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Chapter 7

Polynomial Functions

In Chapter **??**, we have discussed quadratic functions. In this chapter, we will discuss "more complicated" functions. First, we consider a problem that leads to such a function.

Problem A rectangular box without lid is to be made from a square cardboard of sides 18 cm by cutting equal squares from each corner and then folding up the sides. Denote x to be the length, in cm, of the side of each of the squares to be cut off and denote V to be the volume, in cm^3, of the rectangular box. Express V in terms of x.

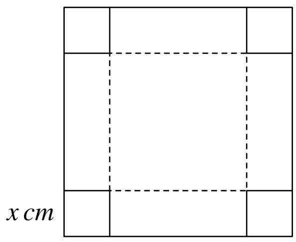
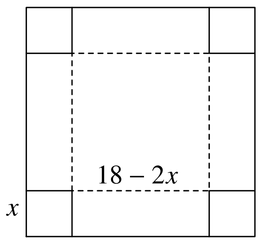
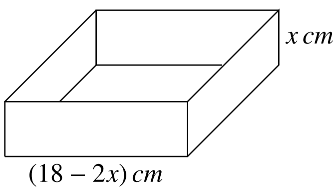


Figure 7.0.1

Explanation Volume of box = base area × height. The base is a square region of side $(18-2x)\,cm$.





Solution $V = (18-2x)^2 \cdot x$ □

Remark We can use *differentiation* to find x such that the volume is maximum. Interested readers may take a course on *Calculus*.

The equality '$V = (18-2x)^2 \cdot x$' describes a function (the volume function). It is a function that we call a *polynomial function* (or more specifically, a *cubic function*). The purpose of this chapter is to study polynomial functions.

- In Section 7.1, we will give a review of polynomials and then we will introduce the concepts of polynomial functions and their degrees.
- In Section 7.2, we will consider addition, subtraction and multiplication of polynomial functions.
- In Section 7.3, we will consider division of polynomials functions.
- In Section 7.4, we will discuss the *Remainder Theorem*, a result that provides a quick way to find the remainder when a polynomial functions is divided by a linear function $(ax-b)$.
- In Section 7.5, we will discuss the *Factor Theorem*, a result that can be used to find linear factor(s) of polynomial functions.
- In Section 7.6, we will describe how to factorize polynomial functions with integer coefficients.

7.1 Basic Concepts of Polynomial Functions

In junior forms, students have already encountered the concept of polynomials.

Terminology A *monomial* is a real number or a product of a real number and variable(s).

Example 7.1.1 Each of the following is a monomial:

(a) 2 (a real number)

(b) $3x$ (product of 3 and x)

(c) x^2 (product of 1 and x and x)

(d) $-4xy^3$ (product of -4 and x and y and y and y)

Terminology A *polynomial* is a monomial or a sum of monomials.

Example 7.1.2 Each of the following is a polynomial:

(a) 0 (a monomial)

(b) $2+3u$ (sum of two monomials 2 and $3u$)

(c) $4-5x-6x^2$ (sum of three monomials 4 and $-5x$ and $-6x^2$)

(d) x^2-yz (sum of two monomials x^2 and $-yz$)

In Example 7.1.2, the polynomials in (b) and (c) are polynomials in one variable. The variable in (b) is u and that in (c) is x. The polynomial in (d) is a polynomial in three variables x, y and z. In this chapter, we will consider polynomials in one variable only. Usually, we use x to represent the variable. The polynomial $4 - 5x - 6x^2$ in Example 7.1.2 (c) is a polynomial in one variable x. It is the sum of the three monomials: 4, $-5x$ and $-6x^2$. Each of these monomial is called a *term* of the polynomial.

- The term 4 is called the *constant term*.
- The terms $-5x$ and $-6x^2$ are called the *x-term* and *x^2-term* respectively and the numbers -5 and -6 are called the *coefficients* of the x-term and x^2-term respectively.

In general, in a polynomial, there are finitely many terms.

- The term that does not involve the variable x is just a real number. It is called the *constant term*.
- Terms that involve the variable x are in the form cx^k, where c is a real number and k is a positive integer (if $k = 1$, the expression cx^1 reduces to cx). The expression x^k is read '*x to the power k*' (k is called the *power* of x). The term in the form cx^k is called the *x^k-term* and the number c is called the *coefficient* of the x^k-term.

In the polynomial $4 - 5x - 6x^2$, the *highest power* of x is 2. The number 2 is called the *degree* of the polynomial. To represent a polynomial of degree 2 in general, we may write

$$a + bx + cx^2$$

where a, b, c are real numbers and $c \neq 0$. Note that the condition $c \neq 0$ is needed, for otherwise, the highest power of x is not 2. To represent a polynomial of degree 3 in general, we may write

$$a + bx + cx^2 + dx^3 \tag{7.1.1}$$

where a, b, c, d are real numbers and $d \neq 0$. Alternatively, we may write

$$a_0 + a_1x + a_2x^2 + a_3x^3 \tag{7.1.2}$$

where a_0, a_1, a_2, a_3 are real numbers and $a_3 \neq 0$. Although the notations a_0, a_1, a_2, a_3 (a letter together with subscripts) look rather complicated, the representation given in (7.1.2) is better than that in (7.1.1) for the following reasons:

- The coefficient of the x^k-term (where $k = 1, 2, 3$) can be recognized easily: it is a_k. Note that the subscript agrees with the power.
- To represent a polynomial with many terms, say to represent a polynomial in which the high-

est power of x is 30, if we use a single letter to denote each coefficient, besides the 26 alphabets, we have to introduce 5 additional letters (note that there are 31 terms); however, if we use subscript notations, we can write

$$a_0 + a_1x + \cdots + a_{30}x^{30}$$

where $a_0, a_1, \ldots, a_{30}$ are real numbers and $a_{30} \neq 0$.

- To represent a polynomial in which the highest power of x is m, where m is an unknown positive integer, using subscript notations, we can write

$$a_0 + a_1x + \cdots + a_mx^m$$

where $a_0, a_1, \ldots, a_m$ are real numbers and $a_m \neq 0$. Note that the coefficient of the x^k-term (where $1 \leq k \leq m$) is a_k and the constant term is a_0.

Given a polynomial, for example $4 - 5x - 6x^2$, it is understood that the variable x can take any value in $\mathbb{R}$ (that is, we can substitute x by any real number). Thus the polynomial $4 - 5x - 6x^2$ describes a function from $\mathbb{R}$ to $\mathbb{R}$; more precisely, it describes the function from $\mathbb{R}$ to $\mathbb{R}$ such that

for every real number a, its image under the function is the real number $4 - 5a - 6a^2$

Such a function is called a *polynomial function*.

Definition 7.1.1 We call ***a polynomial function*** to mean a function from $\mathbb{R}$ to $\mathbb{R}$, denoted by f, satisfying any one of the following conditions:

(1) there exists $c \in \mathbb{R}$ such that

$$\text{for every } x \in \mathbb{R}, \quad f(x) = c$$

(2) there exists $m \in \mathbb{Z}^+$ and there exist $a_0, a_1, \ldots, a_m \in \mathbb{R}$ with $a_m \neq 0$ such that

$$\text{for every } x \in \mathbb{R}, \quad f(x) = a_0 + a_1x + \cdots + a_mx^m \qquad (7.1.3)$$

Explanation $\mathbb{Z}^+$ is the set of all positive integers.

In the expression $a_0 + a_1x + \cdots + a_mx^m$, there are $m + 1$ terms.

- If $m = 1$, the expression reduces to $a_0 + a_1x$.
- If $m = 2$, the expression reduces to $a_0 + a_1x + a_2x^2$.

Remark If f satisfies Condition (1), then f is a constant function. If f satisfies Condition (2), then f is a non-constant function (see Theorem 7.1.1). Thus, Conditions (1) and (2) are mutually exclusive.

Example 7.1.3 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = 123$$

Then f is a polynomial function.

Reason f satisfies Condition (1) in Definition 7.1.1 with $c = 123$.

Example 7.1.4 Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by

$$g(x) = x^2 - 3$$

Then g is a polynomial function.

Reason g satisfies Condition (2) in Definition 7.1.1 with $m = 2$, $a_0 = -3$, $a_1 = 0$ and $a_2 = 1$.

Example 7.1.5 Let $v : \mathbb{R} \to \mathbb{R}$ be the function given by

$$v(x) = (18 - 2x)^2 \cdot x$$

Then v is a polynomial function.

Reason
$$\begin{aligned} v(x) &= (18 - 2x)^2 \cdot x \\ &= (324 - 72x + 4x^2) \cdot x \\ &= 324x - 72x^2 + 4x^3 \end{aligned}$$

v satisfies Condition (2) in Definition 7.1.1 with $m = 3$, $a_0 = 0$, $a_1 = 324$, $a_2 = -72$ and $a_3 = 4$.

The following result means that polynomial functions satisfying Condition (2) in Definition 7.1.1 are non-constant function.

Theorem 7.1.1 Let $m \in \mathbb{Z}^+$ and let $c_0, c_1, \ldots, c_m \in \mathbb{R}$ with $c_m \neq 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = c_0 + c_1 x + \cdots + c_m x^m$$

Then f is a non-constant function.

Explanation A non-constant function means a function that is not a constant function.
To show that f is a non-constant function, it suffices to find two real numbers r and s such that $f(r) \neq f(s)$.

Idea of Proof First we note that $f(0) = c_0$.
Next, note that if x is a real number with large magnitude, then the magnitude of x^m is much larger than that of x^{m-1}. We can choose a large positive real number r such that the magnitude of $c_m r^m$ is larger than the sum of the magnitudes of $c_i x^i$ for $i = 1, \ldots, m-1$, which implies that

$$f(r) = c_0 + (c_1 r + \cdots + c_{m-1} r^{m-1} + c_m r^m) \neq c_0$$

Using the idea described in the idea of proof of Theorem 7.1.1, we see that for a non-constant polynomial function f, the magnitude of $f(x)$ can be arbitrarily large, that is, given any large positive real number R, there exists a real number t such that the magnitude of $f(t)$ is larger than R. In fact, we have a stronger result:

- the magnitude of $f(x)$ is arbitrarily large if x is sufficiently large, that is, given any large positive real number R, there exists a real number s such that

$$\text{magnitude of } f(t) > R \quad \text{whenever } t > s$$

For example, let $f(x) = -x^3 + 2x^2 - 3x + 4$.

- Given $R = 100$, there exists $s = 10$ such that

$$\text{magnitude of } f(t) > 100 \quad \text{whenever } t > 10$$

In fact, we can choose $s = 6$*, or* $s = 5.23$.

- Given $R = 12345$, there exists $s = 100$ such that

$$\text{magnitude of } f(t) > 12345 \quad \text{whenever } t > 100$$

In fact, we can choose $s = 24$*, or* $s = 23.8$.

For another example, let $g(x) = x^4 - 999x^3$.

- Given $R = 10^5$, there exists $s = 1000$ such that

$$\text{(magnitude of) } g(t) > 10^5 \quad \text{whenever } t > 1000$$

In fact, we can choose $s = 999.001$.

- Given $R = 10^{20}$, there exists $s = 200000$ such that

$$\text{(magnitude of) } g(t) > 10^{20} \quad \text{whenever } t > 200000$$

In fact, we can choose $s = 100251$.

The numbers 5.23, 23.8 etc. are obtained by solving polynomial equations using computer softwares (together with knowledge of properties of polynomial functions).

Notation Given a polynomial, for example $4 - 5x - 6x^2$, the expression describes the polynomial function, whose domain and codomain are $\mathbb{R}$, such that

$$\text{for every real number } t, \quad \text{its image under the function is the real number } 4 - 5t - 6t^2$$

Instead of introducing a symbol, sometimes we write

$$4 - 5x - 6x^2$$

to represent this polynomial function. In this way, polynomials are considered to be polynomial functions.

Compare Coefficient Principle Let $n \in \mathbb{Z}^+$. Let $a_0, a_1 \dots, a_n \in \mathbb{R}$ and let $b_0, b_1, \dots, b_n \in \mathbb{R}$. Suppose that

$$a_0 + a_1 x + \cdots a_n x^n = b_0 + b_1 x + \cdots + b_n x^n \tag{7.1.4}$$

Then $a_0 = b_0,\ a_1 = b_1,\ \dots,\ a_n = b_n$.

Explanation Equality (7.1.4) means that the polynomial functions on both sides of the equality sign are equal (the same function).

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = (a_0 - b_0) + (a_1 - b_1)x + \cdots + (a_n - b_n)x^n$$

By the condition given in (7.1.4), we have

$$\text{for every } t \in \mathbb{R}, \quad f(t) = 0$$

Thus the function f is a constant function and so by Theorem 7.1.1,

$$a_1 - b_1 = 0, \quad a_2 - b_2 = 0, \quad \dots, \quad a_n - b_n = 0 \tag{7.1.5}$$

Moreover, since $f(0) = 0$, it follows that

$$a_0 - b_0 = 0 \tag{7.1.6}$$

The required result follows from (7.1.5) and (7.1.6). □

A polynomial function satisfying Condition (2) in Definition 7.1.1 is a non-constant function (by Theorem 7.1.1). Moreover, the positive integer m and the real numbers $a_0, a_1, \dots, a_m$ are unique (by Compare Coefficient Principle).

Terminology Let f be a non-constant polynomial function. The expression

$$a_m x^m + \cdots + a_1 x + a_0 \tag{7.1.7}$$

is called the *standard representation* of f, where $a_0, a_1, \dots, a_m$ are the real numbers described in Condition (2) in Definition 7.1.1.

- The term a_0 is called the *constant term* of (7.1.7).
- For $k = 1, 2, \dots, m$, the term $a_k x^k$ is called the *x^k-term* of (7.1.7).
- For $k = 1, 2, \dots, m$, the number a_k is called the *coefficient* of the x^k-term of (7.1.7).
- The number a_m is called the *leading coefficient* of (7.1.7).

Explanation (7.1.7) refers to the standard representation of f.

Remark The expression $a_0 + a_1 x + \cdots + a_m x^m$ is said to be in *ascending powers of x*. The expression $a_m x^m + \cdots + a_1 x + a_0$ is said to be in *descending powers of x*.

Remark The number a_0 is also called the coefficient of the constant term. Thus the coefficients of the standard representation of f mean the numbers $a_0, a_1, \dots, a_m$.

Example 7.1.6 Let $v : \mathbb{R} \to \mathbb{R}$ be the function given by

$$v(x) = (18 - 2x)^2 \cdot x$$

By the calculation in Example 7.1.5, the standard representation of v is

$$4x^3 - 72x^2 + 324x$$

The leading coefficient of the standard representation of v is 4.

Remark The expression $4x^3 - 72x^2 + 324x$ is understood to be $4x^3 + (-72)x^2 + 324x + 0$.

Remark Sometimes, for simplicity, instead of saying "*the leading coefficient of the standard representation of v*", we simply say "*the leading coefficient of v*" etc.

