-valued functions on X given by

$$g_n = \inf_{k\geq n} f_k \qquad (n \in \mathbb{Z}^+).$$

It is clear that $(g_n)_{n=1}^{\infty}$ is an increasing sequence of non-negative measurable functions.

(1) By definition together with the assumption, we have

$$\text{for every } x \in X, \quad \lim_{n\to\infty} g_n(x) = \liminf_{n\to\infty} f_n(x).$$

Hence by the Monotone Convergence Theorem, we get

$$\lim_{n\to\infty} \mathrm{L}(g_n) = \mathrm{L}\Big(\liminf_{n\to\infty} f_n\Big). \tag{3.37}$$

Note that

$$\text{for every } n \in \mathbb{Z}^+, \quad g_n \leq f_n.$$

Hence by Lemma 3.3.3, we get

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{L}(g_n) \leq \mathrm{L}(f_n) \tag{3.38}$$

from which we obtain

$$\liminf_{n\to\infty} \mathrm{L}(g_n) \leq \liminf_{n\to\infty} \mathrm{L}(f_n). \tag{3.39}$$

The required inequality follows from (3.37) and (3.39).

(2) By assumption, the measure of the set $\{x \in X : \lim_{n\to\infty} g_n(x)\}$ is positive. Hence by Proposition 3.3.6, we have

$$\lim_{n\to\infty} \mathrm{L}(g_n) = \infty. \tag{3.40}$$

The required equality follows from (3.38) and (3.40).

□

The following example illustrates that we may have strict inequality in Fatou's Lemma.

Example Denote $(f_n)_{n=1}^{\infty}$ to be the sequence of functions from $[0, \infty)$ into $\mathbb{R}$ given by

$$f_n(x) = \begin{cases} 1 & \text{if } n-1 \leq x \leq n, \\ 0 & \text{otherwise,} \end{cases} \qquad (n \in \mathbb{Z}^+).$$

Then we have

- $\mathrm{L}\big(\liminf_{n\to\infty} f_n\big) = 0$ since for every $x \in \mathbb{R}$, we have $\lim_{n\to\infty} f_n(x) = 0$;
- $\liminf_{n\to\infty} \mathrm{L}(f_n) = 1$ since for every $n \in \mathbb{Z}^+$, we have $\mathrm{L}(f_n) = 1$.

The following result is another version of the Fatou's Lemma.

Corollary 3.3.13 *Let* $(f_n)_{n=1}^{\infty}$ *be a sequence of non-negative measurable functions having the same domain* X *and let* f *be a non-negative measurable function defined on* X*. Suppose* $(f_n)_{n=1}^{\infty}$ *converges almost everywhere to* f*. Then we have*

$$\mathrm{L}(f) \leq \liminf_{n\to\infty} \mathrm{L}(f_n).$$

Idea There exists a sequence of non-negative measurable functions $(g_n)_{n=1}^{\infty}$ such that $(g_n)_{n=1}^{\infty}$ converges pointwise to f and that for every $n \in \mathbb{Z}^+$, the functions f_n and g_n are equal almost everywhere. Apply the Fatou's Lemma to $(g_n)_{n=1}^{\infty}$.

Proof Denote $(g_n)_{n=1}^{\infty}$ to be the sequence of functions from X into $\mathbb{R}$ given by

$$g_n(x) = \begin{cases} f(x) & \text{if } x \in [f_n \not\to f], \\ f_n(x) & \text{if } x \in [f_n \to f]. \end{cases}$$

Since $[f_n \not\to f]$ is a null set, it follows from the construction of $(g_n)_{n=1}^{\infty}$ that

$$\text{for every } n \in \mathbb{Z}^+, \quad f_n = g_n \ \text{ a.e.}$$

Hence by Corollary 3.3.10, we get

$$\text{for every } n \in \mathbb{Z}^+, \quad \mathrm{L}(f_n) = \mathrm{L}(g_n). \tag{3.41}$$

Note that $(g_n)_{n=1}^{\infty}$ is a sequence of non-negative measurable functions and that

$$\text{for every } x \in X, \quad \lim_{n\to\infty} g_n(x) = f(x).$$

Hence by the Fatou's Lemma and using (3.41), we get

$$\begin{aligned} \mathrm{L}(f) &= \mathrm{L}\Big(\liminf_{n\to\infty} g_n\Big) \\ &\leq \liminf_{n\to\infty} \mathrm{L}(g_n) \\ &= \liminf_{n\to\infty} \mathrm{L}(f_n) \end{aligned}$$

□

Similar to the notation $\mathrm{I}_A(\varphi)$, we have the notation $\mathrm{L}_A(f)$ as given in the following

Notation Let f be a non-negative measurable function and let A be a measurable set with $A \subseteq \operatorname{dom}(f)$. We denote $\mathrm{L}_A(f)$ to be the non-negative extended real number given by

$$\mathrm{L}_A(f) = \begin{cases} 0 & \text{if } A = \emptyset, \\ \mathrm{L}(f|_A) & \text{if } A \neq \emptyset. \end{cases}$$

The following result can be used to give an alternative definition for $\mathrm{L}_A(f)$.

Proposition 3.3.14 *Let f be a non-negative measurable function and let A be a measurable set with $A \subseteq \operatorname{dom}(f)$. Then we have*

$$\mathrm{L}_A(f) = \mathrm{L}(f \cdot \chi_{A,\operatorname{dom}(f)}).$$

Idea Apply the result for I_A.

Proof To prove the result, we consider the following two cases:

(*Case 1*) $A = \emptyset$

In this case, by definition, we have $\mathrm{L}_A(f) = 0$. Note that $f \cdot \chi_{A,\operatorname{dom}(f)} = 0_{\operatorname{dom}(f)}$. Hence we have

$$\mathrm{L}(f \cdot \chi_{A,\operatorname{dom}(f)}) = \mathrm{I}(0_{\operatorname{dom}(f)}) = 0.$$

(*Case 2*) $A \neq \emptyset$

In this case, we have $\mathrm{L}_A(f) = \mathrm{L}(f\big|_A)$. We want to show that the following two inequalities hold:

(1) $\mathrm{L}(f\big|_A) \geq \mathrm{L}(f \cdot \chi_{A,\operatorname{dom}(f)})$;

(2) $\mathrm{L}(f\big|_A) \leq \mathrm{L}(f \cdot \chi_{A,\operatorname{dom}(f)})$.

- To prove (1), by definition, we have to show that

$$\sup\{\mathrm{I}(\varphi) : \varphi \text{ is a non-negative simple function defined on } \operatorname{dom}(f) \text{ and } \varphi \leq f \cdot \chi_{A,\operatorname{dom}(f)}\} \leq \mathrm{L}(f\big|_A).$$

Let φ be a non-negative simple function defined on $\operatorname{dom}(f)$ with $\varphi \leq f \cdot \chi_{A,\operatorname{dom}(f)}$. We want to show that

$$\mathrm{I}(\varphi) \leq \mathrm{L}(f\big|_A).$$

Note that $\varphi\big|_A$ is a non-negative simple function defined on A and $\varphi\big|_A \leq f\big|_A$. Hence by definition, we have

$$\mathrm{I}(\varphi\big|_A) \leq \mathrm{L}(f\big|_A). \tag{3.42}$$

Since φ is non-negative and $\varphi \leq f \cdot \chi_{A,\operatorname{dom}(f)}$, it follows that

$$\varphi = \varphi \cdot \chi_{A,\operatorname{dom}(f)}.$$

Hence by Lemma 3.2.9, we get

$$\mathrm{I}(\varphi) = \mathrm{I}(\varphi \cdot \chi_{A,\operatorname{dom}(f)}) = \mathrm{I}(\varphi\big|_A). \tag{3.43}$$

The required inequality then follows from (3.42) and (3.43).

- To prove (2), by definition, we have to show that

$$\sup\{\mathrm{I}(\varphi) : \varphi \text{ is a non-negative simple function defined on } A \text{ and } \varphi \leq f\big|_A\} \leq \mathrm{L}(f \cdot \chi_{A,\operatorname{dom}(f)}).$$

Let φ be a non-negative simple function defined on A with $\varphi \leq f\big|_A$. We want to show that

$$\mathrm{I}(\varphi) \leq \mathrm{L}(f \cdot \chi_{A,\mathrm{dom}\,(f)}).$$

Denote   to be the simple function (defined on $\mathrm{dom}\,(f)$) given by

$$(x) = \begin{cases} \varphi(x) & \text{if } x \in A, \\ 0 & \text{if } x \in \mathrm{dom}\,(f) \setminus A. \end{cases}$$

Note that $\varphi = \ \big|_A$ and that $\ \cdot \chi_{A,\mathrm{dom}\,(f)} = \ $. Hence by the definition of $\mathrm{I}_A(\ )$ and Lemma 3.2.9, we have

$$\mathrm{I}(\varphi) = \mathrm{I}_A(\ ) = \mathrm{I}(\ \cdot \chi_{A,\mathrm{dom}\,(f)}) = \mathrm{I}(\ ). \tag{3.44}$$

Moreover, since $\varphi \leq f\big|_A$, it follows from the construction of   that

$$\leq f \cdot \chi_{A,\mathrm{dom}\,(f)}.$$

Hence by definition, we have

$$\mathrm{I}(\ ) \leq \mathrm{L}(f \cdot \chi_{A,\mathrm{dom}\,(f)}). \tag{3.45}$$

The required inequality then follows from (3.44) and (3.45).

