Contents

Chapter 6

Quadratic Functions

Problem Mr. Chan wants to enclose a rectangular garden in front of a wall, using a fence 80 m long. How should he fence the garden so that the area of the rectangular garden is the largest?

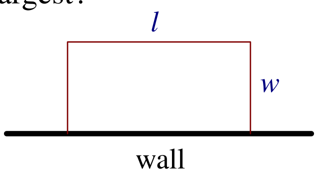


Figure 6.0.1

Explanation The question asks for the dimensions that will maximize the area of the garden. There are many possibilities for the lengths and widths. Table 6.0.2 shows a few possible combinations of l and w (see Figure 6.0.1) together with the corresponding areas of the garden.

l	w	length of fence	area of garden
20 m	30 m	80 m	600 m^2
30 m	25 m	80 m	750 m^2
40 m	20 m	80 m	800 m^2
50 m	15 m	80 m	750 m^2

Table 6.0.2

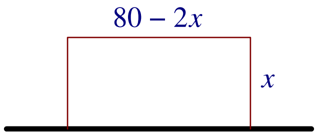
There are infinitely many possible combinations of l and w; it is impossible to list all of them. Note that

(1) the area A depends on both l and w, in fact, $A = l \times w$;

(2) l and w are related, in fact, $l + 2w = 80$ (length of fence available).

We can choose one of the variables l and w to be the independent variable and using (2) we can express the other variable in terms of the independent variable and hence we can express A in terms of the independent variable.

To tackle Let the length of a side perpendicular to the wall be $x\,m$. Since the sum of the three sides of fence is $80\,m$, it follows that the length of the side parallel to the wall is $(80 - 2x)\,m$.



The area A, in m^2, of the garden is given by

$$A = (80 - 2x) \cdot x \tag{6.0.1}$$

Since lengths are positive numbers, it follows that $x > 0$ and $80 - 2x > 0$ which imply that $0 < x < 40$. We want to find x such that A is the largest.

Remark Solution to the problem is given in Example 6.4.2.

Equality (6.0.1) can be written as $A = -2x^2 + 80x$. Such an equality describes a function which is called a quadratic function. In general, a *quadratic function* is a function given by $x \mapsto ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$.

The main purpose of this chapter is to study quadratic functions.

- As we have seen in Chapter **??**, functions and graphs are closely related to each other. In Section 6.1, we will study graphs of quadratic functions.

  As an application of graphs of quadratic functions, in Section 6.2, we will discuss the graphical method for solving quadratic equations in one unknown: $ax^2 + bx + c = 0$.
- In Section 6.3, we will introduce the concepts of maxima and minima of functions. For quadratic functions, information on where the functions attain their maxima or minima can be obtained easily from their graphs.
- In Section 6.4, we will consider some practical problems that lead to finding maxima or minima of quadratic functions.

To discuss graphs of quadratic functions, we will use the concepts of *symmetry* and *translation*. Readers have learnt these concepts before from an intuitive point of view. In Section 6.0, we will give a review using a rigorous approach.

If you find the definitions given in Section 6.0 too tedious, don't worry, you can use your intuition to help you understand the examples.

6.0 Symmetry and Translation

The concept of *symmetry* about a line can be described using a concept called *reflection* about a line in a plane (reflection about a line in space will not be discussed in this chapter).

Reflection about a line Given a line ℓ in a plane, by considering the line ℓ as a mirror, points in the plane that are "in front of" the line have images "behind" the line. More precisely,

- if A is a point in the plane not lying on the line ℓ, then its "mirror image" is the point B on the other side of ℓ such that A and B are equidistant to ℓ and that the line determined by A and B is perpendicular to ℓ;
- if A lies on ℓ, then its "mirror image" is itself.