Definition 7.1.2 Let f be a non-constant polynomial function. We call ***the degree of f***, and write $\deg(f)$, to mean the positive integer m described in Condition (2) in Definition 7.1.1.

Example 7.1.7 (*a*) For the polynomial function g in Example 7.1.4, we have $\deg(g) = 2$.

(*b*) For the polynomial function v in Example 7.1.5, we have $\deg(v) = 3$.

Remark (*a*) Polynomial functions with degrees equal to 1 are called *linear functions*.

(*b*) Polynomial functions with degrees equal to 2 are called *quadratic functions*.

(*c*) Polynomial functions with degrees equal to 3 are called *cubic functions*.

To define the degree of a constant polynomial function, we have to consider two cases: (a) the zero function; (b) other constant functions.

Terminology The *zero function* is the function from $\mathbb{R}$ to $\mathbb{R}$ given by the rule $x \mapsto 0$.

Convention Let f be a non-zero constant function. We define the *degree* of f, denoted by $\deg(f)$, to be the number 0.

Explanation A non-zero constant function means a constant function (from $\mathbb{R}$ to $\mathbb{R}$) that is not the zero function.

Remark The function f can be represented by $f(x) = c$, which can be written as $f(x) = cx^0$. This is a reason why we adopt such a convention. With such a convention, results related to degrees of polynomial functions apply to non-zero constant polynomial functions too (see Theorem 7.2.1, Theorem 7.2.2 and also the remark to the Number of Zero Theorem on Page 37).

Example 7.1.8 For the polynomial function f in Example 7.1.3, we have $\deg(f) = 0$.

Convention We define the *degree* of the zero function to be $-\infty$.

Explanation The notation ∞ (read '*infinity*') is not a number. It can be considered to be "a quantity" that is larger than all counting numbers. The notation $-\infty$ (read '*minus infinity*' or '*negative infinity*') can be considered to be "a quantity" with a magnitude larger than all counting numbers but in the negative direction.

If we add ∞ and $-\infty$ to the real number line, the quantity ∞ (respectively $-\infty$) can be represented by a point that is infinitely far on the right (respectively on the left), farther than any point on the real number line.

Remark We define the order relation between a real number r and ∞ (or $-\infty$) as follows:

$$-\infty < r < \infty$$

We define the sum of a real number r and ∞ (or $-\infty$) as follows:

$$r + \infty = \infty, \qquad r + (-\infty) = -\infty$$

With such conventions, Theorem 7.2.2 can be applied to the case where f or g is/are the zero function.

To close this section, we give an example (Example 7.1.9) to illustrate how to use the *Compare Coefficient Method* (Solution 1). The example can also be done by the *Substitution Method* (Solution 2) or by a combination of both methods (Solution 3).

- In Solution 1, we express the right-side of the given equality using standard representation and apply the Compare Coefficient Principle.
- In Solution 2, we put x equals 1, -2 and 0 respectively to set up a system of three equations (note that the given equality is an equality of functions), then we solve for the knowns a, b and c. The three numbers 1, -2 and 0 are chosen so that the right-side or the left-side of the equality can be found easily.

Example 7.1.9 Suppose that a, b and c are real numbers satisfying the following condition:

$$ax^2 + bx + c = (x - 1)(x + 2)$$

Find the values of a, b and c.

Solution 1 Expanding the right-side of the equality, we get

$$\begin{aligned} ax^2 + bx + c &= (x - 1)(x + 2) \\ ax^2 + bx + c &= x^2 + x - 2 \end{aligned}$$

By the Compare Coefficient Principle,

$$a = 1, \quad b = 1, \quad c = -2$$

□

Solution 2 In the equality $ax^2 + bx + c = (x-1)(x+2)$,

- substitute $x = 1$, we get
$$a + b + c = 0 \tag{7.1.8}$$
- substitute $x = -2$, we get
$$4a - 2b + c = 0 \tag{7.1.9}$$
- substitute $x = 0$, we get $c = -2$.

With $c = -2$, Equation (7.1.8) and Equation (7.1.9) reduce to the following two equations respectively:

$$\begin{aligned} a + b - 2 &= 0 \\ 4a - 2b - 2 &= 0 \end{aligned}$$

Solving, we get $a = 1$ and $b = 1$. □

Solution 3 Comparing the coefficient of the x^2-term of both sides of the given equality

$$ax^2 + bx + c = (x-1)(x+2)$$

we see that $a = 1$ and so the given equality reduces to

$$x^2 + bx + c = (x-1)(x+2)$$

Putting $x = 1$ and $x = -2$ respectively into the above equality, we get

$$\begin{aligned} 1 + b + c &= 0 \\ 4 - 2b + c &= 0 \end{aligned}$$

Solving, we get $b = 1$ and $c = -2$. □

Exercise 7.1

1. For each of the following, the given function f can be written in the form $f(x) = a_0 + a_1x + \cdots + a_mx^m$, where m is a positive integer and $a_0, a_1, \ldots, a_m$ are real numbers with $a_m \neq 0$. Write down the values of m and $a_0, a_1, \ldots, a_m$.

   (a) $f(x) = 3x + 2$ (b) $f(x) = x^2 + 3x - 4$ (c) $f(x) = -x^3 + 7x^2 - 6 - 5$

   (d) $f(x) = x^3 - 1$ (e) $f(x) = x(x-1)$ (f) $f(x) = (2x-3)^2$

2. For each of the following, write down the degree of the given polynomial function f.

   (a) $f(x) = 3x + 2$ (b) $f(x) = x^2 + 3x - 4$ (c) $f(x) = -x^3 + 7x^2 - 6 - 5$

   (d) $f(x) = 12$ (e) $f(x) = x(x-1)$ (f) $f(x) = x(x+1)(x-2)$

3. For each of the following, find the values of the unknown real numbers A and B satisfying the given equality of polynomial functions.

   (a) $A(x+1) + B(x-2) = 5x - 4$

   (b) $A(3x+1) + 3(1-2x) = B$

4. For each of the following, find the values of the unknown real numbers A, B and C satisfying the given equality of polynomial functions.
   (a) $Ax^2 + Bx + C = (2x + 1)(x - 3)$
   (b) $A(x + 1)^2 + B(x + 1) + C = 2x^2 + 7x - 11$
   (c) $A(x - 1)^2 + B(2x + 1) = 3x^2 - 4x + C$

7.2 Addition, Subtraction and Multiplication of Polynomials

In junior forms, readers have learnt how to add, subtract and multiply polynomials. To add (respectively multiply) polynomials means to find the sum (respectively product) of polynomial functions.

Example 7.2.1 Perform the following operation:

$$(x^3 - 5x^2 + 6) + (7x^2 - x + 2)$$

Solution 1
$$\begin{aligned}(x^3 - 5x^2 + 6) + (7x^2 - x + 2) &= x^3 + (-5x^2 + 7x^2) + (-x) + (6 + 2)\\ &= x^3 + 2x^2 - x + 8\end{aligned}$$ □

Solution 2
$$\begin{array}{rrrrrrrr} & x^3 & - & 5x^2 & + & 0x & + & 6\\ (+) & & & 7x^2 & - & x & + & 2\\ \hline & x^3 & + & 2x^2 & - & x & + & 8\end{array}$$

$(x^3 - 5x^2 + 6) + (7x^2 - x + 2) = x^3 + 2x^2 - x + 8$ □

Remark The equality $(x^3 - 5x^2 + 6) + (7x^2 - x + 2) = x^3 + 2x^2 - x + 8$ is an equality of functions. It means the following identity:

$$\text{In } \mathbb{R},\quad (a^3 - 5a^2 + 6) + (7a^2 - a + 2) \equiv a^3 + 2a^2 - a + 8$$

Example 7.2.2 Perform the following operation:

$$(2x^2 - 3x - 4) - (2x^2 - x - 5)$$

Solution 1
$$\begin{aligned}(2x^2 - 3x - 4) - (2x^2 - x - 5) &= 2x^2 - 3x - 4 - 2x^2 + x + 5\\ &= (2x^2 - 2x^2) + (-3x + x) + (-4 + 5)\\ &= -2x + 1\end{aligned}$$ □

Solution 2
$$\begin{aligned}(2x^2 - 3x - 4) - (2x^2 - x - 5) &= (2x^2 - 2x^2) + (-3x - (-x)) + (-4) - (-5))\\ &= (-3x + x) + (-4 + 5)\\ &= -2x + 1\end{aligned}$$ □

Solution 3
$$\begin{array}{rrrrrr} & 2x^2 & - & 3x & - & 4\\ (-) & 2x^2 & - & x & - & 5\\ \hline & & - & 2x & + & 1\end{array}$$

$(2x^2 - 3x - 4) - (2x^2 - x - 5) = -2x + 1$ □

Example 7.2.1 and Example 7.2.2 illustrate that the sum and difference of polynomials functions are polynomial functions. Note that

- in Example 7.2.1, $$\deg(x^3 - 5x^2 + 6) = 3$$ $$\deg(7x^2 - x + 2) = 2$$ $$\deg(\text{sum}) = 3 = \max\{3, 2\}$$
- in Example 7.2.2, $$\deg(2x^2 - 3x - 4) = 2$$ $$\deg(2x^2 - x - 5) = 2$$ $$\deg(\text{difference}) = 1 < \max\{2, 2\}$$

In general, we have the following result concerning the degrees of the sum and difference of polynomial functions.

Theorem 7.2.1 Let p and q be polynomial functions. Then the functions $p + q$ and $p - q$ are polynomial functions. Moreover,

- if $\deg(p) \neq \deg(q)$, then $\deg(p \pm q) = \max\{\deg(p), \deg(q)\}$;
- if $\deg(p) = \deg(q)$, then $\deg(p \pm q) \leq \max\{\deg(p), \deg(q)\}$.

Explanation For real numbers u and v, the notation $\max\{u, v\}$ means the larger of the two numbers (or any one of them if they are equal).

For the case where p and q are non-constant polynomial functions, we can write

$$p(x) = a_0 + a_1 x + \cdots + a_m x^m$$
$$q(x) = b_0 + b_1 x + \cdots + b_n x^n$$

where $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_n$ are real numbers with $a_m \neq 0$ and $a_n \neq 0$. Note that $\deg(p) = m$ and $\deg(q) = n$. To prove the required results (for +), by symmetry, we may assume that $m \leq n$.

- If $m \neq n$, then $m < n$ and so
$$(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_m + b_m)x^m + b_{m+1}x^{m+1} + \cdots + b_n x^n$$
Hence $\deg(p + q) = n = \max\{m, n\}$.
- If $m = n$, then
$$(p + q)(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_m + b_m)x^m$$
Hence $\deg(p + q) \leq m = \max\{m, n\}$ *Note that $(a_m + b_m)$ may be 0.*

Remark For every real number r, we have $\max\{-\infty, r\} = r$.
If p is the zero function and q is a polynomial function with degree n, where $n \geq 0$, then $p + q = q$ and so

$$\deg(p + q) = n = \max\{-\infty, n\} = \max\{\deg(p), \deg(q)\}$$

Example 7.2.3 Perform the following operation:

$$(x^2 - 3x + 2) \cdot (2x - 1)$$

Solution 1

$$\begin{aligned}(x^2 - 3x + 2) \cdot (2x - 1) &= (x^2 - 3x + 2) \cdot 2x + (x^2 - 3x + 2) \cdot (-1) \\ &= (2x^3 - 6x^2 + 4x) + (-x^2 + 3x - 2) \\ &= 2x^3 + (-6x^2 - x^2) + (4x + 3x) + (-2) \\ &= 2x^3 - 7x^2 + 7x - 2\end{aligned}$$

□

Solution 2

$$\begin{array}{lrrrrrrr} & & & x^2 & - & 3x & + & 2 \\ (\times) & & & & & 2x & - & 1 \\ \hline & & - & x^2 & + & 3x & - & 2 \\ & 2x^3 & - & 6x^2 & + & 4x & & \\ \hline & 2x^3 & - & 7x^2 & + & 7x & - & 2 \end{array}$$

$(x^2 - 3x + 2) \cdot (2x - 1) = 2x^3 - 7x^2 + 7x - 2$ □

Example 7.2.3 illustrates that the product of polynomials functions is a polynomial function. Note that

$$\begin{aligned}&\deg(x^2 - 3x + 2) = 2 \\ &\deg(2x - 1) = 1 \\ &\deg(\text{product}) = 3 = 2 + 1\end{aligned}$$

In general, we have the following result concerning the degree of the product of polynomial functions.

Theorem 7.2.2 Let p and q be polynomial functions. Then the function $p \cdot q$ is a polynomial function. Moreover, $\deg(p \cdot q) = \deg(p) + \deg(q)$.

Explanation For the case where p and q are non-constant polynomial functions, we can write

$$\begin{aligned}p(x) &= a_0 + a_1 x + \cdots + a_m x^m \\ q(x) &= b_0 + b_1 x + \cdots + b_n x^n\end{aligned}$$

where $a_0, a_1, \ldots, a_m$ and $b_0, b_1, \ldots, b_n$ are real numbers with $a_m \neq 0$ and $a_n \neq 0$. Note that $(p \cdot q)(x)$ can be written in the form

$$c_0 + c_1 x + \cdots + c_{m+n} x^{m+n}$$

where $c_0, c_1, \ldots, c_{m+n}$ are real numbers with $c_{m+n} = a_m \cdot b_n$. Since $c_{m+n} \neq 0$, it follows that $\deg(p \cdot q) = m + n = \deg(p) + \deg(q)$.

Remark For every real number r, we have $-\infty + r = -\infty$.
If p is the zero function and q is a polynomial function with degree n, where $n \geq 0$, then $p \cdot q = 0 \cdot q = 0$ (the zero function) and so

$$\deg(p + q) = -\infty = -\infty + n = \deg(p) + \deg(q)$$

Example 7.2.4 Consider the polynomial function

$$(x^2 + a)(x - b)$$

where a and b are real numbers. Suppose that the coefficients of the x-term and x^2-term are 3 and 4 respectively. Find the constant term.

Explanation The x-term, x^2-term and constant term refer to that of the standard representation of the given polynomial function.