□

Exercise 3.3

1. Let f be a non-negative measurable function. Suppose that $\mathrm{L}(f) = 0$. Show that $f = 0$ a.e..
2. Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative measurable functions having the same domain X and let f be a non-negative measurable function defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f and that for every $n \in \mathbb{Z}^+$, we have $f_n \leq f$ a.e. Show that $\lim\limits_{n\to\infty} \mathrm{L}(f_n) = \mathrm{L}(f)$.
3. Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative measurable functions having the same domain X and let f be a non-negative measurable function defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f and that $\lim\limits_{n\to\infty} \mathrm{L}(f_n) = \mathrm{L}(f) < \infty$. Show that for every measurable subset A of $\mathbb{R}$ with $A \subseteq X$, we have $\lim\limits_{n\to\infty} \mathrm{L}(f_n \cdot \chi_A) = \mathrm{L}(f \cdot \chi_A)$, where the domain of the characteristic function is X.
4. Let f be a bounded non-negative measurable function. Denote φ to be the function from $[0, \infty)$ into $\mathbb{R}$ given by $\varphi(t) = \mathrm{L}(f \cdot \chi_{[-t,t]\cap \mathrm{dom}\,(f)})$, where the domain of the characteristic function is $\mathrm{dom}\,(f)$. Show that φ is continuous.
5. Let f be a non-negative measurable function and let η be a non-negative real number with $\eta \leq \mathrm{L}(f)$. Show that there exists a measurable subset A of $\mathbb{R}$ with $A \subseteq \mathrm{dom}\,(f)$ such that $\mathrm{L}(f \cdot \chi_A) = \eta$, where the domain of the characteristic function is $\mathrm{dom}\,(f)$.

6. Let f be a non-negative measurable function with $\mathsf{m}\big(\mathrm{dom}\,(f)\big) < \infty$. Show that $\mathrm{L}(f) < \infty$ if and only if $\sum\limits_{n=1}^{\infty} \mathsf{m}[f \geq n] < \infty$.
7. Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative measurable functions having the same domain X and let f be a non-negative measurable function defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges in measure to f. Show that $\mathrm{L}(f) \leq \liminf\limits_{n\to\infty} \mathrm{L}(f_n)$. Give an example to show that strict inequality may occur.

3.4 Integrable Functions

Definition Let f be a measurable function such that $\mathrm{L}(f^+) < \infty$ or $\mathrm{L}(f^-) < \infty$. The *Lebesgue integral of* f, or simply the *integral of* f, denoted by $\displaystyle\int f\,\mathrm{d}\mathsf{m}$, or simply $\displaystyle\int f$, is the extended real number given by

$$\int f\,\mathrm{d}\mathsf{m} = \mathrm{L}(f^+) - \mathrm{L}(f^-).$$

Terminology If f is a measurable function with $\mathrm{L}(f^+) < \infty$ or $\mathrm{L}(f^-) < \infty$, then we say that $\int f$ *exists.*

The following result means that the operator $\int$ is an extension of the operator L.

Proposition 3.4.1 *Let f be a non-negative measurable function. Then we have*

$$\int f\,\mathrm{d}\mathsf{m} = \mathrm{L}(f).$$

Idea Use $f^- = 0$.

Proof Since f is non-negative, it follows that $f^+ = f$ and $f^- = 0_X$. Hence by definition, we have

$$\begin{aligned}\int f\,\mathrm{d}\mathsf{m} &= \mathrm{L}(f^+) - \mathrm{L}(0_X)\\ &= \mathrm{L}(f^+) - 0\\ &= \mathrm{L}(f).\end{aligned}$$

□

Remark Because of the above proposition, results in the last section are usually stated in a form using the notation $\int$.

Proposition 3.4.2 *Let f and g be measurable functions having the same domain. Suppose that $\int f$ exists and that f and g are equal almost everywhere. Then $\int g$ exists and we have*

$$\int f\,\mathrm{d}\mathsf{m} = \int g\,\mathrm{d}\mathsf{m}.$$

Idea Use $[f^+ \neq g^+] \subseteq [f \neq g]$ and $[f^- \neq g^-] \subseteq [f \neq g]$.

Proof To prove the result, it suffices to show that

$$\mathrm{L}(f^+) = \mathrm{L}(g^+) \qquad \text{and} \qquad \mathrm{L}(f^-) = \mathrm{L}(g^-).$$

Since the set $[f \neq g]$ is a null set and

$$[f^+ \neq g^+] \subseteq [f \neq g] \qquad \text{and} \qquad [f^- \neq g^-] \subseteq [f \neq g],$$

it follows that the sets $[f^+ \neq g^+]$ and $[f^- \neq g^-]$ are null sets, that is,

$$f^+ = g^+ \text{ a.e.} \qquad \text{and} \qquad f^- = g^- \text{ a.e.}$$

Hence by Corollary 3.3.10, we have

$$\mathrm{L}(f^+) = \mathrm{L}(g^+) \qquad \text{and} \qquad \mathrm{L}(f^-) = \mathrm{L}(g^-).$$ □

Note that the integral $\int f$ of a measurable function f (with $\int f^+ < \infty$ or $\int f^- < \infty$) is an extended real number; it can be ∞ or $-\infty$. In considering $\int f + \int g$, we may encounter $\infty + (-\infty)$ etc. To avoid this, we introduce the following

Definition A measurable function f is said to be *Lebesgue integrable*, or simply *integrable*, if both $\mathrm{L}(f^+)$ and $\mathrm{L}(f^-)$ are real numbers.

Note

- If f is an integrable function, then $\int f$ exists and is a real number.
- A non-negative measurable function f is integrable if and only if $\mathrm{L}(f) < \infty$.
- A measurable function f is integrable if and only if both f^+ and f^- are integrable.

Lemma 3.4.3 *Let X be a non-empty measurable subset of $\mathbb{R}$. Then for every measurable set A with $A \subseteq X$ and $\mathsf{m}(A) < \infty$, the characteristic function $\chi_{A,X}$ is integrable and we have*

$$\int \chi_{A,X} \, \mathsf{dm} = \mathsf{m}(A).$$

Idea Apply definition.

Proof Let A be a measurable set with $A \subseteq X$ and $\mathsf{m}(A) < \infty$. Note that the function $\chi_{A,X}$ is a non-negative measurable function. By Proposition 3.2.1, we have

$$\mathrm{L}(\chi_{A,X}) = \mathsf{m}(A).$$

The required result then follows from definition. □

The following result can be used to give an alternative definition for integrable functions.

Proposition 3.4.4 *Let f be a measurable function. Then the function f is integrable if and only if*

$$\int |f| \, d\mathsf{m} < \infty.$$

Idea Use $|f| = f^+ + f^-$ and $f^+ \leq |f|$ and $f^- \leq f$.

Proof

($\Longrightarrow$) Suppose f is integrable. Then we have

$$\mathrm{L}(f^+) \in \mathbb{R} \quad \text{and} \quad \mathrm{L}(f^-) \in \mathbb{R}.$$

It follows from Corollary 3.3.8 that

$$\mathrm{L}(|f|) = \mathrm{L}((f^+) + \mathrm{L}(f^-) \in \mathbb{R}.$$

Therefore, we have $\displaystyle\int |f| \, \mathsf{dm} < \infty$.

($\Longleftarrow$) Suppose that $\int |f| < \infty$. To show that f is integrable, we have to show that both $\mathrm{L}(f^+)$ and $\mathrm{L}(f^-)$ are real numbers. For this, it suffices to show that

$$\mathrm{L}(f^+) < \infty \qquad \text{and} \qquad \mathrm{L}(f^-) < \infty$$

since both $\mathrm{L}(f^+)$ and $\mathrm{L}(f^-)$ are non-negative.
Note that f^+ and f^- are non-negative measurable functions and that

$$f^+ \leq |f| \qquad \text{and} \qquad f^- \leq |f|.$$

Hence by Lemma 3.3.3, together with the assumption that $\int |f| < \infty$, we get

$$\mathrm{L}(f^+) \leq \mathrm{L}(|f|) < \infty \qquad \text{and} \qquad \mathrm{L}(f^-) \leq \mathrm{L}(|f|) < \infty.$$

□

Note A measurable function f is integrable if and only if $|f|$ is integrable.