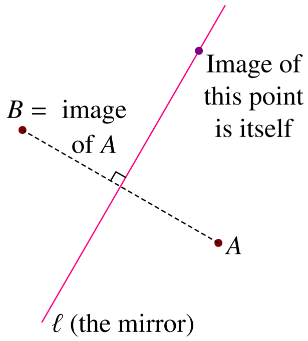


Figure 6.0.1

Explain A and B are equidistant to ℓ means that the distance from A to ℓ is equal to the distance from B to ℓ.

Remark If B is the image of A, then A is the image of B.

Definition 6.0.1 Let $\mathcal{P}$ be a plane and let ℓ be a line on the plane $\mathcal{P}$. We call ***the reflection about ℓ in $\mathcal{P}$*** to mean the function from $\mathcal{P}$ to $\mathcal{P}$ given by the following:

- if A is a point belonging to ℓ, then the image of A is A;
- if A is a point not belonging to ℓ, then the image of A is the point B such that A and B are equidistant to ℓ and Line AB and ℓ are perpendicular.

Remark If the plane is understood, we simply say "*the reflection about ℓ*".

In the following three examples, the plane under consideration is the rectangular coordinate plane.

Example 6.0.1 For the reflection about the x-axis (in the rectangular coordinate plane),

- the image of the point $(2, 1)$ is the point $(2, -1)$;
- the image of the point $(0.2, 2)$ is the point $(0.2, -2)$;
- the image of the point $(-2, 0)$ is the point itself.

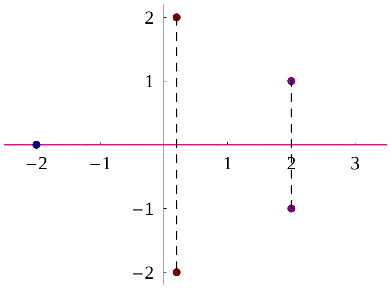


Figure 6.0.2

More generally, the image of the point (a, b) is the point $(a, -b)$.

Example 6.0.2 For the reflection about the y-axis,

- the image of the point $(1, 1)$ is the point $(-1, 1)$;
- the image of the point $(-2, 4)$ is the point $(2, 4)$;
- the image of the point $(0, 0)$ is the point itself.

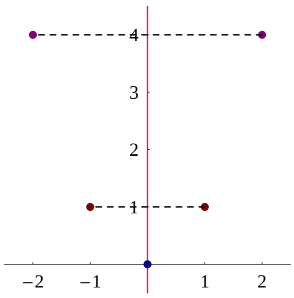


Figure 6.0.3

More generally, the image of the point (a, b) is the point $(-a, b)$.

Example 6.0.3 For the reflection about the vertical line given by $x = 3$,

- the image of the point $(2, 2)$ is the point $(4, 2)$;
- the image of the point $(0, 0.5)$ is the point $(6, 0.5)$;
- the image of the point $(3, 1)$ is the point itself.

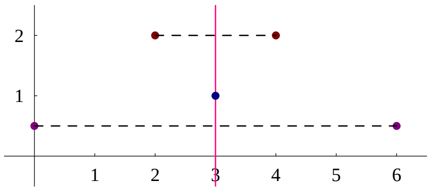


Figure 6.0.4

More generally, the image of the point (a_1, b) is the point (a_2, b), where $a_1 - 3 = 3 - a_2$, that is, $a_1 + a_2 = 6$.

Symmetry about a line For subsets of a plane, there are two concepts of *symmetry about a line*: one is for two subsets and the other is for one subset.

- Two subsets of a plane is said to be *symmetric* about a line in the plane if they are "mirror images" (respect to the line) of each other, that is, for every point belonging to one of the two sets, its image under the reflection about the line is a point belonging to the other set.