Solution

$$\begin{aligned}(x^2 + a)(x - b) &= (x^2 + a) \cdot x - (x^2 + a) \cdot b \\ &= x^3 + ax - bx^2 - ab \\ &= x^3 - bx^2 + ax - ab\end{aligned}$$

From the given coefficients, we have

$$a = 3, \qquad -b = 4$$

Therefore,

$$\begin{aligned}\text{constant term} &= -ab \\ &= -3 \cdot (-4) \\ &= 12\end{aligned}$$

□

Exercise 7.2

1. Perform the following addition of polynomial functions:
   (*a*) $(5x^2 + 8x + 2) + (2x^2 + x + 3)$ (*b*) $(2x^2 - 11x + 7) + (3x^2 + 6x - 9)$
   (*c*) $(x^3 + 2x + 4) + (2x^2 + 5x + 9)$ (*d*) $(2x^3 - 3x + 4) + (3x^2 + x - 6)$
   (*e*) $(4x^3 + 3x^2 - 2x + 1) + (-x^3 + 2x^2 + 4x - 8)$ (*f*) $(1 + 2x - 4x^3) + (4x^3 + 3x^2)$
2. Perform the following subtraction of polynomial functions:
   (*a*) $(x^3 - 2x^2 + 5x + 9) - (3x^2 - 7x + 15)$ (*b*) $(-4x^3 + 2x - 1) - (x^3 - 3x^2 + 3x - 1)$
   (*c*) $(2x^3 - 5x^2 + 11) - (2x^3 - 7x^2 + 4x - 3)$ (*d*) $(3x^4 - 4x^3 - 2x^2 + 1) - (2x^3 + 3x - 4)$
3. Perform the following multiplication of polynomial functions:
   (*a*) $(2x + 3)(4x - 5)$ (*b*) $(x^2 + 3x + 4)(x + 1)$
   (*c*) $(x^2 - 3x + 4)(x - 2)$ (*d*) $(x^2 - 3x - 4)(2x + 3)$
   (*e*) $(x^2 + 3x - 4)(x^2 - 2x + 3)$ (*f*) $(x^2 + 3x - 4)(2x^2 + 3)$
   (*g*) $(x^3 - 2x^2 + 3x - 4)(x - 1)$ (*h*) $(2x^3 - 3x^2 + x - 1)(2x - 3)$
   (*i*) $(2x^3 - 3x^2 + x - 1)(x^2 + 4x - 3)$ (*j*) $(2x^3 - 3x^2 + x - 1)(4x^2 - 3)$

4. For each of the following, find the standard representation of the give polynomial function.
   (*a*) $(x^2 + 5)(x - 3) + (2x - 1)(x + 2)$
   (*b*) $-3(x^3 + x - 2) + (2x^2 + 3x)(3x - 1)$
   (*c*) $(x - 1)(x + 2)(x - 3)$
   (*d*) $(3x^3 + 4x^2 - 5) - x(x + 1)(x - 2)$
   (*e*) $(3x^3 + 4x^2 - 5) - (x + 1)^2$
   (*f*) $(3x^3 + 4x^2 - 5) - (x + 1)(x - 2)^2$
5. Consider the polynomial function $(x^2 + ax - 2a) - (ax^2 + 2x - a)$, where a is a real number. Suppose that the coefficient of the x-term is 5.
   (*a*) Find the value of a.
   (*b*) Find the constant term and the coefficient of the x^2-term.
6. Consider the polynomial function $(x - a)(ax + b)$, where a and b are real numbers. Suppose that the coefficients of the x^2-term and x-term are 2 and 3 respectively.
   (*a*) Find the values of a and b.
   (*b*) Find the constant term.
7. Suppose that a, b and c are real numbers such that $(x + a)(2x - b) = ax^2 + (b - c)x + 6$. Find the values of a, b and c.
8. For each of the following, find a pair of polynomial functions p and q satisfying the given conditions.
   (*a*) $\deg(p) = \deg(q) = 3$ and $\deg(p + q) = 2$
   (*b*) $\deg(p) = \deg(q) = 3$ and $\deg(p + q) = 1$
   (*c*) $\deg(p) = \deg(q) = 4$ and $\deg(p + q) = 4$
9. Let p and q be polynomial functions.
   (*a*) Suppose that $\deg(p) = 3$ and $\deg(p + q) = 4$. What can you tell about $\deg(q)$?
   (*b*) Suppose that $\deg(p) = 3$ and $\deg(p + q) = 3$. What can you tell about $\deg(q)$?
   (*c*) Suppose that $\deg(p) = 3$ and $\deg(p \cdot q) = 4$. What can you tell about $\deg(q)$?
   (*d*) Suppose that $\deg(p \cdot q) = \deg(p)$. What can you tell about the polynomial function q?
   (*e*) Suppose that $\deg(p \cdot q) < \deg(p)$. What can you tell about the polynomial functions p and q?

7.3 Division of Polynomial Functions

Let f and g be polynomial functions, where g is not the zero function. Dividing f by g, we get $f \div g$ or $\frac{f}{g}$, which is a function called *rational function*. One problem concerning rational functions is how to "simplify" the functions.

In primary school, readers have learnt how to simplify fractions.

- One way to is to cancel common factors in the numerator and denominator of the fraction.

  Example $\dfrac{115}{161} = \dfrac{23 \times 5}{23 \times 7} = \dfrac{5}{7}$

- Another way is to express the fraction as "quotient with remainder".

  Example $\dfrac{30}{7} = 4 + \dfrac{2}{7}$ $\dfrac{\text{dividend}}{\text{divisor}} = \text{quotient} + \dfrac{\text{remainder}}{\text{divisor}}$

We can also simplify rational functions by canceling common factors. For this, we have to know how to factorize polynomial functions.

– Factorization of polynomial functions will be discussed in Section 7.6.

– Simplification of rational functions by canceling common factors, together with other manipulation of rational functions, will be discussed in a later chapter.

In this section, we will discuss how to express a rational function as "quotient with remainder".

The following two examples illustrate how to "simplify" $\dfrac{f}{g}$ where g is a monomial.

Example 7.3.1

$$\begin{aligned}\frac{x^3 + 2x^2 - 3x + 4}{x} &= \frac{x^3}{x} + \frac{2x^2}{x} - \frac{3x}{x} + \frac{4}{x} \\ &= x^2 + 2x - 3 + \frac{4}{x}\end{aligned}$$

Remark $x^3 + 2x^2 - 3x + 4$ is called the *dividend*;
x is called the *divisor*;
$x^2 + 2x - 3$ is called the *quotient*;
4 is called the *remainder*.

Note Degree of quotient = 2 = 3 − 1 = (degree of dividend) − (degree of divisor)
Degree of remainder = 0 < 1 = degree of divisor

Example 7.3.2

$$\begin{aligned}\frac{x^3 + 2x^2 - 3x + 4}{x^2} &= \frac{x^3}{x^2} + \frac{2x^2}{x^2} - \frac{3x}{x^2} + \frac{4}{x^2} \\ &= x + 2 - \frac{3}{x} + \frac{4}{x^2}\end{aligned}$$

Alternatively, we can express the rational function $\dfrac{x^3 + 2x^2 - 3x + 4}{x^2}$ in the form "a polynomial + $\dfrac{\text{a polynomial}}{x^2}$".

$$\begin{aligned}\frac{x^3 + 2x^2 - 3x + 4}{x^2} &= \frac{x^3}{x^2} + \frac{2x^2}{x^2} + \frac{-3x}{x^2} + \frac{4}{x^2} \\ &= x + 2 + \frac{-3x + 4}{x^2}\end{aligned}$$

Before giving more examples on division of polynomial functions, we have to know the meaning of *quotient* and *remainder*. For division of integers, instead of writing in fraction form, we can write in integer form. For example, the result $\frac{30}{7} = 4 + \frac{2}{7}$ is equivalent to the following:

$$30 = 7 \times 4 + 2$$

There are many ways to write 30 as (7 × an integer + an integer), that is, to write

$$30 = 7 \times a + b$$

where a and b are integers. However, if we require b to satisfy $0 \leq b < 7$, then 4 (the quotient) and 2 (the remainder) is the unique pair of integers q and r satisfying the following two conditions:

(1) $30 = 7 \times q + r$

(2) $0 \leq r < 7$

Indeed, if a and b is such a pair of integers, then by Condition (1), we have

$$7a + b = 30 = 7 \times 4 + 2$$

from which we get $7(a - 4) = 2 - b$, which implies that

$$(2 - b) \text{ is a multiple of } 7 \tag{7.3.1}$$

By Condition (2), we have

$$0 \leq b < 7 \tag{7.3.2}$$

It follows from (7.3.1) and (7.3.2) that $b = 2$ and hence $a = 4$.

In general, we have the following result on "division of integers".

Theorem 7.3.1 Let m and n be integers with $n > 0$. Then there exists a unique pair of integers q and r satisfying the following two conditions:

- $0 \leq r < n$
- $m = n \cdot q + r$

Explanation For the existence part, the integers q and r can be found using long division. In fact, we have the following formulas:

$$\begin{aligned} q &= \max\{k \in \mathbb{Z} : kn \leq m\} \\ r &= m - n \times \max\{k \in \mathbb{Z} : kn \leq n\} \end{aligned}$$

For the uniqueness part, the idea is similar to the discussion that precedes the theorem.

Remark The condition $m = n \cdot q + r$ is equivalent to the following

$$\frac{m}{n} = q + \frac{r}{n}$$

Terminology For the numbers m, n, q and r described in Theorem 7.3.1,

- the integer q is called the *quotient* when m is divided by n;
- the integer r is called the *remainder* when m is divided by n.

The following result is the polynomial version of Theorem 7.3.1.

Theorem 7.3.2 Let f and g be polynomial functions with $\deg(g) > 0$. Then there exists a unique pair of polynomial functions q and r satisfying the following two conditions:

- $\deg(r) < \deg(g)$
- $f = g \cdot q + r$

Explanation For the existence part, the polynomial functions q and r can be found using long division. Details can be found in the remaining examples in this section. A rigorous proof of the existence of q and r requires the knowledge of *Mathematical Induction*. For the uniqueness part, the idea is to use the following result:

– If d is a polynomial function with $\deg(d) < \deg(g)$ such that d is a *multiple* of g, then d is the zero function. *For the meaning of multiple, see Definition 7.3.2.*

Remark The condition $f = g \cdot q + r$ can be written as

$$f(x) = g(x) \cdot q(x) + r(x)$$

which means that the equality is satisfied for every real number x. It is equivalent to $\frac{f}{g} = q + \frac{r}{g}$, that is,

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

which means that the equality is satisfied for every real number x belonging to the domain of g.

Terminology For the functions f, g, q and r described in Theorem 7.3.2,

- the polynomial function q is called the *quotient* when f is divided by g;
- the polynomial function r is called the *remainder* when f is divided by g.

Similar to that for integers, for the division $\frac{f}{g}$ of a polynomial function f by a polynomial function g with $\deg(g) > 0$, the polynomial function f is called the *dividend* and the polynomial function g the *divisor*. There is a relation between the degree of the quotient q and that of the dividend f and divisor g. In fact, by Theorem 7.2.2, we have

$$\text{degree of quotient} = \text{degree of dividend} - \text{degree of divisor}$$

For the remainder, by the condition on the remainder, we have

$$\text{degree of remainder} < \text{degree of divisor}$$

- If the divisor is a linear function, then its degree is 1 and so the remainder is a polynomial function with degree 0 or $-\infty$, hence the remainder can be written as a real number.
- If the divisor is a quadratic function, then its degree is 2 and so the remainder is a polynomial function with degree 1 or 0 or $-\infty$, hence the remainder can be written in the form $ax + b$, where a and b are real numbers (*can be* 0).

Example 7.3.3 Find the quotient and the remainder when $(x^3 + 5x^2 - 3x + 6)$ is divided by $(x + 2)$.

Explanation The degree of the quotient is 2 and so the quotient can be written as $ax^2 + bx + c$ where $a \neq 0$. The degree of the remainder is less than 1 ans so the remainder can be written as r.

In the solution, we provide three methods to find the quotient and the remainder.

- Method 1 is to find a, then b, and then c and finally r. The idea is to compare leading coefficients.
- Method 2 is called the *Long Division Algorithm*. The idea is the same as that for Method 1, but without introducing the unknowns a, b, c and r.
- Method 3 is the *Compare Coefficient Method*.

Solution Note that the degree of the quotient is 2. Thus the quotient can be written as $ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$.

Note that the degree of the remainder is less than 1 (can be 0 or $-\infty$). Thus the remainder can be written as r where r is a real number.

We want to find a, b, c and r such that

$$x^3 + 5x^2 - 3x + 6 = (x + 2)(ax^2 + bx + c) + r \tag{7.3.3}$$

(Method 1) *Step 1* Comparing the leading coefficients (coefficients of x^3-term) on both sides of (7.3.3), we get $a = 1$.

Step 2 With $a = 1$, Equality (7.3.3) can be written as

$$\begin{aligned} x^3 + 5x^2 - 3x + 6 &= (x + 2)(x^2 + bx + c) + r \\ x^3 + 5x^2 - 3x + 6 &= (x + 2)x^2 + (x + 2)(bx + c) + r \\ (x^3 + 5x^2 - 3x + 6) - (x^3 + 2x^2) &= (x + 2)(bx + c) + r \\ 3x^2 - 3x + 6 &= (x + 2)(bx + c) + r \end{aligned}$$

Comparing the leading coefficients (coefficients of x^2-term) on both sides of the above equality, we get $b = 3$.

(Method 1 cont'd) *Step 3* With $b = 3$, the last equality in Step 2 can be written as

$$
\begin{aligned}
3x^2 - 3x + 6 &= (x+2)(3x+c) + r \\
3x^2 - 3x + 6 &= (x+2)3x + (x+2)c + r \\
(3x^2 - 3x + 6) - (3x^2 + 6x) &= (x+2)c + r \\
-9x + 6 &= (x+2)c + r
\end{aligned}
$$

Comparing the leading coefficients (coefficients of x-term) on both sides of the above equality, we get $c = -9$.

Step 4 With $c = -9$, the last equality in Step 3 can be written as

$$
\begin{aligned}
-9x + 6 &= (x+2)(-9) + r \\
-9x + 6 &= -9x - 18 + r
\end{aligned}
$$

Comparing the constant terms on both sides of the above equality, we get $6 = -18 + r$, which yields $r = 24$.

The quotient is $(x^2 + 3x - 9)$ and the remainder is 24. □

(Method 2) *Step 1* Find a monomial □ such that $(x+2) \cdot \square = x^3 + ?x^2$.

By inspection, we get $\square = x^2$ and we have $(x+2) \cdot x^2 = x^3 + 2x^2$.

$$
\begin{array}{rl}
 & \square \\ \hline
x+2 \big) & x^3 + 5x^2 - 3x + 6 \\
 & x^3 + ?x^2
\end{array}
\qquad
\begin{array}{rl}
 & x^2 \\ \hline
x+2 \big) & x^3 + 5x^2 - 3x + 6 \\
 & x^3 + 2x^2
\end{array}
$$

In writing down the dividend and the divisor, we must arrange the terms in descending powers of x.

Step 2a Taking difference (Second Last Line – Last Line), we get $3x^2 - 3x + 6$. The term 6 will not be used in the next step and so it can be omitted when we perform long division.

$$
\begin{array}{rl}
 & x^2 \\ \hline
x+2 \big) & x^3 + 5x^2 - 3x + 6 \\
 & x^3 + 2x^2 \\ \hline
 & \quad 3x^2 - 3x + 6
\end{array}
\qquad
\begin{array}{rl}
 & x^2 \\ \hline
x+2 \big) & x^3 + 5x^2 - 3x + 6 \\
 & x^3 + 2x^2 \\ \hline
 & \quad 3x^2 - 3x
\end{array}
$$

Step 2b Find a monomial □ such that $(x+2) \cdot \square = 3x^2 + ?x$.