Proposition 3.4.5 *Let f be an integrable function. Then we have*

$$\left| \int f \, \mathsf{dm} \right| \leq \int |f| \, \mathsf{dm}.$$

Idea Apply the triangle inequality for real numbers.

Proof Since f is integrable, it follows that

$$\mathrm{L}(f^+) \in \mathbb{R} \qquad \text{and} \qquad \mathrm{L}(f^-) \in \mathbb{R}.$$

By the definition of $\int f$, we have

$$\int f \, \mathsf{dm} = \mathrm{L}(f^+) - \mathrm{L}(f^-).$$

Taking absolute value and applying the triangle inequality for real numbers, we get

$$\begin{aligned}\left|\int f \, \mathsf{dm}\right| &\leq \mathrm{L}(f^+) + \mathrm{L}(f^-) \\ &= \int |f| \, \mathsf{dm},\end{aligned}$$

where the equality follows from Corollary 3.3.8. □

Remark If we adopt the conventions that $|\infty| = \infty$ and that $|-\infty| = \infty$, then the above result is valid for every measurable function f with $\int f^+ < \infty$ or $\int f^- < \infty$.

Proposition 3.4.6 *Let f be a measurable function. Suppose that f is bounded and that* $\mathsf{m}\big(\mathrm{dom}\,(f)\big) < \infty$. *Then f is integrable.*

Idea Show that $\int |f| < \infty$.

Proof Denote M to be the positive real number (since f is bounded) given by

$$M = \sup\{|f(x)| : x \in \mathrm{dom}\,(f)\} + 1.$$

Since $|f| \leq M \cdot 1_{\mathrm{dom}\,(f)}$, it follows from Lemma 3.3.3, Lemma 3.3.5 and Proposition 3.2.1 that

$$\begin{aligned}\mathrm{L}(|f|) &\leq \mathrm{L}\big(M \cdot 1_{\mathrm{dom}\,(f)}\big) \\ &= M \cdot \mathsf{m}\big(\mathrm{dom}\,(f)\big) < \infty.\end{aligned}$$

Hence by Proposition 3.4.4, the function f is integrable. □

Proposition 3.4.7 *Let f and g be measurable functions having the same domain. Suppose that g is integrable and that $|f| \leq g$* a.e.. *Then f is integrable.*

Idea Show that $\int |f| < \infty$.

Proof Since $|f|$ and g are non-negative measurable functions and $|f| \leq g$ a.e., it follows from Corollary 3.3.11 that

$$\mathrm{L}(|f|) \leq \mathrm{L}(g).$$

Since g is integrable, that is, $\int g \, \mathsf{dm} \in \mathbb{R}$, it follows that

$$\int |f| \, \mathsf{dm} < \infty.$$

Hence by Proposition 3.4.4, the function f is integrable. □

Proposition 3.4.8 *Let f be an integrable function and let $c \in \mathbb{R}$. Then the function cf is integrable and we have*

$$\int cf \, \mathsf{dm} = c \int f \, \mathsf{dm}.$$

Idea Consider $(cf)^+$ and $(cf)-$. For example, if $c > 0$, then $(cf)^+ = c \cdot f^+$ and $(cf)- = c \cdot f^-$.

Proof We want to show that both $\mathrm{L}\big((cf)^+\big)$ and $\mathrm{L}\big((cf)^-\big)$ are real numbers (hence $\int cf$ exists and the function cf is integrable) and that the equality holds. For this, we consider the following three cases:

(*Case 1*) $c = 0$

In this case, the functions $(cf)^+$ and $(cf)^-$ are identically 0 and so

$$\mathrm{L}\big((cf)^+\big) = 0 \in \mathbb{R} \quad \text{and} \quad \mathrm{L}\big((cf)^-\big) = 0 \in \mathbb{R}.$$

Hence by definition and using the condition that f is integrable ($\int f \in \mathbb{R}$), we get

$$\int (cf) \, \mathsf{dm} = 0 = c \int f \, \mathsf{dm}.$$

(*Case 2*) $c > 0$

In this case, we have

$$(cf)^+ = c \cdot f^+ \quad \text{and} \quad (cf)^- = c \cdot f^-.$$

By Proposition 3.3.5 together with the condition that f is integrable, we get

$$\mathrm{L}\big((cf)^+\big) = c \cdot \mathrm{L}(f^+) \in \mathbb{R} \quad \text{and} \quad \mathrm{L}\big((cf)^- big) = c \cdot \mathrm{L}(f^-) \in \mathbb{R}.$$

Hence by definition, we have

$$\begin{aligned} \int (cf) \, \mathsf{dm} &= \mathrm{L}\big((cf)^+\big) - \mathrm{L}\big((cf)^-\big) \\ &= c \cdot \mathrm{L}(f^+) - c \cdot \mathrm{L}(f^-) \\ &= c\big(\mathrm{L}(f^+) - \mathrm{L}(f^-)\big) \\ &= c \int f \, \mathsf{dm}. \end{aligned}$$

(*Case 3*) $c < 0$

In this case, we have

$$(cf)^+ = (-c) \cdot f^- \quad \text{and} \quad (cf)^- = (-c) \cdot f^+.$$

By Proposition 3.3.5 together with the condition that f is integrable, we get

$$\mathrm{L}\big((cf)^+\big) = (-c) \cdot \mathrm{L}(f^-) \in \mathbb{R} \quad \text{and} \quad \mathrm{L}\big((cf)^-\big) = (-c) \cdot \mathrm{L}(f^+) \in \mathbb{R}.$$

Hence by definition, we have

$$\begin{aligned}\int cf\,\mathsf{dm} &= (-c)\cdot \mathrm{L}(f^-) - (-c)\cdot \mathrm{L}(f^+)\\ &= c\cdot \big(\mathrm{L}(f^+) - \mathrm{L}(f^-)\big)\\ &= c\int f\,\mathsf{dm}.\end{aligned}$$

□

Remark The equality given in the above proposition is still valid if f is not integrable but $\int f$ exists and c is a non-zero real number,

In view of Theorem 3.3.7, it is natural to guess that if f and g are integrable functions, then $\int (f+g) = \int f + \int g$. However, we cannot apply definition since $(f+g)^+$ may not be equal to $f^+ + g^+$ etc. To prove the result, we first establish the following lemma.

Lemma 3.4.9 *Let f be an integrable function. Suppose that g and h are non-negative integrable functions and $f = g - h$. Then we have*

$$\int f\,\mathsf{dm} = \int g\,\mathsf{dm} - \int h\,\mathsf{dm}.$$

Idea Use $f^+ + h = f^- + g$.

Proof We want to show that

$$\mathrm{L}(f^+) - \mathrm{L}(f^-) = \int g\,\mathsf{dm} - \int h\,\mathsf{dm},$$

or equivalently that

$$\mathrm{L}(f^+) + \mathrm{L}(h) = \mathrm{L}(f^-) + \mathrm{L}(g)$$

since $\mathrm{L}(f^-)$ and $\mathrm{L}(h)$ are real numbers.

Note that $f = f^+ - f^-$. Hence by the assumption $f = g - h$, we get

$$f^+ + h = f^- + g.$$

Since all the terms in the above equality are non-negative measurable functions, by applying Theorem 3.3.7 to the left-side and the right-side of the above equality, we get

$$\mathrm{L}(f^+) + \mathrm{L}(h) = \mathrm{L}(f^-) + \mathrm{L}(g).$$

□

Theorem 3.4.10 *Let f and g be integrable functions. Then the function $f+g$ is integrable and we have*

$$\int (f+g)\,\mathsf{dm} = \int f\,\mathsf{dm} + \int g\,\mathsf{dm}.$$

Idea The functions $(f^+ + g^+)$ and $(f^- + g^-)$ are non-negative and integrable and $f + g = (f^+ + g^+) - (f^- + g^-)$.

Proof Since f and g are integrable, it follows from Proposition 3.4.4 that

$$\int |f| \,\mathsf{dm} < \infty \quad \text{and} \quad \int |g| \,\mathsf{dm} < \infty.$$

Hence by Corollary 3.3.9, we have

$$\int |f+g| \,\mathsf{dm} \leq \int |f| \,\mathsf{dm} + \int |g| \,\mathsf{dm} < \infty$$

and so by Proposition 3.4.4, the function $f+g$ is integrable.