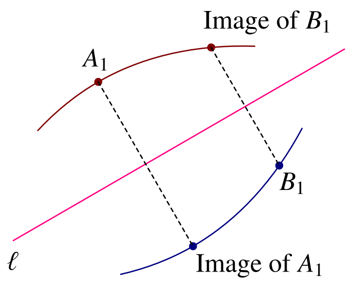


Figure 6.0.5

Definition 6.0.2 Let $\mathcal{P}$ be a plane. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subsets of $\mathcal{P}$ and let ℓ be a line that is a subset of $\mathcal{P}$. We say that ***$\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric about ℓ in $\mathcal{P}$*** to mean that the following condition is satisfied:

- for every point belonging to $\mathcal{S}_1$, its image under the reflection about ℓ in $\mathcal{P}$ belongs to $\mathcal{S}_2$, and vice versa.

Remark If the plane is understood (usually it is determined by $\mathcal{S}_1$ and $\mathcal{S}_2$), we simply say "*$\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric about ℓ*".

In the following three examples, the plane under consideration is the rectangular coordinate plane.

Example 6.0.4 Denote $\mathcal{S}_1$ to be the line segment with endpoints $(1, 1)$ and $(3, 1)$. Denote $\mathcal{S}_2$ to be the line segment with endpoints $(1, -1)$ and $(3, -1)$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric about the x-axis.

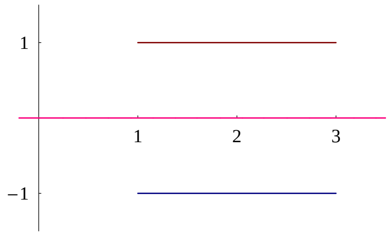


Figure 6.0.6

Example 6.0.5 Denote $\mathcal{S}_1$ to be the ray starting at the origin in the direction of $(1, 1)$. Denote $\mathcal{S}_2$ to be the ray starting at the origin in the direction of $(-1, 1)$. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric about the y-axis.

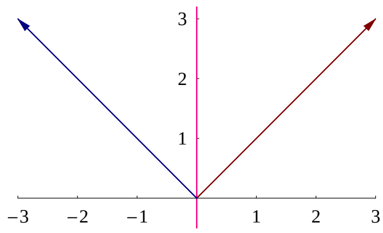


Figure 6.0.7

Example 6.0.6 Denote $\mathcal{S}_1$ and $\mathcal{S}_2$ to be the upper half and lower half, respectively, of the circle with center at the point $(2, 1)$ and radius equal to 1. Then $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric about the horizontal line given by $y = 1$.

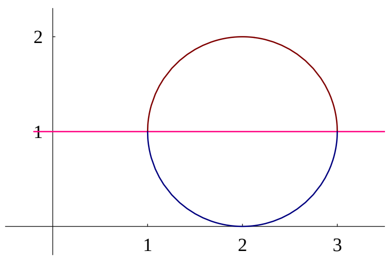


Figure 6.0.8

- A subset of a plane is said to be *symmetric* about a line in the plane if it is the "mirror image" (respect to the line) of itself, that is, for every point belonging to the set, its image under the reflection about the line is also a point belonging to the set.

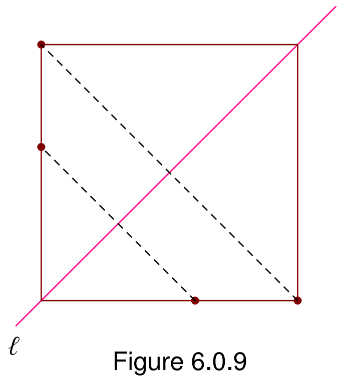


Figure 6.0.9

Definition 6.0.3 Let $\mathcal{P}$ be a plane. Let $\mathcal{S}$ be a subset of $\mathcal{P}$ and let ℓ be a line that is a subset of $\mathcal{P}$. We say that ***$\mathcal{S}$ is symmetric about ℓ in $\mathcal{P}$*** to mean that the following condition is satisfied:

- For every point belonging to $\mathcal{S}$, its image under the reflection about ℓ in $\mathcal{P}$ belongs to $\mathcal{S}$.