By inspection, we get $\square = 3x$ and we have $(x+2) \cdot 3x = 3x^2 + 6x$.

$$
\begin{array}{rl}
 & x^2 + \square \\ \hline
x+2 \big) & x^3 + 5x^2 - 3x + 6 \\
 & x^3 + 2x^2 \\ \hline
 & \quad 3x^2 - 3x \\
 & \quad 3x^2 + ?x
\end{array}
\qquad
\begin{array}{rl}
 & x^2 + 3x \\ \hline
x+2 \big) & x^3 + 5x^2 - 3x + 6 \\
 & x^3 + 2x^2 \\ \hline
 & \quad 3x^2 - 3x \\
 & \quad 3x^2 + 6x
\end{array}
$$

(Method 2 cont'd) *Step 3a* Taking difference (Second Last Line − Last Line), remembering that the constant term 6 in Second Last Line is hidden, we get $-9x+6$.

$$
\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & & & x^2 & + & 3x & & \\
\cline{2-8}
x+2 & x^3 & + & 5x^2 & - & 3x & + & 6 \\
\multicolumn{1}{r}{} & x^3 & + & 2x^2 & & & & \\
\cline{2-8}
\multicolumn{1}{r}{} & & & 3x^2 & - & 3x & & \\
\multicolumn{1}{r}{} & & & 3x^2 & + & 6x & & \\
\cline{4-8}
\multicolumn{1}{r}{} & & & & & -9x & + & 6
\end{array}
$$

Step 3b Find a monomial □ such that $(x+2)\cdot\square = -9x + ?$.
By inspection, we get $\square = -9$ and we have $(x+2)\cdot -9 = -9x - 18$.

$$
\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & & & x^2 & + & 3x & + & \square \\
\cline{2-8}
x+2 & x^3 & + & 5x^2 & - & 3x & + & 6 \\
\multicolumn{1}{r}{} & x^3 & + & 2x^2 & & & & \\
\cline{2-8}
\multicolumn{1}{r}{} & & & 3x^2 & - & 3x & & \\
\multicolumn{1}{r}{} & & & 3x^2 & + & 6x & & \\
\cline{4-8}
\multicolumn{1}{r}{} & & & & & -9x & + & 6 \\
\multicolumn{1}{r}{} & & & & & -9x & + & ?
\end{array}
\qquad
\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & & & x^2 & + & 3x & - & 9 \\
\cline{2-8}
x+2 & x^3 & + & 5x^2 & - & 3x & + & 6 \\
\multicolumn{1}{r}{} & x^3 & + & 2x^2 & & & & \\
\cline{2-8}
\multicolumn{1}{r}{} & & & 3x^2 & - & 3x & & \\
\multicolumn{1}{r}{} & & & 3x^2 & + & 6x & & \\
\cline{4-8}
\multicolumn{1}{r}{} & & & & & -9x & + & 6 \\
\multicolumn{1}{r}{} & & & & & -9x & - & 18
\end{array}
$$

Step 4 Taking difference (Second Last Line − Last Line), we get 24.

$$
\begin{array}{r|rrrrrrr}
\multicolumn{1}{r}{} & & & x^2 & + & 3x & - & 9 \\
\cline{2-8}
x+2 & x^3 & + & 5x^2 & - & 3x & + & 6 \\
\multicolumn{1}{r}{} & x^3 & + & 2x^2 & & & & \\
\cline{2-8}
\multicolumn{1}{r}{} & & & 3x^2 & - & 3x & & \\
\multicolumn{1}{r}{} & & & 3x^2 & + & 6x & & \\
\cline{4-8}
\multicolumn{1}{r}{} & & & & & -9x & + & 6 \\
\multicolumn{1}{r}{} & & & & & -9x & - & 18 \\
\cline{6-8}
\multicolumn{1}{r}{} & & & & & & & 24
\end{array}
\tag{7.3.4}
$$

The quotient is $(x^2 + 3x - 9)$ and the remainder is 24. □

Remark We may omit the explanations in the steps and simply write the calculation shown in (7.3.4). Readers may compare it with long division of integers.

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & 4 & 5 & 2 \\
\cline{2-5}
12 & 5 & 4 & 3 & 1 \\
\multicolumn{1}{r}{} & 4 & 8 & & \\
\cline{2-5}
\multicolumn{1}{r}{} & & 6 & 3 & \\
\multicolumn{1}{r}{} & & 6 & 0 & \\
\cline{3-5}
\multicolumn{1}{r}{} & & & 3 & 1 \\
\multicolumn{1}{r}{} & & & 2 & 4 \\
\cline{4-5}
\multicolumn{1}{r}{} & & & & 7
\end{array}
\qquad
\begin{array}{lcl}
\text{Quotient} & = & 452 \\
\text{Remainder} & = & 7
\end{array}
$$

(Method 3) In Equality (7.3.3), expanding the right-side, we get

$$\begin{aligned}
x^3 + 5x^2 - 3x + 6 &= (x+2)(ax^2 + bx + c) + r \\
x^3 + 5x^2 - 3x + 6 &= (x+2)\cdot ax^2 + (x+2)\cdot bx + (x+2)\cdot c + r \\
x^3 + 5x^2 - 3x + 6 &= ax^3 + 2ax^2 + bx^2 + 2bx + cx + 2c + r \\
x^3 + 5x^2 - 3x + 6 &= ax^3 + (2a+b)x^2 + (2b+c)x + (2c+r)
\end{aligned}$$

By the Compare Coefficient Principle,

$$\begin{aligned}
1 &= a \\
5 &= 2a + b \\
-3 &= 2b + c \\
6 &= 2c + r
\end{aligned}$$

With $a = 1$, the second equation reduces to $5 = 2 + b$, which yields $b = 3$.
With $b = 3$, the third equation reduces to $-3 = 6 + c$, which yields $c = -9$.
With $c = -9$, the last equation reduces to $6 = -18 + r$, which yields $r = 24$.

The quotient is $(x^2 + 3x - 9)$ and the remainder is 24. □

In the rest of this section, we will give more examples on finding quotients and remainders using the long division method. First we give a few examples in which the divisors are linear functions.

Example 7.3.4 Find the quotient and the remainder when $(x^2 - 13)$ is divided is $(x - 3)$.

Explanation Note that $x^2 - 13 = x^2 + 0x - 13$. In writing long division, we add the term $0x$ or leave a space for the term.

Solution

$$\begin{array}{r|rrr l}
 & x & + & 3 & \\
\hline
x - 3 & x^2 + 0x & - & 13 & \\
 & x^2 - 3x & & & (x-3)\cdot x = x^2 - 3x \\
\hline
 & 3x & - & 13 & \\
 & 3x & - & 9 & (x-3)\cdot 3 = 3x - 9 \\
\hline
 & & & -4 &
\end{array}$$

The quotient is $(x + 3)$ and the remainder is -4. □

Remark In performing long division for polynomial functions, the variable is not important. What we need are the coefficients (and their positions). The following is a shorthand for the above long division.

$$\begin{array}{rr|rrr}
 & & & 1 & 3 \\
\hline
1 & -3 & 1 & 0 & -13 \\
 & & 1 & -3 & \\
\hline
 & & & 3 & -13 \\
 & & & 3 & -9 \\
\hline
 & & & & -4
\end{array}$$

Example 7.3.5 Find the quotient and the remainder when $(x^3 + 3x^2 - 7x + 15)$ is divided is $(x + 5)$.

Solution

$$
\begin{array}{rrrrrrrl}
 & & & x^2 & - & 2x & + & 3 \\
\cline{2-8}
x + 5 \,\big) & x^3 & + & 3x^2 & - & 7x & + & 15 \\
 & x^3 & + & 5x^2 & & & & \qquad (x+5)\cdot x^2 = x^3 + 5x^2 \\
\cline{2-8}
 & & & -2x^2 & - & 7x & & \\
 & & & -2x^2 & - & 10x & & \qquad (x+5)\cdot -2x = -2x^2 - 10x \\
\cline{4-8}
 & & & & & 3x & + & 15 \\
 & & & & & 3x & + & 15 \qquad (x+5)\cdot 3 = 3x + 15 \\
\cline{6-8}
 & & & & & & & 0
\end{array}
$$

The quotient is $(x^2 - 2x + 3)$ and the remainder is 0. □

Remark The result means that $x^3 + 3x^2 - 7x + 15 = (x^2 - 2x + 3)(x + 5)$.

For integers, we have the concepts of *divisible*, *factor* and *multiple*. For polynomial functions, we also have such concepts.

Definition 7.3.1 Let f and g be polynomial functions with $\deg(g) > 0$. We say that ***f is divisible by g*** to mean that when f is divided by g, the remainder is 0.

Example 7.3.6 By Example 7.3.5, the polynomial function $(x^3 + 3x^2 - 7x + 15)$ is divisible by $(x + 5)$.

Definition 7.3.2 Let f and g be polynomial functions.

- We say that ***g is a factor of f*** to mean that there exists a polynomial function h such that $g \cdot h = f$.
- We say that ***f is a multiple of g*** to mean that g is a factor of f.

Remark For the case where $\deg(g) > 0$, we have the following:
– g is a factor of f if and only if f is divisible by g.

Example 7.3.7 By Example 7.3.5, we have $x^3 + 3x^2 - 7x + 15 = (x^2 - 2x + 3)(x + 5)$. Thus

- $(x + 5)$ is a linear factor of $(x^3 + 3x^2 - 7x + 15)$.
- $(x^2 - 2x + 3)$ is a quadratic factor of $(x^3 + 3x^2 - 7x + 15)$.
- $(x^3 + 3x^2 - 7x + 15)$ is a multiple of $(x + 5)$.
- $(x^3 + 3x^2 - 7x + 15)$ is a multiple of $(x^2 - 2x + 3)$.

Remark A *linear factor* (respectively *quadratic factor*) of a polynomial function f means a linear function (respectively quadratic function) that is a factor of f.

Example 7.3.8 Find the quotient and the remainder when $(6x^3 + 7x^2 - x + 9)$ is divided is $(2x - 1)$.

Solution

$$
\begin{array}{r|rrrrrrr}
 & & & 3x^2 & + & 5x & + & 2 \\ \hline
2x - 1 & 6x^3 & + & 7x^2 & - & x & + & 9 \\
 & 6x^3 & - & 3x^2 & & & & \\ \hline
 & & & 10x^2 & - & x & & \\
 & & & 10x^2 & - & 5x & & \\ \hline
 & & & & & 4x & - & 9 \\
 & & & & & 4x & - & 2 \\ \hline
 & & & & & & & -7
\end{array}
$$

$(2x - 1) \cdot 3x^2 = 6x^3 - 3x^2$

$(2x - 1) \cdot 5x = 10x^2 - 5x$

$(2x - 1) \cdot 2 = 4x - 2$

The quotient is $(3x^2 + 5x + 2)$ and the remainder is -7. □

Next we give two examples in which the divisors are quadratic functions.

Example 7.3.9 Find the quotient and the remainder when $(x^3 - 5x^2 + 3x - 6)$ is divided is $(x^2 + 3x - 2)$.

Explanation The degree of the quotient is 1 and so the quotient can be written as $(ax + b)$, where $a \neq 0$.
The degree of the remainder is less than 2 and so the remainder can be written as $(cx + d)$, where c and d can be 0.

Solution

$$
\begin{array}{r|rrrrrrr}
 & & & & & x & - & 8 \\ \hline
x^2 + 3x - 2 & x^3 & - & 5x^2 & + & 3x & - & 6 \\
 & x^3 & + & 3x^2 & - & 2x & & \\ \hline
 & & & -8x^2 & + & 5x & - & 6 \\
 & & & -8x^2 & - & 24x & + & 16 \\ \hline
 & & & & & 29x & - & 22
\end{array}
$$

$(x^2 + 3x - 2) \cdot x = x^3 + 3x^2 - 2$

$(x^2 + 3x - 2) \cdot (-8) = -8x^2 - 24x + 16$

The quotient is $(x - 8)$ and the remainder is $(29x - 22)$. □

For division of polynomial, sometimes we have to work with rational numbers even the coefficients of the terms in the dividend and divisor are integers. This is illustrated by the following example.

Example 7.3.10 Find the quotient and remainder when $(x^3 + 4x - 7)$ is divided is $(2x^2 - x - 3)$.

Explanation The procedure is the same as that in the last example.
In order to get the required coefficients, we have to use fractions for the coefficients in the quotient.

Solution

$$\begin{array}{rrrrrrr}
 & & & & \frac{1}{2}x & + & \frac{1}{4} \\
\cline{2-7}
2x^2 - x - 3 \,\big) & x^3 & + & 0x^2 & + \; 4x & - & 7 \\
 & x^3 & - & \frac{1}{2}x^2 & - \; \frac{3}{2}x & & \\
\cline{2-7}
 & & & \frac{1}{2}x^2 & + \; \frac{11}{2}x & - & 7 \\
 & & & \frac{1}{2}x^2 & - \; \frac{1}{4}x & - & \frac{3}{4} \\
\cline{4-7}
 & & & & \frac{23}{4}x & - & \frac{25}{4}
\end{array}$$

$(2x^2 - x - 3) \cdot \frac{1}{2}x = x^3 - \frac{1}{2}x^2 - \frac{3}{2}$

$(2x^2 - x - 3) \cdot \frac{1}{4} = \frac{1}{2}x^2 - \frac{1}{4}x - \frac{3}{4}$

The quotient is $(\frac{1}{2}x + \frac{1}{4})$ and the remainder is $(\frac{23}{4}x - \frac{25}{4})$. □

Example 7.3.11 Suppose k is a real number such that when $(x^2 + kx - 3)$ is divided by $(x - 2)$, the remainder is 9. Find the value of k.

Explanation We can use long division to obtain the remainder in terms of k and then find k using the condition on the remainder.
Alternatively, we can use the compare coefficient method, noting that the quotient can be written as $ax + b$, where $a \neq 0$.

Solution 1

$$\begin{array}{rrrrr}
 & & & x & + \; (k+2) \\
\cline{2-5}
x - 2 \,\big) & x^2 & + & kx & - \; 3 \\
 & x^2 & - & 2x & \\
\cline{2-5}
 & & & (k+2)x & - \; 3 \\
 & & & (k+2)x & - \; 2(k+2) \\
\cline{4-5}
 & & & & 2(k+2) - 3
\end{array}$$

$(x - 2) \cdot x = x^2 - 2x$

$(x - 2) \cdot (k + 2) = (k + 2)x - 2(k + 2)$

It is given that the remainder is 9. Thus

$$2(k + 2) - 3 = 9$$

Hence $k = 4$. □

Remark Using the Remainder Theorem (see Example 7.4.3 in Section 7.4), we can express the remainder in terms of k without performing long division. Hence we can find the value of k quickly.