Since $f+g = (f^+ - f^-) + (g^+ - g^-)$, it follows that

$$f+g = (f^+ + g^+) - (f^- + g^-). \tag{3.46}$$

By Theorem 3.3.7, we have

$$\mathrm{L}(f^+ + g^+) = \mathrm{L}(f^+) + \mathrm{L}(g^+) \qquad \text{and} \qquad \mathrm{L}((f^- + g^-) = \mathrm{L}(f^-) + \mathrm{L}(g^-). \tag{3.47}$$

Since f and g are integrable, it follows that $\mathrm{L}(f^+)$, $\mathrm{L}(f^-)$, $\mathrm{L}(g^+)$ and $\mathrm{L}(g^-)$ are real numbers and so the (non-negative measurable) functions $f^+ + g^+$ and $f^- + g^-$ are integrable.

In view of (3.46), by Lemma 3.4.9 and (3.47), we get

$$\begin{aligned} \int (f+g) \,\mathsf{dm} &= \int (f^+ + g^+) \,\mathsf{dm} - \int (f^- + g^-) \,\mathsf{dm} \\ &= \big(\mathrm{L}(f^+) + \mathrm{L}(g^+)\big) - \Big(\mathrm{L}(f^-) + \mathrm{L}(g^-)\Big) \\ &= \big(\mathrm{L}(f^+) - \mathrm{L}(f^-)\big) + \big(\mathrm{L}(g^+) - \mathrm{L}(g^-)\big) \\ &= \int f \,\mathsf{dm} + \int g \,\mathsf{dm}. \end{aligned}$$

□

Corollary 3.4.11 *Let* $\varphi = \sum_{i=1}^{n} c_i \cdot \chi_{A_i, X}$ *be a simple function, where* $c_1, \ldots, c_n$ *are real numbers,* X *is a non-empty measurable subset of* $\mathbb{R}$ *and* $A_1, \ldots, A_n$ *are measurable sets that are subsets of* $\operatorname{dom}(f)$ *such that for every* $i = 1, \ldots, n$, *we have* $\mathsf{m}(A_i) < \infty$. *Then the function* φ *is integrable and we have*

$$\int \varphi \,\mathsf{dm} = \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i).$$

Idea The operator $\int$ is linear.

Proof Below, the domains of the characteristic functions are understood to be X.

By Lemma 3.4.3, we have

$$\text{for every } i = 1, \ldots, n, \qquad \chi_{A_i} \text{ is integrable and } \int \chi_{A_i} \,\mathsf{dm} = \mathsf{m}(A_i).$$

Hence by Proposition 3.4.8, we have

$$\text{for every } i = 1, \ldots, n, \qquad c_i \cdot \chi_{A_i} \text{ is integrable and } \int c_i \cdot \chi_{A_i} \,\mathsf{dm} = c_i \times \mathsf{m}(A_i).$$

Therefore, by Theorem 3.4.10 and induction, the function φ is integrable and we have

$$\begin{aligned} \int \varphi \,\mathsf{dm} &= \sum_{i=1}^{n} \int c_i \cdot \chi_{A_i} \,\mathsf{dm} \\ &= \sum_{i=1}^{n} c_i \times \mathsf{m}(A_i) \end{aligned}$$

□

Corollary 3.4.12 *Let f and g be integrable functions. Then the function $f - g$ is integrable and we have*

$$\int (f - g) \,\mathsf{dm} = \int f \,\mathsf{dm} - \int g \,\mathsf{dm}.$$

Idea Use $f - g = f + (-1)g$.

Proof This result is an immediate consequence of Theorem 3.4.10 and Proposition 3.4.8 since

$$f - g = f + (-1)g.$$

□

Proposition 3.4.13 *Let f and g be integrable functions having the same domain. Suppose that $f \le g$ a.e.. Then we have*

$$\int f \,\mathsf{dm} \le \int g \,\mathsf{dm}.$$

Proof Since both functions f and g are integrable, it follows from Corollary 3.4.12 that the function $g - f$ is integrable and that

$$\int (g - f) \,\mathsf{dm} = \int g \,\mathsf{dm} - \int f \,\mathsf{dm}.$$

To prove the required inequality, it is sufficient (and also necessary) to prove that

$$\int (g - f) \,\mathsf{dm} \ge 0.$$

Denote h to be the (measurable) function from the common domain of f and g into $\mathbb{R}$ given by

$$h(x) = \begin{cases} g(x) - h(x) & \text{if } x \in [g \ge f], \\ 0 & \text{if } x \in [g < f]. \end{cases}$$

Since $f \le g$ a.e., it follows from the construction of h that

$$h = g - f \text{ a.e..}$$

Hence by Proposition 3.4.2, we get

$$\begin{aligned}\int (g-f)\,\mathsf{dm} &= \int h\,\mathsf{dm}\\ &= \mathrm{L}(h) \geq 0.\end{aligned}$$

□

Dominated Convergence Theorem *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f be a measurable function defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f and that there exists an integrable function g defined on X such that for every $n \in \mathbb{Z}^+$, we have $|f_n| \leq g$ a.e.. Then we have*

$$\lim_{n\to\infty} \int f_n\,\mathsf{dm} = \int f\,\mathsf{dm},$$

where $\left(\int f_n\right)_{n=1}^{\infty}$ is a sequence of real numbers and $\int f$ is a real number.

Idea If $(f_n)_{n=1}^{\infty}$ converges pointwise to f and for every $n \in \mathbb{Z}^+$, $|f_n| \leq g$, then the sequence of non-negative measurable functions $(2g - |f_n - f|)_{n=1}^{\infty}$ converges pointwise to the function $2g$. Apply the Fatou's Lemma.

Proof Since for every $n \in \mathbb{Z}^+$, we have $|f_n| \leq g$ a.e., it follows from Proposition 3.4.7 that for every $n \in \mathbb{Z}^+$, the function f_n is integrable and hence $\left(\int f_n\right)_{n=1}^{\infty}$ is a sequence of real numbers. Note that $|f| \leq g$ a.e. and so by the same reason, we see that $\int f$ is a real number.

To prove the equality $\lim_{n\to\infty} \int f_n\,\mathsf{dm} = \int f\,\mathsf{dm}$, by Corollary 3.4.12, it is sufficient (and also necessary) to prove that

$$\lim_{n\to\infty} \int (f_n - f)\,\mathsf{dm} = 0,$$

or equivalently that

$$\lim_{n\to\infty} \left| \int (f_n - f)\,\mathsf{dm} \right| = 0.$$

For this, by Proposition 3.4.5, it suffices to prove that

$$\limsup_{n\to\infty} \int |f_n - f|\,\mathsf{dm} = 0. \tag{3.48}$$

Denote A to be the subset of X given by

$$A = [f_n \not\to f] \cup \bigcup_{n=1}^{\infty} [f_n > g].$$

By assumption, the set $[f_n \not\to f]$ is a null set and for every $n \in \mathbb{Z}^+$, the set $[f_n > g]$ is a null set. Hence by the countable subadditivity of the Lebesgue measure, the set A is a null set.

Denote $(h_n)_{n=1}^{\infty}$ to be the sequence of functions from X into $\mathbb{R}$ given by

$$h_n(x) = \begin{cases} 2g(x) & \text{if } x \in A, \\ 2g(x) - |f_n(x) - f(x)| & \text{if } x \in X \setminus A, \end{cases} \qquad (n \in \mathbb{Z}^+).$$

It is clear from construction that $(h_n)_{n=1}^{\infty}$ is a sequence of non-negative measurable functions and that

$$\text{for every } x \in X, \qquad \lim_{n\to\infty} h_n(x) = 2g(x).$$

Hence by the Fatou's Lemma (Corollary 3.3.13), we get

$$\int 2g \,\mathsf{dm} \le \liminf_{n\to\infty} \int h_n \,\mathsf{dm}. \tag{3.49}$$

Since A is a null set, it follows that

$$\text{for every } n \in \mathbb{Z}^+, \qquad h_n = 2g - |f_n - f| \ \text{ a.e..}$$

Hence by Proposition 3.4.2 and Corollary 3.4.12, we have

$$\text{for every } n \in \mathbb{Z}^+, \qquad \int h_n \,\mathsf{dm} = \int 2g \,\mathsf{dm} - \int |f_n - f| \,\mathsf{dm}$$

from which we obtain

$$\liminf_{n\to\infty} \int h_n \,\mathsf{dm} = \int 2g \,\mathsf{dm} - \limsup_{n\to\infty} \int |f_n - f| \,\mathsf{dm}. \tag{3.50}$$

Combining (3.49) and (3.50), we get

$$\int 2g \,\mathsf{dm} \le \int 2g \,\mathsf{dm} - \limsup_{n\to\infty} \int |f_n - f| \,\mathsf{dm}$$

which yields the required equality (3.48). □

Definition Let f be a real-valued function.