Remark If the plane is understood (usually it is determined by $\mathcal{S}$), we simply say "$\mathcal{S}$ *is symmetric about ℓ*".

Remark The line ℓ is called an *axis of symmetry* of $\mathcal{S}$ (see Terminology 6.0.4).

Note that a set is symmetric about a line means that the two parts of the set on the two sides of the line are symmetric about the line.

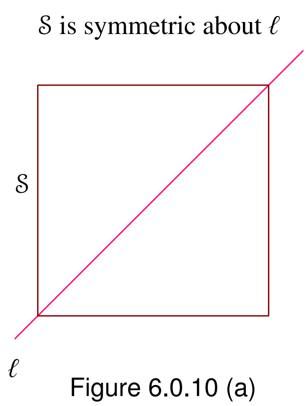


Figure 6.0.10 (a)

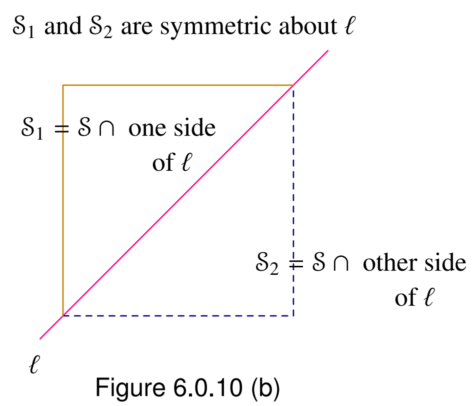


Figure 6.0.10 (b)

Terminology 6.0.4 Let $\mathcal{P}$ be a plane and let $\mathcal{S}$ be a subset of $\mathcal{P}$. We call ***an axis of symmetry of $\mathcal{S}$ in $\mathcal{P}$*** to mean a line, denoted by ℓ, such that ℓ is a subset of $\mathcal{P}$ and $\mathcal{S}$ is symmetric about ℓ in $\mathcal{P}$.

Remark If the plane is understood (usually it is determined by $\mathcal{S}$), we simply call "*an axis of symmetry of* $\mathcal{S}$".

Remark The plural of '*axis*' is '*axes*'.

In the following three examples, the plane under consideration is the rectangular coordinate plane.

- In the first example, the given set has one and only one axis of symmetry.
- In the second example, the given set has two axes of symmetry.
- In the third example, the given set has infinitely many axes of symmetry.

Example 6.0.7 Denote $\mathcal{V}$ to be the rotated V-shaped figure formed by the line segment with endpoints $(1,1)$ and $(3,0)$ and the line segment with endpoints $(1,-1)$ and $(3,0)$.

- The set $\mathcal{V}$ is symmetric about the x-axis. In other words, the x-axis is an axis of symmetry of $\mathcal{V}$ (see Figure 6.0.11).
- In fact, the x-axis is the only axis of symmetry of $\mathcal{V}$. That is, $\mathcal{V}$ is not symmetric about any line different from the x-axis.

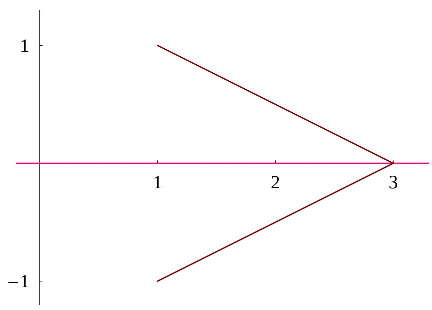


Figure 6.0.11

Example 6.0.8 Denote $\mathcal{R}$ to be the rectangle with vertices at the points $(2, 0)$, $(2, 1)$, $(0, 1)$ and the origin.

- The set $\mathcal{R}$ is symmetric about the vertical line given by $x = 1$.
  In other words, the vertical line given by $x = 1$ is an axis of symmetry of $\mathcal{R}$ [see Figure 6.0.12 (a)].
- Besides the vertical line given by $x = 1$, the horizontal line given by $y = \frac{1}{2}$ is also an axis of symmetry of $\mathcal{R}$ [see Figure 6.0.12 (b)].