Solution 2 Note that the quotient can be written as $ax+b$, where a and b are real numbers with $a \neq 0$. Since the remainder is 9, it follows that

$$\begin{aligned} x^2 + kx - 3 &= (x-2)(ax+b) + 9 \\ x^2 + kx - 3 &= ax^2 - 2ax + bx - 2b + 9 \\ x^2 + kx - 3 &= ax^2 + (b-2a)x + (9-2b) \end{aligned}$$

Comparing coefficients, we get

$$\begin{aligned} 1 &= a \\ k &= b - 2a \\ -3 &= 9 - 2b \end{aligned}$$

From the third equation, we get $b = 6$.

Hence $k = 6 - 2 \times 1 = 4$. □

Exercise 7.3

1. For each of the following divisions, find the quotient and remainder.

   (*a*) $(5x-6) \div (x+1)$ (*b*) $(6x+5) \div (2x-1)$

   (*c*) $(x^2+3x-4) \div (x-2)$ (*d*) $(2x^2-5x-7) \div (x+3)$

   (*e*) $(3x^2+4) \div (x-5)$ (*f*) $(4x^2-7) \div (2x+3)$

   (*g*) $(6x^2+7x-8) \div (3x-5)$ (*h*) $(1-12x^2) \div (1-3x)$

   (*i*) $(x^3+2x^2-3x+4) \div (x+5)$ (*j*) $(2x^3-8x+11) \div (x-2)$

   (*k*) $(2x^3-3x^2+17x-7) \div (2x+3)$ (*l*) $(2x^3+3x^2+6x-7) \div (2x+3)$

   (*m*) $(3x+45) \div x$ (*n*) $(x^2+2x-3) \div x$

   (*o*) $(2x^2+4x-7) \div 2x$ (*p*) $(2x^2+4x-6) \div 2x$

2. For each of the following divisions, find the quotient and remainder.

   (*a*) $(2x^2+5x+6) \div (x^2+3x-4)$ (*b*) $(2x^2+5x+6) \div (x^2+4)$

   (*c*) $(2x^2+5x+6) \div (x^2+3x)$ (*d*) $(2x^2+6) \div (x^2+4)$

   (*e*) $(2x^2+5x) \div (x^2+4)$ (*f*) $(x^3+3x^2-x+4) \div (x^2+x-1)$

   (*g*) $(2x^3-5x^2+6x+3) \div (x^2-4x-7)$ (*h*) $(2x^3+5x^2-3x-1) \div (2x^2+x-1)$

   (*i*) $(2x^3+4x^2-3x+1) \div (2x^2-1)$ (*j*) $(2x^3+6x^2-5x-3) \div (2x^2-1)$

   (*k*) $(2x^3+3x^2-4x+5) \div x^2$ (*l*) $(2x^3+3x^2-4x) \div 2x^2$

3. For each of the following, express the quotient and the remainder in terms of k.

   (*a*) $(x^2-3x+k) \div (x+2)$ (*b*) $(x^2+kx-4) \div (x-5)$

   (*c*) $(x^3+kx-7) \div (x+2)$ (*d*) $(5x^2+kx+k^2) \div (x+k)$

4. For each of the following, find the unknown real number k satisfying the given condition.
   (a) $(x^3 + 3x^2 + k)$ is divisible by $(x + 2)$.
   (b) $(x^3 + kx^2 - 5)$ is divisible by $(x - 3)$.
   (c) When $(x^2 + 2x + k)$ is divided by $(x + 1)$, the remainder is -2.
   (d) When $(x^2 - kx + 2k)$ is divided by $(x - k)$, the remainder is 3.
5. Suppose a and b are real numbers such that when $(2x^3 + 7x^2 + ax + b)$ is divided by $(2x + 1)$, the quotient is $(x^2 + 3x - 4)$ and the remainder is -5. Find the values of a and b.

7.4 Remainder Theorem

The Remainder Theorem provides a quick way to find the remainder when a polynomial function f is divided by a linear function. First, we consider the case where the linear factor is in the form $(x-b)$. When f is divided by $(x-b)$, the quotient q is a polynomial function with $\deg(q) = \deg(f) - 1$ and the remainder r is a polynomial function with $\deg(r) < 1$ and so r is a constant function. We may write

$$f(x) = q(x) \cdot (x - b) + r \tag{7.4.1}$$

where r is a real number (which represents a constant function). Note that Equality (7.4.1) holds for every real number x.

- If we substitute, for example, $x = 0$ in (7.4.1), we get

  $$f(0) = q(0) \cdot (0 - b) + r$$

  from which we get $r = f(0) + b \cdot q(0)$. This doesn't help much—to find r, we have to know the quotient q.

- However, if we substitute $x = b$ in (7.4.1), we get

  $$\begin{aligned} f(b) &= q(b) \cdot (b - b) + r \\ f(b) &= q(b) \cdot 0 + r \\ f(b) &= r \end{aligned}$$

Thus we can find the remainder without knowing the quotient.

Remainder Theorem (Special Version) Let f be a polynomial function and let b be a real number. Then when f is divided by the linear function $(x - b)$, the remainder is $f(b)$.

Explanation $f(b)$ is a real number; it is also considered to be a constant function (a polynomial function with degree 0 or $-\infty$).

Example 7.4.1 For each of the following, find the remainder when $(x^3 - 2x + 5)$ is divided by given linear function.

(a) $x - 2$

(b) $x + 3$

Solution Let $f(x) = x^3 - 2x + 5$.

(a) By the Remainder Theorem,

$$\begin{aligned}\text{the required remainder} &= f(2)\\ &= 2^3 - 2 \times 2 + 5\\ &= 9\end{aligned}$$

(b) Note that $x + 3 = x - (-3)$. By the Remainder Theorem,

$$\begin{aligned}\text{the required remainder} &= f(-3)\\ &= (-3)^3 - 2 \times (-3) + 5\\ &= -16\end{aligned}$$

□

Remainder Theorem (General Version) Let f be a polynomial function and let a and b be real numbers with $a \neq 0$. Then when f is divided by the linear function $(ax - b)$, the remainder is $f\left(\frac{b}{a}\right)$.

Proof Denote the quotient by q and the remainder by r. By the definition of quotient and remainder, we have

$$f(x) = q(x) \times (ax - b) + r$$

Putting $x = \frac{b}{a}$, we get

$$\begin{aligned}f\left(\frac{b}{a}\right) &= q\left(\frac{b}{a}\right) \times \left(a \cdot \frac{b}{a} - b\right) + r\\ f\left(\frac{b}{a}\right) &= q\left(\frac{b}{a}\right) \times 0 + r\\ f\left(\frac{b}{a}\right) &= r\end{aligned}$$

□

Remark When a polynomial f is divided by $(ax + b)$, the remainder is $f(\frac{-b}{a})$. If you can't remember the formula clearly, you can always consider the following:

$$f(x) = q(x) \cdot (ax + b) + r$$

and substitute x equals a suitable number so that $ax + b = 0$.

Example 7.4.2 Find the remainder when $(9x^3 + 6x^2 - 2x + 1)$ is divided by $(3x + 2)$.

Solution Let $f(x) = 9x^3 + 6x^2 - 2x + 1$. Note that $3x + 2 = 3x - (-2)$. By the Remainder Theorem,

$$\begin{aligned} \text{the required remainder} &= f\left(\frac{-2}{3}\right) \\ &= 9 \cdot \left(\frac{-2}{3}\right)^3 + 6 \cdot \left(\frac{-2}{3}\right)^2 - 2 \cdot \left(\frac{-2}{3}\right) + 1 \\ &= \frac{-8}{3} + \frac{8}{3} + \frac{4}{3} + 1 \\ &= \frac{7}{3} \end{aligned}$$

□

The question in following example is the same as that in Example 7.3.11. Now we have the Remainder Theorem in hand, we can find the value of k easily.

Example 7.4.3 Suppose k is a real number such that when $(x^2 + kx - 3)$ is divided by $(x - 2)$, the remainder is 9. Find the value of k.

Solution Let $f(x) = x^2 + kx - 3$. By the Remainder Theorem, when $(x^2 + kx - 3)$ is divided by $(x - 2)$, the remainder is $f(2)$. Thus

$$\begin{aligned} f(2) &= 9 && \text{Given remainder} \\ 2^2 + k \cdot 2 - 3 &= 9 \\ 2k &= 9 - 4 + 3 \end{aligned}$$

Hence $k = 4$. □

Example 7.4.4 Suppose a and b are real numbers such that when $(x^3 + 3x^2 + ax + b)$ is divided by $(x + 1)$ and $(x - 2)$, the remainders are -7 and 5 respectively. Find the values of a and b.

Solution Let $f(x) = x^3 + 3x^2 + ax + b$.

- By the Remainder Theorem, when $(x^3 + 3x^2 + ax + b)$ is divided by $(x + 1)$, the remainder is $f(-1)$. Note that

$$\begin{aligned} f(-1) &= (-1)^3 + 3 \cdot (-1)^2 + a \cdot (-1) + b \\ &= -1 + 3 - a + b \\ &= 2 - a + b \end{aligned}$$

It is given that the remainder is -7. Thus

$$2 - a + b = -7 \tag{7.4.2}$$

(soln cont'd)

- By the Remainder Theorem, when $(x^3 + 3x^2 + ax + b)$ is divided by $(x - 2)$, the remainder is $f(2)$. Note that

$$\begin{aligned} f(2) &= 2^3 + 3 \cdot 2^2 + a \cdot 2 + b \\ &= 8 + 12 + 2a + b \\ &= 20 + 2a + b \end{aligned}$$

It is given that the remainder is 5. Thus

$$20 + 2a + b = 5 \qquad (7.4.3)$$

$$\begin{aligned} 9 &= a - b \\ 15 &= -2a - b \\ \hline -6 &= 3a \end{aligned}$$

Solving (7.4.2) and (7.4.3), we get $a = -2$ and $b = -11$. □

Example 7.4.5 Let f be a polynomial function. Suppose that when f is divided by $(x+2)$ and $(2x - 1)$, the remainders are -7 and 3 respectively. Find the remainder when f is divided by $(x + 2)(2x - 1)$.

Explanation Since the degree of $(x + 2)(2x - 1)$ is 2, it follows that the required remainder has degree ≤ 1 and so it can be written as $ax + b$. To find the unknowns a and b, we set up a system of two equations using the given information.

Solution Denote q and r be the quotient and the remainder respectively when f is divided by $(x + 2)(2x - 1)$. Since the degree of $(x + 2)(2x - 1)$ is 2, it follows that r can be written as $ax + b$.

By the definition of quotient and remainder,

$$f(x) = (x + 2)(2x - 1) \cdot q(x) + (ax + b) \qquad (7.4.4)$$

- Since the remainders is -7 when f is divided by $(x+2)$, it follows from from the Remainder Theorem that

$$f(-2) = -7 \qquad (7.4.5)$$

In (7.4.4), putting $x = -2$, we get

$$\begin{aligned} f(-2) &= (-2 + 2)(2 \cdot (-2) - 1) \cdot q(-2) + (a \cdot (-2) + b) \\ &= 0 \cdot (-5) \cdot q(-2) - 2a + b \\ &= -2a + b \end{aligned}$$

Hence by (7.4.5),

$$-2a + b = -7 \qquad (7.4.6)$$

(soln cont'd)

- Since the remainders is 3 when f is divided by $(2x-1)$, it follows from from the Remainder Theorem that

$$f(\tfrac{1}{2}) = 3 \tag{7.4.7}$$

In (7.4.4), putting $x = \frac{1}{2}$, we get

$$\begin{aligned} f(\tfrac{1}{2}) &= (\tfrac{1}{2}+2)\cdot(2\cdot\tfrac{1}{2}-1)\cdot q(\tfrac{1}{2}) + (a\cdot\tfrac{1}{2}+b) \\ &= \tfrac{1}{2}a + b \end{aligned}$$

Hence by (7.4.7),

$$\begin{aligned} -7 &= -2a + b \\ 3 &= \tfrac{1}{2}a + b \\ \hline -10 &= -\tfrac{5}{2}a \end{aligned}$$

$$\tfrac{1}{2}a + b = 3 \tag{7.4.8}$$

Solving (7.4.6) and (7.4.8), we get $a = 4$ and $b = 1$. □

Exercise 7.4

1. Let f be a polynomial function. For each of the following, find the remainder in the form f(a number) when f is divided by the given linear function.

   (*a*) $x-6$ (*b*) $x+7$ (*c*) $3x-5$ (*d*) $7x+8$

   (*e*) $-x+2$ (*f*) $-3x-2$ (*g*) x (*h*) $-3x$

   (*i*) $ax+b$ (*j*) $ax-b$ (*k*) $b-ax$

   In (i), (j) and (k), a and b are real numbers with $a \neq 0$.

2. Let $f(x) = x^3 + ax^2 - 2ax - 3$, where a is a non-zero real number. For each of the following, find the remainder in terms of a when f is divided by the given linear function.

   (*a*) $x-3$ (*b*) $x+1$ (*c*) $-x+2$

   (*d*) $x-a$ (*e*) $x+a$ (*f*) $ax+2$

3. For each of the following divisions, use the Remainder Theorem to find the remainder.

   (*a*) $(x^2-3x+7)\div(x-2)$ (*b*) $(x^2-3x+7)\div(x+3)$

   (*c*) $(x^3+5x^2-6x-7)\div(x+1)$ (*d*) $(x^3+5x^2-6x-7)\div(2-x)$

   (*e*) $(4x^3-3x+4)\div(2x-1)$ (*f*) $(4x^3-3x+4)\div(2x+3)$

   (*g*) $(6x^3+9x^2+1)\div(3x+1)$ (*h*) $(6x^3+9x^2+1)\div(3x-2)$

   (*i*) $(x^{123}-4x+5)\div(x+1)$ (*j*) $(x^{32}-5x^{31}+6x-1)\div(x-5)$

   (*k*) $(4x^{12}+x^{11}-3x)\div(4x+1)$ (*l*) $(13x^7-12x^5+11x^3+9x-8)\div(56x)$

4. For each of the following, find the unknown real number a satisfying the given condition.
   (*a*) When $(3x^2 - 4x + a)$ is divided by $(x - 2)$, the remainder is -7.
   (*b*) When $(2x^3 + ax + 1)$ is divided by $(x + 1)$, the remainder is 3.
   (*c*) $(x^4 - ax^2 + 20)$ is divisible by $(x - 2)$.
   (*d*) When $(ax^3 + 4x^2 + 3)$ is divided by $(2x + 1)$, the remainder is 4.
   (*e*) When $(x^2 + 2x - 3)$ is divided by $(x - a)$, the remainder is a^2.
5. Suppose a is a real number such that when $(2x^2 + 3x - 5)$ is divided by $(x + a)$, the remainder is 4. Find the possible values of a.
6. Suppose a and b are real numbers such that when $(x^3 + ax^2 + bx)$ is divided by $(x - 1)$ and $(x + 2)$, the remainders are -1 and 2 respectively. Find the values of a and b.
7. Let f be a polynomial function. Suppose that when f is divided by $(x - 2)$, the remainder is 3 and that f is divisible by $(x + 3)$. Find the remainder when f is divided by $(x - 2)(x + 3)$.
8. Let $f(x) = x^2 - 4x + 7$. Find the smallest possible value of the remainder when f is divided by $(ax + b)$, where a and b are real numbers with $a \neq 0$.