- Suppose that A is a non-empty measurable set with $A \subseteq \text{dom}\,(f)$ and that the function $f\big|_A$ is Lebesgue integrable. Then we say that f *is (Lebesgue) integrable over* A and we call the value $\int f\big|_A \,\mathsf{dm}$ the *(Lebesgue) integral of* f *over* A and denote it by $\int_A f \,\mathsf{dm}$.
- We define $\int_{\emptyset} f \,\mathsf{dm} = 0$.

Remark

- A function g is integrable means that g is a measurable function and that the Lebesgue integral of g exists and is a real number.
- If f is an integrable function, then for every non-empty measurable set A with $A \subseteq \text{dom}\,(f)$, the function f is integrable over A (see Proposition 3.4.14).

Similar to the result $\mathrm{L}_A(f) = \mathrm{L}(f \cdot \chi_{A,\text{dom}\,(f)})$, we have the following

Proposition 3.4.14 *Let f be an integrable function. Then for every measurable set A with $A \subseteq$ dom (f), we have*

$$\int_A f \, \mathsf{dm} = \int f \cdot \chi_{A,\mathrm{dom}\,(f)} \, \mathsf{dm},$$

where the numbers in the equality are real numbers.

Idea For the case where $A \neq \emptyset$, use $f = f^+ - f^-$.

Proof To prove the result, we consider the following two case:

(*Case 1*) $A = \emptyset$

In this case, we have $\int_A f \, \mathsf{dm} = 0$. Note that $f \cdot \chi_{A,\mathrm{dom}\,(f)} = 0_{\mathrm{dom}\,(f)}$. Hence we have

$$\int f \cdot \chi_{A,\mathrm{dom}\,(f)} \, \mathsf{dm} = \mathrm{L}(0_{\mathrm{dom}\,(f)}) = 0.$$

(*Case 2*) $A \neq \emptyset$

In this case, we want to show that both $\int f\big|_A \, \mathsf{dm}$ and $\int f \cdot \chi_{A,\mathrm{dom}\,(f)} \, \mathsf{dm}$ exist and are real numbers and that

$$\int f\big|_A \, \mathsf{dm} = \int f \cdot \chi_{A,\mathrm{dom}\,(f)} \, \mathsf{dm}. \tag{3.51}$$

Hence f is Lebesgue integrable over A and the required equality follows since $\int_A f \, \mathsf{dm} = \int f\big|_A \, \mathsf{dm}$.

Applying Proposition 3.3.14 to the non-negative measurable functions f^+ and f^-, we get

$$\mathrm{L}_A(f^+) = \mathrm{L}(f^+ \cdot \chi_{A,\mathrm{dom}\,(f)}) \qquad \text{and} \qquad \mathrm{L}_A(f^-) = \mathrm{L}(f^- \cdot \chi_{A,\mathrm{dom}\,(f)}). \tag{3.52}$$

Moreover, the numbers in the two equalities in (3.52) are real numbers. This is because

$$|f^+ \cdot \chi_{A,\mathrm{dom}\,(f)}| \leq |f| \qquad \text{and} \qquad |f^- \cdot \chi_{A,\mathrm{dom}\,(f)}| \leq |f|$$

and the function $|f|$ is integrable.

- Since

$$(f \cdot \chi_{A,\mathrm{dom}\,(f)})^+ = f^+ \cdot \chi_{A,\mathrm{dom}\,(f)} \qquad \text{and} \qquad (f \cdot \chi_{A,\mathrm{dom}\,(f)})^- = f^- \cdot \chi_{A,\mathrm{dom}\,(f)},$$

it follows that $\int f \cdot \chi_{A,\mathrm{dom}\,(f)} \, \mathsf{dm}$ exists and

$$\int f \cdot \chi_{A,\mathrm{dom}\,(f)} \, \mathsf{dm} = \mathrm{L}(f^+ \cdot \chi_{A,\mathrm{dom}\,(f)}) - \mathrm{L}(f^- \cdot \chi_{A,\mathrm{dom}\,(f)}) \in \mathbb{R}. \tag{3.53}$$

- Since
$$\left(f\big|_A\right)^+ = f^+\big|_A \qquad \text{and} \qquad \left(f\big|_A\right)^- = f^-\big|_A,$$
it follows that $\int f\big|_A \,\mathsf{dm}$ exists and
$$\int f\big|_A \,\mathsf{dm} = \mathrm{L}_A(f^+) - \mathrm{L}_A(f^-) \in \mathbb{R}. \tag{3.54}$$

The required equality in (3.51) then follows from (3.52) (3.53) and (3.54).

□

The following result is a simple consequence of the Dominated Convergence Theorem.

Proposition 3.4.15 *Let f be an integrable function and let $(A_n)_{n=1}^{\infty}$ be a disjoint sequence of measurable sets that are subsets of* dom (f)*. Then we have*
$$\int_{\bigcup\limits_{n=1}^{\infty} A_n} f \,\mathsf{dm} = \sum_{n=1}^{\infty} \int_{A_n} f \,\mathsf{dm}.$$

Idea Consider the sequence of integrable functions $\left(\sum\limits_{k=1}^{n} f \cdot \chi_{A_k}\right)_{n=1}^{\infty}$.

Proof In the proof below, the domains of the characteristic functions are understood to be dom (f).

Denote $(g_n)_{n=1}^{\infty}$ to be the sequence of integrable functions from dom (f) into $\mathbb{R}$ given by
$$g_n = \sum_{k=1}^{n} f \cdot \chi_{A_k}, \qquad (n \in \mathbb{Z}^+),$$
and denote g to be the integrable function from dom (f) into $\mathbb{R}$ given by
$$g = f \cdot \chi_{\bigcup\limits_{k=1}^{\infty} A_k}.$$
Since $(g_n)_{n=1}^{\infty}$ converges pointwise to g and for every $n \in \mathbb{Z}^+$, we have $|g_n| \leq |f|$ and $|f|$ is an integrable function, it follows from the Dominated Convergence Theorem that
$$\lim_{n\to\infty} \int g_n \,\mathsf{dm} = \int g \,\mathsf{dm},$$
which, by Theorem 3.4.10 and induction, yields
$$\lim_{n\to\infty} \sum_{k=1}^{n} \int f \cdot \chi_{A_k} \,\mathsf{dm} = \int f \cdot \chi_{\bigcup\limits_{k=1}^{\infty} A_k} \,\mathsf{dm}.$$
Hence by Proposition 3.4.14, we get
$$\sum_{k=1}^{\infty} \int_{A_k} f \,\mathsf{dm} = \int_{\bigcup\limits_{k=1}^{\infty} A_k} f \,\mathsf{dm}.$$

□

The following result is also a simple consequence of the Dominated Convergence Theorem. It will be used to prove Theorem 3.4.17.

Lemma 3.4.16 *Let f be an integrable function. Then for every $\epsilon > 0$, there exists a bounded integrable function g defined on* $\operatorname{dom}(f)$ *such that*

$$\int |f - g| \, \mathsf{dm} < \epsilon.$$

Idea The sequence of functions $\left(f - f \cdot \chi_{[|f| \leq n]}\right)_{n=1}^{\infty}$ is dominated by $2|f|$ and converges pointwise to the zero function.

Proof Denote $(g_n)_{n=1}^{\infty}$ to be the sequence of functions from $\operatorname{dom}(f)$ into $\mathbb{R}$ given by

$$g_n = f \cdot \chi_{[|f| \leq n]} \qquad (n \in \mathbb{Z}^+).$$

It is clear that for every $n \in \mathbb{Z}^+$, the function g_n is bounded and integrable.

Let $\epsilon > 0$. We want to show that there exists $n_0 \in \mathbb{Z}^+$ such that

$$\int |g_{n_0} - f| \, \mathsf{dm} < \epsilon.$$

Note that the sequence of integrable functions $\left(|g_n - f|\right)_{n=1}^{\infty}$ converges pointwise to the function $0_{\operatorname{dom}(f)}$, that for every $n \in \mathbb{Z}^+$, we have $|g_n - f| \leq 2|f|$ and that the function $2|f|$ is integrable. By the Dominated Convergence Theorem, we have

$$\lim_{n \to \infty} \int |g_n - f| \, \mathsf{dm} = \int 0_{\operatorname{dom}(f)} \, \mathsf{dm} = 0.$$

Hence the required result follows. □

The following result describes a property, known as the *absolute continuity*, of Lebesgue integrals.