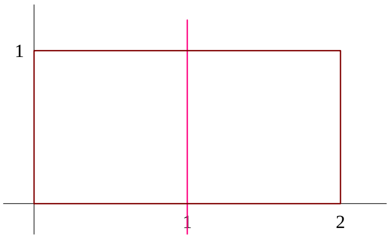


Figure 6.0.12 (a)

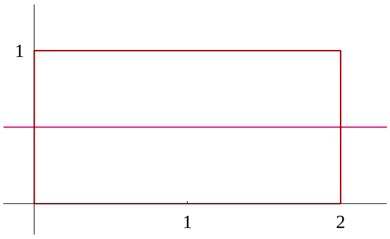


Figure 6.0.12 (b)

Example 6.0.9 Denote $\mathcal{C}$ to be circle with center at the point $(2, 1)$ and radius equal to 1.

- The set $\mathcal{C}$ is symmetric about the line given by $y = 1$.
  In other words, the line given by $y = 1$ is an axis of symmetry of $\mathcal{R}$ [see Figure 6.0.13 (a)].
- In fact, every line passing through the center of the circle $\mathcal{C}$ is an axis of symmetry of $\mathcal{C}$.

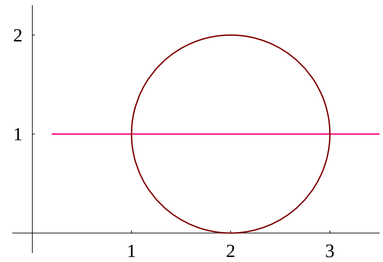


Figure 6.0.13 (a)

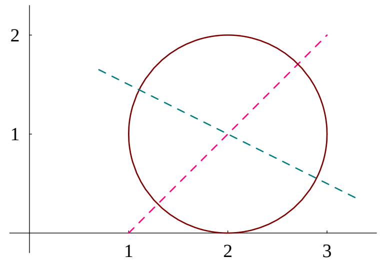


Figure 6.0.13 (b)

Translations and Translates A *translation* is a function from a plane (or space) to itself such that every point in the plane (or space) is moved by a fixed distance in a fixed direction. Instead of describing translations in general, we will discuss the concept of the *translate* of a subset of the the rectangular coordinate plane in four directions: up, down, left and right.

Given a subset $\mathcal{S}$ of the rectangular coordinate plane, the set obtained by moving the points in $\mathcal{S}$ to the right by p units (where p is a positive real number) is called the *translate* of $\mathcal{S}$ by p-unit right.

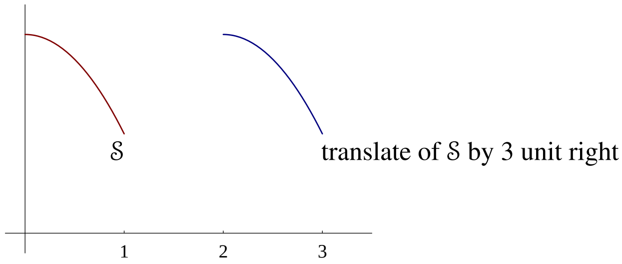


Figure 6.0.14

Definition 6.0.5 Let $\mathcal{S}$ be a subset of the rectangular coordinate plane. Let p be a positive real number.