7.5 Factor Theorem

The Factor Theorem, in some sense, is a special case of the Remainder Theorem (in which the remainder is 0). The result is stated in the form "*p if and only if q*" which means that "*if q is true, then p is true and vice versa*".

Factor Theorem Let f be a polynomial function and let b be a real number. Then $(x - b)$ is a factor of f if and only if $f(b) = 0$.

Explanation The theorem consists of two results: (1) If $f(b) = 0$, then $(x - b)$ is a factor of f.
(2) If $(x - b)$ is a factor of f, then $f(b) = 0$.

Proof (1) Suppose that $f(b) = 0$. Then by the Remainder Theorem,

$$f(x) = (x - b) \cdot q(x) + 0$$

where q is the quotient when f is divided by $(x - b)$. Hence

$$f(x) = (x - b) \cdot q(x)$$

and so $(x - b)$ is a factor of f.

(2) Suppose that $(x - b)$ is a factor of f. Then there exists a polynomial function g such that

$$f(x) = (x - b) \cdot g(x)$$

Putting $x = b$, we get

$$f(b) = (b - b) \cdot g(b) = 0 \cdot g(b) = 0$$

□

The Factor Theorem can be used in the following ways:

- To determine whether $(x - b)$ is a factor of f, we can check whether $f(b) = 0$ or not (see Example 7.5.1 and Example 7.5.2).
- Suppose it is given that $(x-b)$ is a factor of f, then we know that $f(b) = 0$ (see Example 7.5.3).

Example 7.5.1 Let $f(x) = x^3 - 4x^2 + x + 6$. For each of the following, determine whether the given linear function is a factor of f.

(a) $x - 2$ (b) $x + 3$

Solution (a) Note that
$$\begin{aligned} f(2) &= 2^3 - 4 \cdot 2^2 + 2 + 6 \\ &= 8 - 16 + 2 + 6 \\ &= 0 \end{aligned}$$

By the Factor Theorem, the linear function $(x - 2)$ is a factor of f.

(b) Note that
$$\begin{aligned} f(-3) &= (-3)^3 - 4 \cdot (-3)^2 + (-3) + 6 \\ &= -27 - 36 - 3 + 6 \\ &= -60 \\ &\neq 0 \end{aligned}$$

By the Factor Theorem, the linear function $(x + 3)$ is not a factor of f.

□

The Factor Theorem can also be used to determine whether $(ax-b)$ is a factor of f or not (where a and b are real numbers with $a \neq 0$) This is because

$$ax - b = a \cdot (x - \frac{b}{a})$$

and so $(ax - b)$ is a factor of f if and only if $(x - \frac{b}{a})$ is a factor of f.

Factor Theorem (Alternative Version) Let f be a polynomial function and let a and b be real numbers with $a \neq 0$. Then $(ax - b)$ is a factor of f if and only if $f(\frac{b}{a}) = 0$.

Example 7.5.2 Let $f(x) = 6x^3 + x^2 - 5x - 2$. For each of the following, determine whether the given linear function is a factor of f.

(a) $2x - 1$ (b) $3x + 2$

Solution (a) Note that

$$\begin{aligned} f(\tfrac{1}{2}) &= 6 \cdot (\tfrac{1}{2})^3 + (\tfrac{1}{2})^2 - 5 \cdot \tfrac{1}{2} - 2 \\ &= \frac{3}{4} + \frac{1}{4} - \frac{5}{2} - 2 \\ &= -\frac{7}{2} \\ &\neq 0 \end{aligned}$$

By the Factor Theorem, the linear function $(2x - 1)$ is not a factor of f.

(b) Note that

$$\begin{aligned} f(\tfrac{-2}{3}) &= 6 \cdot (\tfrac{-2}{3})^3 + (\tfrac{-2}{3})^2 - 5 \cdot (\tfrac{-2}{3}) - 2 \\ &= -\frac{16}{9} + \frac{4}{9} + \frac{10}{3} - 2 \\ &= 0 \end{aligned}$$

By the Factor Theorem, the linear function $(3x + 2)$ is a factor of f. □

Example 7.5.3 Let $f(x) = 2x^3 - 3x^2 - 8x + k$, where k is a real number. Suppose that $(2x + 1)$ is a factor of f. Find the value of k.

Solution Since $(2x + 1)$ is a factor of f, it follows from the Factor Theorem that

$$f(\tfrac{-1}{2}) = 0$$

that is,

$$\begin{aligned} 2 \cdot (\tfrac{-1}{2})^3 - 3 \cdot (\tfrac{-1}{2})^2 - 8 \cdot (\tfrac{-1}{2}) + k &= 0 \\ -\frac{1}{4} - \frac{3}{4} + 4 + k &= 0 \\ -1 + 4 + k &= 0 \end{aligned}$$

Hence $k = -3$. □

Exercise 7.5

1. Let $f(x) = x^3 + 4x^2 - 7x - 10$. For each of the following, determine whether the given linear function is a factor of f.

(a) $x - 1$	(b) $x + 1$	(c) $x - 2$	(d) $x + 2$
(e) $x - 3$	(f) $x + 3$	(g) $x - 4$	(h) $x + 4$
(i) $x - 5$	(j) $x + 5$	(k) $x - \sqrt{2}$	(l) x

2. Let $f(x) = 2x^4 + 3x^3 - 3x^2 - 2x$. For each of the following, determine whether the given linear function is a factor of f.

(*a*) $x - 1$	(*b*) $x + 1$	(*c*) $x - 2$	(*d*) $x + 2$
(*e*) $2x - 1$	(*f*) $2x + 1$	(*g*) $3x - 1$	(*h*) $2x + 3$

3. For each of the following, find the unknown real number a satisfying the given condition.
   (*a*) $(x - 3)$ is a factor of $(x^3 - 2x^2 + ax + 5)$.
   (*b*) $(x + 2)$ is a factor of $(x^4 - ax^3 - 6)$.
   (*c*) $(3x - 2)$ is a factor of $(3x^3 - 11x^2 + 21x + 2a)$.
   (*d*) $(2x - 1)$ is a factor of $(ax^3 - 4x^2 + 3x - 1)$

7.6 Factorization of Polynomial Functions

In this section, we will discuss how to factorize polynomial functions with integer coefficients. First we consider an example which illustrates how to factorize a cubic function if we know a linear factor of the function.

Example 7.6.1 Let $f(x) = x^3 - 5x^2 + 8x - 4$.

(*a*) Show that $(x - 2)$ is a factor of f.

(*b*) Find the quotient when f is divided by $(x - 2)$.

(*c*) Factorize f as a product of three linear functions.

Solution (*a*) Note that

$$\begin{aligned} f(2) &= 2^3 - 5 \cdot 2^2 + 8 \cdot 2 - 4 \\ &= 8 - 20 + 16 - 4 \\ &= 0 \end{aligned}$$

It follows from the Factor Theorem that $(x - 2)$ is a factor of f.

(*b*) By Long Division:

$$\begin{array}{r|rrrr} & x^2 & -\,3x & +\,2 & \\ \hline x - 2 & x^3 - 5x^2 & +\,8x & -\,4 & \\ & x^3 - 2x^2 & & & \\ \hline & -3x^2 & +\,8x & & \\ & -3x^2 & +\,6x & & \\ \hline & & 2x & -\,4 & \\ & & 2x & -\,4 & \\ \hline \end{array}$$

The quotient is $(x^2 - 3x + 2)$.

(*c*)

$$\begin{aligned} x^3 - 5x^2 + 8x - 4 &= (x - 2)(x^2 - 3x + 2) && \text{By (b)} \\ &= (x - 2)(x - 1)(x - 2) \end{aligned}$$

□

Remark We may also write $x^3 - 5x^2 + 8x - 4 = (x - 1)(x - 2)^2$.
There are many possible answers. For example: $(2 - x)(1 - x)(x - 2)$

7.6.1 Zeros of Polynomial Functions

To factorize a polynomial function f as a product of linear functions (if it is possible), the first step is to *guess* a real number a such that $f(a) = 0$, which by the Factor Theorem, implies that $(x - a)$ is a factor of f. If we randomly pick a real number a and calculate the value $f(a)$, the chance that $f(a) = 0$ is very very small (to be precise, the *probability* is 0). This is because the number of solutions to $f(x) = 0$ is finite. In fact, there is a relation between the number of solutions and the degree of f.

Terminology 7.6.1 Let f be a function from a subset of $\mathbb{R}$ to $\mathbb{R}$.

- We call ***a zero of f*** to mean a real number, denoted by r, such that $f(r) = 0$.
- For the equation

$$f(x) = 0 \tag{7.6.1}$$

we call ***a solution to Equation (7.6.1)*** or ***a root to Equation (7.6.1)*** to mean a real number, denoted by r, such that $f(r) = 0$.

Remark A solution to Equation (7.6.1) means a zero of f.

Example 7.6.2 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = x^3 - 5x^2 + 8x - 4$$

By Example 7.6.1,

$$f(x) = (x - 1)(x - 2)^2$$

Therefore, by the Product Zero Theorem,

- the number 1 is a zero of f;
- the number 2 is a zero of f;
- every real number different from 1 and 2 is not a zero f.

Thus, the zeros of f are 1 and 2. In other words, the solutions (or roots) to the equation $f(x) = 0$ are 1 and 2.

Remark Note that

- the degree of f is 3;
- the number of zeros of f is 2.

The number of zeros of f is not bigger than the degree of f.

There is a general result (*Number of Zero Theorem*) concerning the degree of a non-constant polynomial function f and the number of zeros of f. Before stating the result, we consider some special cases.

(*Case 1*) If the degree of f is 1, then the polynomial function f can be written as $f(x) = ax + b$, where a and b are real numbers with $a \neq 0$. It is clear that f has exactly one zero.

(*Case 2*) If the degree of f is 2, then the polynomial function f can be written as $f(x) = ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$. Using the discriminant $b^2 - 4ac$, we can determine how many solutions there are for the quadratic equation $f(x) = 0$: there may be no solution, one solution or two solutions. Thus, the number of zeros of f is at most 2.

(*Case 3*) If the degree of f is 3, then the polynomial function f can be written as

$$f(x) = ax^3 + bx^2 + cx + d$$

where a, b, c, d are real numbers with $a \neq 0$.

- If f has no zero, then the number of zeros of f is less than 3.
- If f has a zero, say denoted by r, then by the Factor Theorem, $(x - r)$ is a factor of f and so there exists a polynomial function q such that

  $$f(x) = (x - r) \cdot q(x)$$

  Note that the degree of q is 2 (because the degree of f is 3). From the discussion in Case 2, we see that the number of zeros of q is at most 2. Hence the number of zeros of f is at most 3. This is because if s is a zero of f, then $s = r$ or s is a zero of q.

In any case, the number of zeros of f is at most 3.

(*Case 4*) If the degree of f is 4, then using the argument in Case 3, we see that the number of zero of f is at most 4.

Number of Zero Theorem Let f be a non-constant polynomial function. Then

$$\text{number of zeros of } f \leq \deg(f)$$

Explanation A non-constant polynomial function means a polynomial function with degree ≥ 1. From the discussion that precedes the theorem, readers may convince themselves of the validity of the result. A vigorous proof of the result requires Mathematical Induction.

Remark If f is a non-zero constant polynomial function, then the degree of f is 0 and the number of zeros of f is also 0.
If f is the zero function, then f has infinitely many zeros. In fact, every real number is a zero of f.

Suppose that a and b are different integers that are factors of an integer n. It may happen that $a \cdot b$ is not a factor of n.

Example 4 and 6 are factors of 12, but 4×6 is not a factor of 12.

For polynomial functions, it may also happen that g and h are different polynomial functions that are factors of a polynomial function f but $g \cdot h$ is not a factor of h.

Example Let $f(x) = (x-1)(x-2)(x-3)$. Let $g(x) = (x-1)(x-2)$ and let $h(x) = (x-2)(x-3)$. Then g and h are factors of f but $g \cdot h$ is not a factor of f.

However, the following result tells that if $(x-\alpha)$ and $(x-\beta)$, where $\alpha \neq \beta$, are factors of f, then $(x-\alpha)(x-\beta)$ is a factor of f.

General Factor Theorem Let f be a non-constant polynomial function. Let α and β be real numbers with $\alpha \neq \beta$. Suppose that α and β are zeros of f. Then $(x-\alpha)(x-\beta)$ is a factor of f.

Proof Since α is a zero of f, it follows from the Factor Theorem that $(x-\alpha)$ is a factor of f. Thus there exists a polynomial function g such that

$$f(x) = (x-\alpha) \cdot g(x) \tag{7.6.2}$$

Since $f(\beta) = 0$ and $\alpha \neq \beta$, it follows from (7.6.2) that

$$g(\beta) = 0$$

that is, β is a zero of g which, by the Factor Theorem, implies that $(x-\beta)$ is a factor of g. Thus there exists a polynomial function h such that

$$g(x) = (x-\beta) \cdot h(x) \tag{7.6.3}$$

In view of (7.6.2) and (7.6.3), we see that

$$f(x) = (x-\alpha) \cdot (x-\beta) \cdot h(x)$$

Therefore $(x-\alpha)(x-\beta)$ is a factor of f. □

Remark The above result can be generalized to the following result which can be proved using the idea of the above proof together with Mathematical Induction.

– Let f be a non-constant polynomial function. Let $\alpha_1, \ldots, \alpha_n$ be distinct real numbers. Suppose that $\alpha_1, \ldots, \alpha_n$ are zeros of f. Then $(x-\alpha_1)\cdots(x-\alpha_n)$ is a factor of f.

The following result tells that if f is a non-constant polynomial function such that the number of zeros of f is equal to the degree of f, then f can be factorized as a product of linear factors together with a real number (a constant function). The result can be proved using the General Factor Theorem (see its remark) together with the Compare Coefficient Principle. Below we prove the result using the Number of Zero Theorem.

Theorem 7.6.1 Let f be a non-constant polynomial function. Denote m to be the degree of f. Let $\alpha_1, \ldots, \alpha_m$ be distinct real numbers. Suppose that $\alpha_1, \ldots, \alpha_m$ are zeros of f. Then

$$f(x) = c \cdot (x - \alpha_1) \cdots (x - \alpha_m)$$

where c is the leading coefficient of the standard representation of f.