Theorem 3.4.17 *Let f be an integrable function. Then for every $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable set A with $A \subseteq \operatorname{dom}(f)$ and $\mathsf{m}(A) < \delta$, we have*

$$\int_A |f| \, \mathsf{dm} < \epsilon.$$

Idea If f is bounded, the result is obvious. If f is unbounded, there exists a bounded integrable function g such that $\int |f - g|$ is small.

Proof We divide the proof into two steps. In Step 1, we prove the result for the case where f is bounded. In Step 2, we prove the result in general.

(*Step 1*) Let g be a bounded integrable function. We want to show that for every $\epsilon > 0$, there exists $\delta > 0$ such that for every measurable set A with $A \subseteq \mathrm{dom}\,(g)$ and $\mathsf{m}(A) < \delta$, we have

$$\int_A |g| \,\mathsf{dm} < \epsilon.$$

Let $\epsilon > 0$. Denote δ to be the positive real number given by

$$\delta = \frac{\epsilon}{\sup\big(\mathrm{range}\,(|g|)\big) + 1}.$$

Then for every measurable set A with $A \subseteq \mathrm{dom}\,(g)$ and $\mathsf{m}(A) < \delta$, we have (using Proposition 3.4.14)

$$\begin{aligned}
\int_A |g| \,\mathsf{dm} &= \int |g| \cdot \chi_{A,\mathrm{dom}\,(f)} \,\mathsf{dm} \\
&\leq \int \big(\sup\big(\mathrm{range}\,(|g|)\big) + 1\big) \cdot \chi_{A,\mathrm{dom}\,(f)} \,\mathsf{dm} \\
&= \big(\sup\big(\mathrm{range}\,(|g|)\big) + 1\big) \times \mathsf{m}(A) \\
&< \big(\sup\big(\mathrm{range}\,(|g|)\big) + 1\big) \times \delta \\
&= \epsilon.
\end{aligned}$$

(*Step 2*) Let $\epsilon > 0$. By Lemma 3.4.16 and using $\frac{\epsilon}{2} > 0$, we see that there exists a bounded integrable function g defined on $\mathrm{dom}\,(f)$ such that

$$\int |g - f| \,\mathsf{dm} < \frac{\epsilon}{2}. \tag{3.55}$$

Applying Step 1 to the bounded integrable function g and using $\frac{\epsilon}{2} > 0$, we see that there exists $\delta > 0$ such that

$$\text{for every measurable set } A \text{ with } A \subseteq \mathrm{dom}\,(g) \text{ and } \mathsf{m}(A) < \delta, \qquad \int_A |g| \,\mathsf{dm} < \tfrac{\epsilon}{2}. \tag{3.56}$$

Hence for every measurable set A with $A \subseteq \mathrm{dom}\,(f)$ and $\mathsf{m}(A) < \delta$, we have (using Proposition 3.4.14)

$$\begin{aligned}
\int_A |f| \,\mathsf{dm} &= \int \big(|f - g + g| \cdot \chi_{A,\mathrm{dom}\,(f)}\big) \,\mathsf{dm} \\
&\leq \int \big(|f - g| \cdot \chi_{A,\mathrm{dom}\,(f)}\big) \,\mathsf{dm} + \int \big(|g| \cdot \chi_{A,\mathrm{dom}\,(f)}\big) \,\mathsf{dm} \\
&\leq \int |f - g| \,\mathsf{dm} + \int \big(|g| \cdot \chi_{A,\mathrm{dom}\,(f)}\big) \,\mathsf{dm} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon,
\end{aligned}$$

where the last inequality follows from (3.55) and (3.56).

□

Exercise 3.4

1. Let f be an integrable function. Show that $\lim_{n\to\infty} \mathsf{m}[|f| \geq n] = 0$.
2. Let f and g be integrable functions having the same domain X. Suppose that $\mathsf{m}(X) > 0$ and that $g < f$ a.e.. Show that $\int g < \int f$.
3. Let f be an integrable function. Show that
   (a) for every $\epsilon > 0$, there exists a simple function φ defined on dom (f) such that $\int |f - \varphi| < \epsilon$;
   (b) for every $\epsilon > 0$, there exists a continuous function g defined on dom (f) such that $\int |f - g| < \epsilon$.

   Suppose in addition, the domain of f is a non-degenerate interval. Show that for every $\epsilon > 0$, there exists a step function h defined on dom (f) such that $\int |f - h| < \epsilon$.
4. Let f be an integrable function. Suppose that for every measurable set A with $A \subseteq$ dom (f), we have $\int_A f = 0$. Show that $f = 0$ a.e..
5. Let f be an integrable function defined on $[0, 1]$. Suppose that for every real number c with $0 \leq c \leq 1$, we have $\int_{[0,c]} f = 0$. Show that $f = 0$ a.e..
6. Let f be an integrable function defined on $\mathbb{R}$. Show that $\lim_{t\to 0} \int |f_t - f| \, \mathsf{dm} = 0$, where $\{f_t\}_{t\in\mathbb{R}}$ is the family of functions from $\mathbb{R}$ into $\mathbb{R}$ given by $f_t(x) = f(x - t)$.
7. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X and let f be an integrable function defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges uniformly to f and $\mathsf{m}(X) < \infty$.
   (a) Show that there exists $n_0 \in \mathbb{Z}^+$ such that for every $n \geq n_0$, the function f_n is integrable.
   (b) Show that $\lim_{n\to\infty} \int f_n = \int f$.
8. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X. Suppose that there exists an integrable function g defined on X such that for every $n \in \mathbb{Z}^+$, we have $|f_n| \leq g$ a.e.. Show that $\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f_n \leq \int \limsup_{n\to\infty} f_n$.
9. Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative integrable functions having the same domain X. Suppose that $\lim_{n\to\infty} \int f_n = 0$. Show that $(f_n)_{n=1}^{\infty}$ converges in measure to the function 0_X.
10. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable functions having the same domain X, where $\mathsf{m}(X) < \infty$. Show that $(f_n)_{n=1}^{\infty}$ converges in measure to the function 0_X if and only if $\lim_{n\to\infty} \int \frac{|f_n|}{1+|f_n|} = 0$.
11. Let $(f_n)_{n=1}^{\infty}$ be a sequence of integrable functions having the same domain X and let f be an integrable function defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f. Show that $\lim_{n\to\infty} \int |f - f_n| = 0$ if and only if $\lim_{n\to\infty} \int |f_n| = \int |f|$.

3.5 Riemann Integrals and Lebesgue Integrals

In this chapter, we will show that if a function f is Riemann integrable over a closed interval $[a, b]$, then it is Lebesgue integrable over the interval and we have $\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathsf{dm}$. Hence, we can apply results for Lebesgue integrable functions, for example, the Dominated Convergence Theorem, to consider Riemann integrals.

Below we will state a criterion for Riemann integrability. Instead of using upper sums and lower sums, we use Lebesgue integrals for step functions. Note that if φ is a step function defined on a closed interval $[a, b]$, then there exists a subdivision $(x_i)_{i=1}^n$ of $[a, b]$ and a family $\{\alpha_i\}_{i=1}^n$ of real numbers such that

$$\text{for every } i = 1, \ldots, n, \quad \text{for every } x \in (x_{i-1}, x_i), \qquad \varphi(x) = \alpha_i.$$

Hence φ can be written as

$$\varphi = \sum_{i=1}^n \alpha_i \cdot \chi_{(x_{i-1},x_i)} + \sum_{i=0}^n \varphi(x_i) \cdot \chi_{\{x_i\}},$$

where the domains of the characteristic functions are understood to be $[a, b]$. Therefore, by Corollary 3.4.11, we have

$$\begin{aligned} \int \varphi\,\mathsf{dm} &= \sum_{i=1}^n \alpha_i \times \mathsf{m}(x_{i-1}, x_i) + \sum_{i=0}^n \varphi(x_i) \times \mathsf{m}\{x_i\} \\ &= \sum_{i=1}^n \alpha_i \times (x_i - x_{i-1}). \end{aligned}$$

The following criterion can be used as an alternative definition for Riemann integrability/integral. The proof of the result in a form using upper sums and lower sums can be found in many books on Mathematical Analysis.

Theorem 3.5.0 *Let f be a bounded real-valued function defined on a closed interval $[a, b]$ where $a < b$. Then f is Riemann integrable if and only if*

$$\sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_{[a,b]} \text{ and } \varphi \le f\right\} = \inf\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_{[a,b]} \text{ and } \varphi \ge f\right\}, \tag{3.57}$$

where $\mathcal{S}_{[a,b]}$ denotes the set of all step functions defined on $[a, b]$. Moreover, if (3.57) holds, then we have

$$\int_a^b f(x)\,\mathrm{d}x = \sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_{[a,b]} \text{ and } \varphi \le f\right\}.$$

Similar to Theorem 3.5.0, we have the following criterion for measurability.