Explanation $\alpha_1, \ldots, \alpha_m$ are distinct means that $\alpha_i \neq \alpha_j$ whenever $i \neq j$.

Proof Let g be the polynomial function given by

$$g(x) = f(x) - c \cdot (x - \alpha_1) \cdots (x - \alpha_m)$$

(1) It is clear that the numbers $\alpha_1, \ldots, \alpha_m$ are zeros of g.

(2) Since c is the leading coefficient of the standard representation of f, it follows that the degree of g is less than m.

In view of (1) and (2), it follows from the Number of Zero Theorem that g is a constant function. Moreover, the function g is the zero function (since it has zeros). Hence

$$f(x) = c \cdot (x - \alpha_1) \cdots (x - \alpha_m)$$

□

Exercise 7.6.1

1. For each of the following, find a polynomial function f with degree equal to 3 and satisfying the given condition.

   (a) Number of zeros of $f = 3$

   (b) Number of zeros of $f = 2$

   (c) Number of zeros of $f = 1$

   It can be shown that every cubic function has at least one zero.

2. For each of the following, find a polynomial function f with degree equal to 4 and satisfying the given condition.

   (a) Number of zeros of $f = 4$

   (b) Number of zeros of $f = 3$

   (c) Number of zeros of $f = 2$

   (d) Number of zeros of $f = 1$

   (e) Number of zeros of $f = 0$

7.6.2 Factorization of Quadratic Functions

In this section, first we give two examples to illustrate how to factorize quadratic functions using Theorem 7.6.1; and then we give a summary for factorization of quadratic functions. The results given in the summary can be obtained using the ideas discussed in the two examples.

Example 7.6.3 Factorize $12x^2 - 11x - 15$.

Explanation The question is to write $(12x^2 - 11x - 15)$ as $(ax + b)(cx + d)$ where a, b, c, d are integers. This can be done using inspection; however, it may be time consuming.

Solution Let $f(x) = 12x^2 - 11x - 15$.

Solving $f(x) = 0$ by the Quadratic Formula, we get

$$x = \frac{-(-11) \pm \sqrt{11^2 - 4 \cdot 12 \cdot (-15)}}{2 \cdot 12} = \frac{11 \pm \sqrt{841}}{24} = \frac{11 \pm 29}{24} = \frac{5}{3} \text{ or } \frac{-3}{4}$$

Thus $\frac{5}{3}$ and $\frac{-3}{4}$ are the zeros of f.

Note that the leading coefficient of $(12x^2 - 11x - 15)$ is 12. Hence

$$\begin{aligned} 12x^2 - 11x - 15 &= 12 \cdot \left(x - \frac{5}{3}\right) \cdot \left(x - \frac{-3}{4}\right) && \text{By Theorem 7.6.1} \\ &= 3 \cdot 4 \cdot \left(x - \frac{5}{3}\right) \cdot \left(x - \frac{-3}{4}\right) && \text{Write } 12 = 3 \cdot 4 \\ &= 3\left(x - \frac{5}{3}\right) \cdot 4\left(x - \frac{-3}{4}\right) \end{aligned}$$

Thus, $12x^2 - 11x - 15 = (3x - 5)(4x + 3)$. □

Example 7.6.4 Factorize $(x^2 + 3x - 7)$ in the form $(x - \alpha)(x - \beta)$, where α and β are real numbers.

Explanation Usually, in factorization, we want to have factors with integer coefficients. In this question, we allow the coefficients to be irrational numbers.

Solution Let $f(x) = x^2 + 3x - 7$.

Solving $f(x) = 0$ by the Quadratic Formula, we get

$$x = \frac{-3 \pm \sqrt{3^2 - 4 \cdot 1 \cdot (-7)}}{2 \cdot 1} = \frac{-3 \pm \sqrt{37}}{2}$$

Thus $\frac{-3 + \sqrt{37}}{2}$ and $\frac{-3 - \sqrt{37}}{2}$ are the zeros of f.

Note that the leading coefficient of $(x^2 + 3x - 7)$ is 1. Hence

$$x^2 + 3x - 7 = \left(x - \frac{-3 + \sqrt{37}}{2}\right)\left(x - \frac{-3 - \sqrt{37}}{2}\right)$$

□

Summary for Factorization of Quadratic Functions

Let $f(x) = ax^2 + bx + c$, where a, b, c are real numbers with $a \neq 0$.

- If $b^2 - 4ac > 0$, then f can be factorized in the form $f(x) = k(x - \alpha)(x - \beta)$, where k is a real number and α and β are distinct real numbers.
- If $b^2 - 4ac = 0$, then f can be factorized in the form $f(x) = k(x - \alpha)^2$, where k and α are real numbers.
- If $b^2 - 4ac < 0$, then f cannot be factorized in the form $f(x) = k(x - \alpha)(x - \beta)$, where k, α and β are real numbers.

For the case where a, b, c are integers, we have the following:

- The function f can be factorized in the form $f(x) = (px + q)(rx + s)$, where p, q, r, s are integers if and only if $b^2 - 4ac$ is a perfect square (that is, the square of an integer).

Exercise 7.6.2

1. For each of the following, use Theorem 7.6.1 to factorize the given quadratic function in the form $(ax + b)(cx + d)$ where a, b, c, d are real numbers.

   (a) $x^2 - 4x - 96$ (b) $x^2 + 9x - 90$

   (c) $6x^2 - 5x - 6$ (d) $12x^2 - x - 20$

2. For each of the following, factorize the given quadratic function in the form $k(x - \alpha)(x - \beta)$ where α, β and k are real numbers.

   (a) $x^2 + 3x - 5$ (b) $2x^2 + 3x - 4$

3. Show that the quadratic function $2x^2 + 3x + 7$ cannot be factorized in the form $k(x - \alpha)(x - \beta)$ where α, β and k are real numbers.

4. For each of the following, determine whether the given quadratic function can be factorized in the form $(ax + b)(cx + d)$, where a, b, c, d are integers.

   (a) $60x^2 + 32x - 15$ (b) $60x^2 + 8x - 33$

7.6.3 Factorization of Higher Degree Polynomial Functions

For factorization problems, usually we are given a polynomial function f with integer coefficients and we want to rewrite f as a product of polynomial functions with integer coefficients whose degrees are less than that of f (if possible, as a product of linear functions with integer coefficients).

By the Factor Theorem, a linear function $(ax + b)$, where a and b are integers with $a \neq 0$, is a factor of f if and only if the rational number $\frac{b}{a}$ is a zero of f. Although there are infinitely many rational numbers, most of them are not zeros of f. In fact, if $\frac{b}{a}$ is a zero of f and a, b are *relatively prime*, then a and b must satisfy the conditions given in the *Rational Zero Theorem* (see Page 44).

Definition 7.6.2 Let m and n be integers. We say that ***n is a factor of m*** to mean that there exists an integer, denoted by k, such that $m = k \cdot n$.

Explanation For the case where $n \neq 0$, the integer n is a factor of m means that $\frac{m}{n}$ is an integer. For the case where $n > 0$, the integer n is a factor of m means that when m is divided by n, the remainder is 0.

Remark 1 and -1 are factors of every integer.

Example 7.6.5

- 2 is a factor of 6.

  Reason $\frac{6}{2} = 3$ is an integer.

- -3 is a factor of 6.

  Reason $\frac{6}{-3} = -2$ is an integer.

- 4 is not a factor of 6.

  Reason $\frac{6}{4}$ is not an integer.

In fact, the factors of 6 are 1, -1, 2, -2, 3, -3, 6 and -6.

Definition 7.6.3 Let a and b be integers. We say that ***a and b are relatively prime*** to mean that the largest common factor of a and b is 1.

Explanation The largest (or highest) common factor of a and b means the largest integer that is a factor of both a and b.

Example 7.6.6 The integers 4 and 15 are relatively prime.

Reason The factors of 4 are 1, -1, 2, -2, 4 and -4.

The factors of 15 are 1, -1, 3, -3, 5, -5, 15 and -15.

The common factors of 4 and 15 are 1 and -1.

The largest common factor of 4 and 15 is 1.

Example 7.6.7 The integers 4 and −6 are not relatively prime.

Reason The factors of 4 are 1, −1, 2, −2, 4 and −4.

The factors of −6 are 1, −1, 2, −2, 3, −3, 6 and −6.

The common factors of 4 and −6 are 1, −1, 2 and −2.

The largest common factor of 4 and −6 is 2.

Note that every rational number can be written in the form $\frac{b}{a}$, where a and b are relatively prime integers. For example $\frac{24}{15}$ can be written as $\frac{8}{5}$. The *Rational Zero Theorem* tells that if a rational number $\frac{b}{a}$, where a and b are relatively prime integers, is a zero of a polynomial function with integer coefficients, then

- a is a factor of the leading coefficient;
- b is a factor of the constant term.

Readers may have already used a version of the Rational Zero Theorem to factorize quadratic expressions (functions) with integer coefficients. For example, to factorize $5x^2 + 7x - 6$, we try to find integers a, b, c and d such that

$$5x^2 + 7x - 6 = (ax + b)(cx + d)$$

Suppose that there exist such integers a, b, c and d. Then

- by comparing the coefficient of the x^2-term, we get $a \cdot c = 5$, which implies that a is a factor of 5, that is, a is a factor of the leading coefficient of $(5x^2 + 7x - 6)$;
- by comparing the constant term, we get $b \cdot d = -6$, which implies that b is a factor of −6, that is, b is a factor of the constant term of $(5x^2 + 7x - 6)$.

Details of the inspection method (try and error)

We may assume that both a and c are positive. By the condition on a and c and using symmetry, we try $a = 1$ and $c = 5$, that is, try

$$5x^2 + 7x - 6 = (x + b)(5x + d)$$

By the condition on b, we try $b = 1, -1, 2, -2, 3, -3, 6, -6$ together with the corresponding values of d ($d = -6, 6, -3, 3, -2, 2, -1, 1$ respectively).

Rational Zero Theorem Let m be a positive integer and let $c_0, c_1, \ldots, c_m$ be integers with $c_m \neq 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial function given by

$$f(x) = c_m x^m + \cdots + c_1 x + c_0$$

Let a and b be relatively prime integers with $a \neq 0$. Suppose that $\frac{b}{a}$ is a zero of f. Then

(1) a is a factor of c_m;

(2) b is a factor of c_0.

Proof The number $\frac{b}{a}$ is a zero of f means that $f(\frac{b}{a}) = 0$, that is,

$$c_m \cdot \left(\frac{b}{a}\right)^m + \cdots + c_1 \cdot \left(\frac{b}{a}\right) + c_0 = 0$$

Multiplying both sides of the above equality by a^m, we get

$$c_m b^m + c_{m-1} b^{m-1} a + \cdots + c_1 b a^{m-1} + c_0 a^m = 0 \tag{7.6.4}$$

(1) From (7.6.4), we get

$$c_m b^m = -c_{m-1} b^{m-1} a - \cdots - c_1 b a^{m-1} - c_0 a^m$$

Since a is a factor of the right-side of the above equality, it follows that

$$a \text{ is a factor of } c_m b^m$$

Hence a is a factor of c_m. This is because a and b are relatively prime which implies that the common factors of a and b^m are 1 and -1.

(2) From (7.6.4), we get

$$c_m b^m + c_{m-1} b^{m-1} a + \cdots + c_1 b a^{m-1} = -c_0 a^m$$

Since b is a factor of the left-side of the above equality, it follows that

$$b \text{ is a factor of } c_0 a^m$$

Hence b is a factor of c_0. This is because a and b are relatively prime which implies that the common factors of b and a^m are 1 and -1. □

In applying the Rational Zero Theorem, we may simply try the cases where $a > 0$. This is because every rational number can be written in a form where the denominator is positive. In the following two examples, the given polynomial functions are quadratic functions. Instead of solving $f(x) = 0$ by factorization (using inspection) or by the quadratic formula, we apply the Rational Zero Theorem to find the zeros of f.

Example 7.6.8 Let $f(x) = x^2 - 2x - 15$.

To look for rational numbers that are zeros of f, we try $\frac{b}{a}$, where

$$a > 0, \quad a \text{ is a factor of } 1 \quad \text{and} \quad b \text{ is a factor of } -15$$

Thus we try $a = 1$ and $b = 1, -1, 3, -3, 5, -5, 15, -15$.

Direct substitution gives

$$f(1) = -14 \qquad f(-1) = -12 \qquad f(3) = -12 \qquad f(-3) = 0$$

$$f(5) = 0 \qquad f(-5) = \qquad f(15) = \qquad f(-15) =$$

Thus the zeros of f are -3 and 5.

Remark Once we find two zeros of f, there is no need to check the other number(s). This is because f has at most two zeros by the Number of Zero Theorem.

Example 7.6.9 Let $f(x) = 2x^2 - 7x + 3$.

To look for rational numbers that are zeros of f, we try $\frac{b}{a}$, where

$$a > 0, \quad a \text{ is a factor of } 2 \quad \text{and} \quad b \text{ is a factor of } 3$$

Thus we try $a = 1, 2$ and $b = 1, -1, 3, -3$. The possible combinations of $\frac{b}{a}$ are:

$$1, \quad -1, \quad 3, \quad -3, \quad \frac{1}{2}, \quad \frac{-1}{2}, \quad \frac{3}{2}, \quad \frac{-3}{2}$$

Direct substitution gives

$$f(3) = 0 \quad \text{and} \quad f(\tfrac{1}{2}) = 0$$

Thus the zeros of f are 3 and $\frac{1}{2}$.

Remark We omit writing the values of $f(\frac{b}{a})$ that are not equal to 0.

Example 7.6.10 Let $f(x) = x^3 - 7x - 6$. Factorize f as a product of three linear functions with integer coefficients.

Explanation To find linear factor(s) with integer coefficients, we apply the Factor Theorem and the Rational Zero Theorem.

- In Solution 1, we find *all* zeros of f that are rational numbers. Because there are three such zeros and the degree of f is 3, we can apply Theorem 7.6.1.
- In Solution 2, we find *one* zero of f that is a rational number. Then we proceed as in Example 7.6.1.

Solution 1 First we find the zeros of f that are rational numbers. By the Rational Zero Theorem, we try $\frac{b}{a}$, where $a = 1$ and $b = 1, -1, 2, -2, 3, -3, 6, -6$. Direct substitution gives

$$f(-1) = 0, \qquad f(-2) = 0 \qquad \text{and} \qquad f(3) = 0$$

Thus the function f has three zeros, namely -1, -2 and 3. Note that the degree of f is 3. Moreover, the leading coefficient of the standard representation of f is 1. Hence

$$\begin{aligned} f(x) &= 1 \cdot (x - (-1))(x - (-2))(x - 3) && \text{By Theorem 7.6.1} \\ &= (x + 1)(x + 2)(x - 3) \end{aligned}$$

□

Solution 2 First we find a zero of f that is a rational number. By the Rational Zero Theorem, we try $\frac{b}{a}$, where $a = 1$ and $b = 1, -1, 2, -2, 3, -3, 6, -6$.
Direct substitution gives $f(-1) = 0$.
By the Factor Theorem, $(x + 1)$ is a factor of $f(x)$.
By Long Division,

$$\begin{array}{r|rrrr} & & x^2 & -\ x & -\ 6 \\ \hline x+1 & x^3 & +\ 0x^2 & -\ 7x & -\ 6 \\ & x^3 & +\ x^2 & & \\ \hline & & -x^2 & -\ 7x & \\ & & -x^2 & -\ x & \\ \hline & & & -6x & -\ 6 \\ & & & -6x & -\ 6 \\ \hline & & & & 0 \end{array}$$

Thus
$$\begin{aligned} f(x) &= (x + 1)(x^2 - x - 6) \\ &= (x + 1)(x - 3)(x + 2) \end{aligned}$$

□

It should be pointed out that the method in Solution 1 to Example 7.6.10 works only for the case where the number of zeros is equal to the degree of the given polynomial function.

- For the polynomial function f given in Example 7.6.1, there are only two zeros of f, namely, 1 and 2. From this, we know that $(x-1)$ and $(x-2)$ are factors of $f(x)$. There are two possibilities (note that the leading coefficient is 1)

  $$\text{either} \quad f(x) = (x-1)^2(x-2) \quad \text{or} \quad f(x) = (x-1)(x-2)^2$$

  To determine which case it is, we may compare the constant term (for example).

In the next two examples, we will use the method in Solution 2 to Example 7.6.10 to factorize the given polynomial functions. Below we describe the steps to find a rational zero of a non-constant polynomial function f with integer coefficients (a *rational zero* of f means a rational number that is a zero of f).

(*Step 1*) List the positive factors of the leading coefficient.

(*Step 2*) List the factors (positive or negative) of the constant term.

(*Step 3*) List all possible combinations of $\frac{b}{a}$, where a and b are numbers obtained in Step 1 and Step 2 respectively.

(*Step 4*) Find the value of $f(\frac{b}{a})$ until 0 is obtained, where $\frac{b}{a}$ is a number obtained in Step 3.

Remark If the leading coefficient is 1, Step 1 and Step 3 are redundant and Step 4 is to find the value of $f(b)$ until 0 is obtained, where b is a number obtained in Step 2.

Example 7.6.11 Factorize $2x^3 + 3x^2 - 11x - 6$.

Solution Let $f(x) = 2x^3 + 3x^2 - 11x - 6$.

First, we find a rational number that is a zero of f.

Note that positive factors of the leading coefficient 2 are 1 and 2;

factors of the constant term 6 are 1, −1, 2, −2, 3, −3, 6 and −6

Rational numbers that can be zeros of f are

$$1,\ -1,\ 2,\ -2,\ 3,\ -3,\ 6,\ -6$$

$$\frac{1}{2},\ -\frac{1}{2},\ \frac{3}{2},\ -\frac{3}{2}$$

No need to repeat $\frac{2}{2} = 1$ etc.

Direct substitution gives $f(2) = 0$.

By the Factor Theorem, $(x-2)$ is a factor of f.

$$\begin{array}{rrrrrr} & & 2 & 7 & 3 & \\ \hline 1 & -2 \,) & 2 & 3 & -11 & -6 \\ & & 2 & -4 & & \\ \hline & & & 7 & -11 & \\ & & & 7 & -14 & \\ \hline & & & & 3 & -6 \\ & & & & 3 & -6 \\ \hline \end{array}$$

By Long Division, when $(2x^3+3x^2-11x-6)$ is divided by $(x-2)$, the quotient is $(2x^2 + 7x + 3)$.

Thus
$$\begin{aligned} 2x^3 + 3x^2 - 11x - 6 &= (x-2)(2x^2 + 7x + 3) \\ &= (x-2)(x+3)(2x+1) \end{aligned}$$
□

Example 7.6.12 Factorize $x^4 - 9x^2 - 4x + 12$.

Explanation Note that the leading coefficient is 1. Thus rational numbers that are zeros of f must be integers.

Solution Let $f(x) = x^4 - 9x^2 - 4x + 12$.

We want to find a rational number that is a zero of f. For this, we try

$$1, \quad -1, \quad 2, \quad -2, \quad 3, \quad -3, \quad 4, \quad -4, \quad 6, \quad -6, \quad 12, \quad -12$$

Direct substitution gives $f(1) = 0$.

By the Factor Theorem, $(x - 1)$ is a factor of f.

By Long Division, when $(x^4 - 9x^2 - 4x + 12)$ is divided by $(x - 1)$, the remainder is $(x^3 + x^2 - 8x - 12)$.

Let $g(x) = x^3 + x^2 - 8x - 12$.

We want to find a rational number that is a zero of g. For this, we try

$$1, \quad -1, \quad 2, \quad -2, \quad 3, \quad -3, \quad 4, \quad -4, \quad 6, \quad -6, \quad 12, \quad -12$$

Direct substitution gives $g(-2) = 0$.

By the Factor Theorem, $(x + 2)$ is a factor of g.

By Long Division, when $(x^3 + x^2 - 8x - 12)$ is divided by $(x + 2)$, the quotient is $(x^2 - x - 6)$.

Hence
$$\begin{aligned} x^4 - 9x^2 - 4x + 12 &= (x-1)(x^3 + x^2 - 8x - 12) \\ &= (x-1)(x+2)(x^2 - x - 6) \\ &= (x-1)(x+2)(x+2)(x-3) \end{aligned}$$

□

$$\begin{array}{rr|rrrrr} & & & 1 & 1 & -8 & -12 \\ \hline 1 & -1 & 1 & 0 & -9 & -4 & 12 \\ & & 1 & -1 & & & \\ \hline & & & 1 & -9 & & \\ & & & 1 & -1 & & \\ \hline & & & & -8 & -4 & \\ & & & & -8 & 8 & \\ \hline & & & & & -12 & 12 \\ & & & & & -12 & 12 \\ \hline \end{array} \qquad \begin{array}{rr|rrrr} & & & 1 & -1 & -6 \\ \hline 1 & 2 & 1 & 1 & -8 & -12 \\ & & 1 & 2 & & \\ \hline & & & -1 & -8 & \\ & & & -1 & -2 & \\ \hline & & & & -6 & -12 \\ & & & & -6 & -12 \\ \hline \end{array}$$

Remark After getting a zero of f (the number 1), we may continue to find another zero of f. Direct substitution gives $f(-2) = 0$. Thus by the General Factor Theorem, the quadratic function $(x - 1)(x + 2) = x^2 + x - 2$ is a factor of f. Using long division for $f(x) \div (x^2 + x - 2)$, we get

$$f(x) = (x^2 + x - 2) \cdot (x^2 - x - 6)$$

from which we can further factorize $f(x)$.

Exercise 7.6.3

1. For each of the following, find the rational numbers that are zeros of the given polynomial function.
   - (*a*) $x^3 - 2x^2 - 5x + 6$
   - (*b*) $x^3 + x^2 - 8x - 12$
   - (*c*) $x^3 + 3x^2 + 4x + 4$
   - (*d*) $x^3 - 2x^2 - 7x - 2$
   - (*e*) $2x^3 - x^2 - x - 3$
   - (*f*) $8x^3 + 12x^2 + 6x + 1$
2. Let $f(x) = x^3 + 2x^2 - 5x - 6$.
   - (*a*) Show that $f(-1) = 0$.
   - (*b*) Factorize f.
3. Let $f(x) = x^3 + x^2 + ax + 3$, where a is a real number. Suppose that $(x + 3)$ is a factor of f.
   - (*a*) Find the value of a.
   - (*b*) Factorize f.
4. Let $f(x) = x^3 + ax^2 + x + b$, where a and b are real numbers. Suppose that both $(x - 2)$ and $(x + 1)$ are factors of f.
   - (*a*) Find the values of a and b.
   - (*b*) Factorize f.
5. For each of the following, factorize the given polynomial function.
   - (*a*) $x^3 + 2x^2 - x - 2$
   - (*b*) $x^3 - x^2 - x + 1$
   - (*c*) $x^3 - 3x + 2$
   - (*d*) $x^3 - x^2 + x - 1$
   - (*e*) $x^3 + 4x^2 - 7x - 10$
   - (*f*) $2x^3 + x^2 - 2x - 1$
   - (*g*) $3x^3 + 5x^2 + x - 1$
   - (*h*) $5x^3 - 8x^2 + x + 2$
   - (*i*) $2x^3 + x^2 - 12x + 9$
   - (*j*) $15 + 32x + 3x^2 - 2x^3$
   - (*k*) $x^4 + 3x^3 - 3x^2 - 11x - 6$
   - (*l*) $2x^4 - 7x^3 - 2x^2 + 13x + 6$
6. Let $f(x) = x^3 - 3x^2 + 4$.
   - (*a*) Factorize f.
   - (*b*) Solve the equation $f(x) = 0$.
7. Let f be a polynomial function. Suppose that $f(3) = 24$ and that when f is divided by $(x - 1)$, the quotient is $(x^2 + ax - 2)$ and the remainder is -8.
   - (*a*) Find the value of a.
   - (*b*) Find the remainder when f is divided by $(x + 1)$.
   - (*c*) Factorize f.
   - (*d*) Solve the equation $f(x) = 0$.
8. Find a polynomial function f having the properties that both $(x + 1)$ and $(x - 1)$ are factors of f, that when f is divided by x, the remainder is 2 and that when f is divided by $(x - 2)$, the remainder is 3.
9. Let $f(x) = x^m + 1$, where m is a positive integer. For what values of m does f has a linear factor?

7.6.4 More on Zeros of Polynomial Functions

Given a polynomial function f with integer coefficients, it may happen that f does not have any rational zero; in this case, the function f does not have any linear factor in the form $ax + b$, where a and b are integers. However, it may have linear factor(s) with irrational coefficients. The following is an example of a cubic function that does not have any rational zero but can be "factorized" as a product of three linear functions.

Example 7.6.13 Let $f(x) = x^3 - 3x - 1$.

- Since $f(1) \neq 0$ and $f(-1) \neq 0$, it follows from the Rational Zero Theorem that there does not exist any rational number r such that $f(r) = 0$. Hence f does not have any factor in the form $ax + b$, where a and b are integers with $a \neq 0$.
- It can be shown that f has three zeros (the graph of f is shown in Figure 7.6.1. There is
  - a zero, denoted by α, between -2 and -1;
  - a zero, denoted by β, between -1 and 0;
  - a zero, denoted by γ, between 1 and 2.

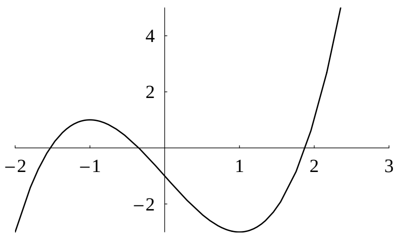


Figure 7.6.1

By Theorem 7.6.1 (noting that the leading coefficient is 1), the function f can factorized as a product of three linear functions:

$$f(x) = (x - \alpha)(x - \beta)(x - \gamma)$$

The following is an example of a polynomial function that does not have any linear factor but can be factorized as a product of quadratic functions with integer coefficients.

Example 7.6.14 Let $f(x) = x^4 + 2x^2 + 1$.

- Since $f(1) \neq 0$ and $f(-1) \neq 0$, it follows from the Rational Zero Theorem that there does not exist any rational number r such that $f(r) = 0$. Hence f does not have any factor in the form $ax + b$, where a and b are integers with $a \neq 0$.
- It is clear that for every real number t, we have $f(t) \geq 1$. Hence the function f does not have any zero and so f does not have any linear factor.
- However, we can factorize f as a product of quadratic functions having integer coefficients:

$$\begin{aligned} f(x) &= x^4 + 2x^2 + 1 \\ &= (x^2)^2 + 2 \cdot x^2 \cdot 1 + 1^2 \\ &= (x^2 + 1)^2 && \text{Perfect Square Identity} \end{aligned}$$

Remark If complex numbers are allowed, we can factorize f as a product of linear functions:

$$f(x) = (x + \mathrm{i})^2(x - \mathrm{i})^2$$

Some Results on Polynomial Equations An equation in the form $a_n x^n + \cdots + a_1 x + a_0 = 0$, where $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$, is called a *polynomial equation of degree n*.

- Polynomial equations of degree 2 are quadratic equations. Solutions to a quadratic equation can be expressed in terms of the coefficients using arithmetic operations together with taking square roots (quadratic formula).
- Polynomial equations of degree 3 are called *cubic* equations. Solutions to a cubic equation can be expressed in terms of the coefficients using arithmetic operations together with taking square roots and cube roots.
- Solutions to a polynomial equation of degree 4 can be expressed in terms of the coefficients using arithmetic operations together with taking roots.
- However, there is no formula (involving only arithmetic operations and taking roots) for solutions to polynomial equations of degree $n \geq 5$.

There are polynomial equations without any (real) solution. However, if we allow complex numbers as solutions, then solutions always exist. The *Fundamental Theorem of Algebra* tells that every polynomial equation (with real or complex coefficients) has a complex solution, that is, there exists at least one complex number satisfying the polynomial equation.

The following result, called the *Integer Zero Theorem*, is a simple consequence of the Rational Zero Theorem. Readers may have already discovered the result themselves.

Integer Zero Theorem Let f be a non-constant polynomial function with integer coefficients and leading coefficient 1. Suppose that r is a rational number and that r is a zero of f. Then r is an integer.

Proof We may write r as $\frac{b}{a}$, where a and b are relatively prime integers and $a \neq 0$. It follows from the Rational Zero Theorem that $a = 1$ or $a = -1$. In any case, the number $r = \frac{b}{a}$ is an integer. □

Remark From the Rational Zero Theorem, we have the following:

- The integer r divides the constant term of f.

To close this chapter, we apply the Integer Zero Theorem to prove that the number $\sqrt{2}$ is not a rational number, that is, it cannot be written as $\frac{m}{n}$, where m and n are integers. First, we prove the following:

Lemma 7.6.2 Let $f(x) = x^2 - 2$. Then there does not exist any rational number r such that $f(r) = 0$.

Proof Note that for every integer k, we have $f(k) \neq 0$. The result follows from the Integer Zero Theorem. □

Corollary 7.6.3 The real number $\sqrt{2}$ is an irrational number.

Proof By definition, the number $\sqrt{2}$ is a solution to the equation $x^2 - 2 = 0$. However, Lemma 7.6.2 tells that solutions to the equation $x^2 - 2 = 0$ are not rational numbers. Hence the number $\sqrt{2}$ is an irrational number. □

Remark The above result was discovered by ancient Greeks (the Pythagorean school). They proved the result using a method called *Proof by Contradiction*.