Theorem 3.5.1 *Let f be a bounded real-valued function defined a (non-empty) measurable set X, where $\mathsf{m}(X) < \infty$. Then f is measurable (hence Lebesgue integrable) if and only if*

$$\sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_X^* \text{ and } \varphi \leq f\right\} = \inf\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_X^* \text{ and } \varphi \geq f\right\}, \tag{3.58}$$

where $\mathcal{S}_X^*$ denotes the set of all simple functions defined on X. Moreover, if (3.58) *holds, then we have*

$$\int f\,\mathsf{dm} = \sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_X^* \text{ and } \varphi \leq f\right\}.$$

Remark Since f is bounded, it follows that there exist $\varphi, \ \in \mathcal{S}_X^*$ such that $\varphi \leq f \leq \ $.

Idea For the "only if part", for every $\epsilon > 0$, there exist simple functions φ and such that $f - \epsilon \leq \varphi \leq f \leq \ \leq f + \epsilon$. For the "if part", there exist $(\varphi_n)_{n=1}^\infty$ and $(\ _n)_{n=1}^\infty$ such that for every $n \in \mathbb{Z}^+$, $\varphi_n \leq f \leq \ $ and that $\lim\limits_{n\to\infty} \int \varphi_n = \lim\limits_{n\to\infty} \int \ _n$.

Proof We will prove the "if and only if" part. The "moreover" part is obtained in the proof of the "only if" part.

($\Longrightarrow$) Suppose that f is measurable. Since f is bounded and $\mathsf{m}(X) < \infty$, it follows from Proposition 3.4.6 that f is Lebesgue integrable. By Proposition 3.4.13, we have

$$\sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_X^* \text{ and } \varphi \leq f\right\} \ \leq \ \int f\,\mathsf{dm} \ \leq \ \inf\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_X^* \text{ and } \varphi \geq f\right\}. \tag{3.59}$$

To show that

$$\sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_X^* \text{ and } \varphi \leq f\right\} \ = \ \int f\,\mathsf{dm} \ = \ \inf\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_X^* \text{ and } \varphi \geq f\right\},$$

it is sufficient (and also necessary) to show that for every $\epsilon > 0$, there exist $\varphi, \ \in \mathcal{S}_X^*$ with $\varphi \leq f \leq \ $ such that

$$\int \varphi\,\mathsf{dm} \geq \int f\,\mathsf{dm} - \epsilon \quad \text{and} \quad \int \ \ \mathsf{dm} \leq \int f\,\mathsf{dm} + \epsilon.$$

Let $\epsilon > 0$.

- By Proposition 2.4.1 (using $\epsilon \cdot \mathsf{m}(X)^{-1} > 0$), there exists $\varphi \in \mathcal{S}_X^*$ such that

$$\text{for every } x \in X, \qquad f(x) - \epsilon \cdot \mathsf{m}(X)^{-1} \leq \varphi(x) \leq f(x),$$

  which implies that $\varphi \leq f$ and that

$$\int \varphi\,\mathsf{dm} \geq \int \left(f - \epsilon \cdot \mathsf{m}(X)^{-1} \cdot 1_X\right)\mathsf{dm} = \int f\,\mathsf{dm} - \epsilon.$$

- Similarly, applying Proposition 2.4.1 to $-f$ and using $\epsilon \cdot \mathsf{m}(X)^{-1} > 0$, we see that there exists $\varphi_1 \in \mathcal{S}_X^*$ such that

$$\text{for every } x \in X, \qquad -f(x) - \epsilon \cdot \mathsf{m}(X)^{-1} \leq \varphi_1(x) \leq -f(x).$$

Denote $\quad = -\varphi_1$. Then we have $\quad \in \mathcal{S}_X^*$ and $f \leq \quad$ and

$$\int \quad \mathsf{dm} \leq \int \Big(f + \epsilon \cdot \mathsf{m}(X)^{-1} \cdot 1_X\Big)\, \mathsf{dm} = \int f \, \mathsf{dm} + \epsilon.$$

($\Longleftarrow$) Suppose that (3.58) holds. Then there exists two sequences $(\varphi_n)_{n=1}^{\infty}$ and $(\quad {}_n)_{n=1}^{\infty}$ in $\mathcal{S}_X^*$ with

$$\text{for every } n \in \mathbb{Z}^+, \qquad \varphi_n \leq f \leq \quad {}_n, \tag{3.60}$$

such that

$$\lim_{n\to\infty} \int \varphi_n \, \mathsf{dm} = \lim_{n\to\infty} \int \quad {}_n \, \mathsf{dm}. \tag{3.61}$$

Denote g and h to be the measurable (and hence integrable since they are bounded and $\mathsf{m}(X) < \infty$) functions from X into $\mathbb{R}$ given by

$$g = \sup_{n\in\mathbb{Z}^+} \varphi_n, \qquad h = \inf_{n\in\mathbb{Z}^+} \quad {}_n.$$

By (3.60) and the construction of g and h, we have

$$g \leq f \leq h. \tag{3.62}$$

By Proposition 3.4.13 and the construction of g and h, we have

$$\text{for every } n \in \mathbb{Z}^+, \qquad \int \varphi_n \, \mathsf{dm} \leq \int g \, \mathsf{dm} \quad \text{and} \quad \int h \, \mathsf{dm} \leq \int \quad {}_n \, \mathsf{dm},$$

hence by (3.61), we get

$$\int g \, \mathsf{dm} = \int h \, \mathsf{dm},$$

which, by Corollary 3.4.12, yields

$$\int (h - g) \, \mathsf{dm} = 0.$$

Since the function $h - g$ is non-negative, it follows from Proposition 3.3.4 that

$$h - g = 0 \quad \text{a.e.}. \tag{3.63}$$

Note that by (3.62), we have

$$0_X \leq f - g \leq h - g,$$

which implies that

$$[f - g \neq 0] \;=\; [f > g] \;\subseteq\; [h > g] \;=\; [h - g \neq 0].$$

By (3.63), the set $[h - g \neq 0]$ is a null set and so the set $[f - g \neq 0]$ is also a null set, that is,

$$f = g \quad \text{a.e.}.$$

Since g is measurable, it follows from Theorem 2.2.18 that f is measurable.

□

Theorem 3.5.2 *Let f be a real-valued function that is Riemann integrable over $[a,b]$ where $a<b$ and $[a,b] \subseteq \operatorname{dom}(f)$. Then f is Lebesgue integrable over $[a,b]$ and we have*

$$\int_a^b f(x)\,\mathrm{d}x = \int_{[a,b]} f\,\mathsf{dm}.$$

Idea Step functions are simple functions. Use criteria for Riemann integrability and Lebesgue integrability.

Proof In the proof, we denote $\mathcal{S}_{[a,b]}$ to be the set of all step functions defined on $[a,b]$ and denote $\mathcal{S}^*_{[a,b]}$ to be the set of all simple functions defined on $[a,b]$.

Since f is Riemann integrable over $[a,b]$, it follows that the function $f\big|_{[a,b]}$ is bounded. Moreover, by Theorem 3.5.0, we have

$$\sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_{[a,b]} \text{ and } \varphi \le f\big|_{[a,b]}\right\} = \int_a^b f(x)\,\mathrm{d}x = \inf\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}_{[a,b]} \text{ and } \varphi \ge f\big|_{[a,b]}\right\}.$$

Since $\mathcal{S}_{[a,b]} \subseteq \mathcal{S}^*_{[a,b]}$, it follows that

$$\sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}^*_{[a,b]} \text{ and } \varphi \le f\big|_{[a,b]}\right\} = \int_a^b f(x)\,\mathrm{d}x = \inf\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}^*_{[a,b]} \text{ and } \varphi \ge f\big|_{[a,b]}\right\}. \tag{3.64}$$

Hence by Theorem 3.5.1, the function $f\big|_{[a,b]}$ is Lebesgue integrable (that is, the function f is Lebesgue integrable over $[a,b]$) and we have

$$\int f\big|_{[a,b]}\,\mathsf{dm} = \sup\left\{\textstyle\int \varphi\,\mathsf{dm} : \varphi \in \mathcal{S}^*_{[a,b]} \text{ and } \varphi \le f\big|_{[a,b]}\right\}. \tag{3.65}$$

The required equality then follows from (3.64) and (3.65) since $\displaystyle\int_{[a,b]} f\,\mathsf{dm} = \int f\big|_{[a,b]}\,\mathsf{dm}$ by definition. □

Example 3.5.1 $\displaystyle\lim_{n\to\infty}\int_0^1 \mathrm{e}^{-nx^2}\cos\pi x\,\mathrm{d}x = 0.$

Remark The sequence of functions $\left(\mathrm{e}^{-nx^2}\cos\pi x\right)_{n=1}^{\infty}$ converges pointwise in $[0,1]$ to the function $\chi_{\{0\}}$. Although the convergence is not uniform, we can apply the Dominated Convergence Theorem.

Proof In the proof, the domains of the characteristic functions are understood to be $[0,1]$.

Denote $(f_n)_{n=1}^{\infty}$ to be the sequence of (continuous) functions from $[0,1]$ into $\mathbb{R}$ given by

$$f_n(x) = \mathrm{e}^{-nx^2}\cos\pi x, \quad (n \in \mathbb{Z}^+).$$

By Theorem 3.5.2, we have

$$\text{for every } n \in \mathbb{Z}^+, \qquad \int_0^1 \mathrm{e}^{-nx^2}\cos\pi x\,\mathrm{d}x = \int f_n\,\mathsf{dm}. \tag{3.66}$$

Note that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to the function $0_{[0,1]}$ and that for every $n \in \mathbb{Z}^+$, we have $|f_n| \leq 1_{[0,1]}$ and the function $1_{[0,1]}$ is a Lebesgue integrable function. Hence by the Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int f_n \, \mathsf{dm} = \int 0_{[0,1]} \, \mathsf{dm} = 0. \tag{3.67}$$

The required equality then follows from (3.66) and (3.67). □

3.6 Integration of Measurable Extended Real-valued Functions

In this section, we describe briefly integration of measurable extended real-valued functions. Most of the results in Section 3.3 and Section 3.4 are also valid for extended real-valued functions. Some results for integration of non-negative measurable extended real-valued functions, for example, the Monotone Convergence Theorem and the Fatou's Lemma, can be simplified. Results for Lebesgue integrable extended real-valued functions are essentially the same as that for real-valued functions. This is because integrable extended real-valued functions are finite almost everywhere. We will state without proof a few results related to this matter.

To consider integration of measurable extended real-valued functions, similar to Section 3.3, we consider non-negative functions first. The definition for $\mathrm{L}(f)$ can also be applied to non-negative measurable extended real-valued functions.

Definition Let f be a non-negative measurable extended real-valued function. We denote $\mathrm{L}(f)$ to be the extended real-number given by

$$\mathrm{L}(f) = \sup\Big\{\mathrm{I}(\varphi) : \varphi \text{ is a non-negative simple function defined on } \operatorname{dom}(f) \text{ and } \varphi \leq f\Big\}$$

Properties of $\mathrm{L}(f)$ remain valid for measurable extended real-valued functions. To be more specific, Lemma 3.3.2, Lemma 3.3.3, Proposition 3.3.4, Proposition 3.3.5, Theorem 3.3.7, Corollary 3.3.8, Corollary 3.3.9, Corollary 3.3.10 and Corollary 3.3.11 remain valid if "*measurable function(s)*" is replaced by "*measurable extended real-valued function(s)*".

The Monotone Convergence Theorem (and its generalization Theorem 3.3.12) and Proposition 3.3.6 can be combined together. Note that if $(f_n)_{n=1}^{\infty}$ is a (almost everywhere) increasing sequence of non-negative extended real-valued function, then there exists an extended real-valued function f such that $(f_n)_{n=1}^{\infty}$ converges to f pointwise (almost everywhere).

Monotone Convergence Theorem *Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative measurable extended real-valued functions having the same domain X such that for every $n \in \mathbb{Z}^+$, we have $f_n \le f_{n+1}$* a.e.. *Then there exists a non-negative measurable extended real-valued function f defined on X such that $(f_n)_{n=1}^{\infty}$ converges to f almost everywhere. Moreover, we have*

$$\lim_{n\to\infty} \mathrm{L}(f_n) = \mathrm{L}(f).$$

Note If f is a non-negative measurable extended real-valued function such that

$$\mathsf{m}\big(\{x \in \mathrm{dom}\,(f) : f(x) = \infty\}\big) > 0,$$

then we have $\mathrm{L}(f) = \infty$.

For the Fatou's Lemma, there is no need to consider two cases. This is because for a sequence $(f_n)_{n=1}^{\infty}$ of (non-negative) extended real-valued functions, the function $\liminf f_n$ always exists.

Fatou's Lemma *Let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative measurable extended real-valued functions having the same domain X. Then we have*

$$\lim_{n\to\infty} \mathrm{L}(f_n) = \infty.$$

In particular, Corollary 3.3.13 remains valid if if "*measurable function(s)*" is replaced by "*measurable extended real-valued function(s)*".

The notation $\mathrm{L}_A(f)$ can also be defined for non-negative measurable extended real-valued functions and Proposition 3.3.14 remains valid if "*measurable function*" is replaced by "*measurable extended real-valued function*".

The notation $\int f$ can also be defined for measurable extended real-valued functions. and Proposition 3.4.1 and Proposition 3.4.2 remain valid if "*measurable function*" is replaced by "*measurable extended real-valued function*".

Definition Let f be a measurable extended real-valued function such that $\mathrm{L}(f^+) < \infty$ or $\mathrm{L}(f^-) < \infty$. The *Lebesgue integral of f*, or simply the *integral of f*, denoted by $\displaystyle\int f \,\mathsf{dm}$, or simply $\displaystyle\int f$, is the extended real number given by

$$\int f \,\mathsf{dm} = \mathrm{L}(f^+) - \mathrm{L}(f^-).$$

The definition of Lebesgue integrable extended real-valued function is the same as that for real-valued function.

Definition A measurable extended real-valued function f is said to be *Lebesgue integrable*, or simply *integrable*, if both $\mathrm{L}(f^+)$ and $\mathrm{L}(f^-)$ are real numbers.

Most of the results for Lebesgue integrable (real-valued) functions are also valid for Lebesgue integrable extended real-valued functions. For example, the following is the extended real-valued version of the Dominated Convergence Theorem.

Dominated Convergence Theorem *Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable extended real-valued functions having the same domain X and let f be a measurable extended real-valued function defined on X. Suppose that $(f_n)_{n=1}^{\infty}$ converges almost everywhere to f and that there exists an integrable extended real-valued function g defined on X such that for every $n \in \mathbb{Z}^+$, we have $|f_n| \leq g$* a.e.. *Then we have*

$$\lim_{n\to\infty} \int f_n \,\mathsf{dm} = \int f \,\mathsf{dm},$$

where $\left(\int f_n\right)_{n=1}^{\infty}$ is a sequence of real numbers and $\int f$ is a real number.

To prove results for Lebesgue integrable extended real-valued functions, we may use the same ideas as that for Lebesgue integrable real-valued functions. Alternatively, we may apply Proposition 3.6.2 to replace extended real-valued functions by real-valued functions. This is because of the following

Proposition 3.6.1 *Let f be an Lebesgue integrable extended real-valued function. Then f is finite almost everywhere.*

Proposition 3.6.2 *Let f be a Lebesgue integrable extended real-valued function. Denote f_r to be the function from* $\operatorname{dom}(f)$ *into* $\mathbb{R}$ *given by*

$$f_r(x) = \begin{cases} f(x) & \text{if } f(x) \in \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.68}$$

Then the set

$$\{x \in \operatorname{dom}(f) : f(x) \neq f_r(x)\}$$

is a null set. Moreover, the function f_r is Lebesgue integrable and we have

$$\int f \,\mathsf{dm} = \int f_r \,\mathsf{dm}.$$

Remark To make Theorem 3.4.10 (for example) meaningful for Lebesgue integrable extended real-valued functions, we have to adopt some conventions, for example, we may define $\infty + (-\infty) = 0$ etc. Alternatively, we may consider integration of functions that are defined almost everywhere.

Definition Let X be a non-empty measurable subset of $\mathbb{R}$. Suppose f is a function with $\operatorname{dom}(f) \subseteq X$ such that the set $X \setminus \operatorname{dom}(f)$ is a null set. Then we say that f is *defined almost everywhere on X.*

Proposition 3.6.3 *Let f be a real-valued or extended real-valued function that is defined almost everywhere on a non-empty measurable subset X of $\mathbb{R}$ and let g be an extension of f to X. Then we have*

(1) *f is measurable if and only if g is measurable;*

(2) *f is Lebesgue integrable if and only if g is Lebesgue integrable.*

Moreover, if f is Lebesgue integrable, then we have

$$\int f \, \mathsf{dm} = \int g \, \mathsf{dm}.$$