- We call ***the translate of $\mathcal{S}$ by p-unit up*** to mean the following set:

  $$\{(x, y) \in \mathbb{R}^2 : \text{there exists } (a, b) \in \mathcal{S} \text{ such that } x = a \text{ and } y = b + p\}$$

- We call ***the translate of $\mathcal{S}$ by p-unit down*** to mean the following set:

  $$\{(x, y) \in \mathbb{R}^2 : \text{there exists } (a, b) \in \mathcal{S} \text{ such that } x = a \text{ and } y = b - p\}$$

- We call ***the translate of $\mathcal{S}$ by p-unit right*** to mean the following set:

  $$\{(x, y) \in \mathbb{R}^2 : \text{there exists } (a, b) \in \mathcal{S} \text{ such that } x = a + p \text{ and } y = b\}$$

- We call ***the translate of $\mathcal{S}$ by p-unit left*** to mean the following set:

  $$\{(x, y) \in \mathbb{R}^2 : \text{there exists } (a, b) \in \mathcal{S} \text{ such that } x = a - p \text{ and } y = b\}$$

Explanation The set $\{(x, y) \in \mathbb{R}^2 : \text{there exists } (a, b) \in \mathcal{S} \text{ such that } x = a \text{ and } y = b + p\}$ is the subset of the rectangular coordinate plane consisting of all the points that are obtained by adding p to the second coordinate of the points in $\mathcal{S}$.

Example 6.0.10 Denote $\mathcal{S}$ to be the line segment with endpoints $(0,0)$ and $(1,1)$.

- The translate of $\mathcal{S}$ by 2-unit up is the line segment with endpoints $(0,2)$ and $(1,3)$.
- The translate of $\mathcal{S}$ by 1-unit down is the line segment with endpoints $(0,-1)$ and $(1,0)$.
- The translate of $\mathcal{S}$ by 1-unit right is the line segment with endpoints $(1,0)$ and $(2,1)$.
- The translate of $\mathcal{S}$ by 2-unit left is the line segment with endpoints $(-2,0)$ and $(-1,1)$.

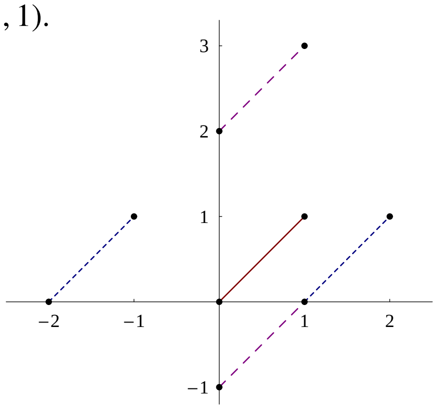


Figure 6.0.14

By considering translates of translates in the four directions, we can move a subset of the rectangular coordinate plane to any position. The following example illustrates how to do that.

Example 6.0.11 Denote $\mathcal{C}$ to be the circle with center at the origin and radius equal to 1.

- The translate of $\mathcal{C}$ by 3-unit right is the circle with center at the point $(3,0)$ and radius equal to 1.
- The translate of (the translate of $\mathcal{C}$ by 3-unit right) by 2-unit up is the circle with center at the point $(3,2)$ and radius equal to 1.

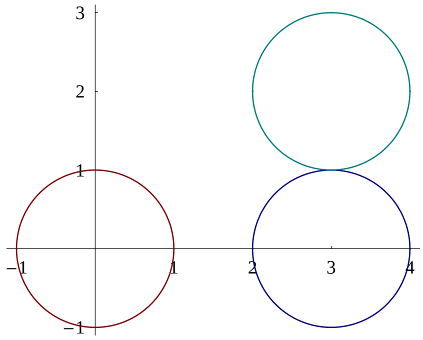


Figure 6.0.15

Remark For simplicity, the translate of (the translate of $\mathcal{C}$ by 3-unit right) by 2 unit up will be called *the translate of* $\mathcal{C}$ *by* 3*-unit right and* 2*-unit up* etc.

Exercise 6.0

1. Denote $A = (0, -1)$, $B = (1, 0)$, $C = (3, -3)$ and $D = (4, -2)$ and denote ℓ to be the horizontal line given by $y + 1 = 0$.
   - (a) In the rectangular coordinate plane shown in Figure 6.0.16, locate the points A, B, C and D and sketch the line ℓ.
   - (b) For each of the points A, B, C and D, find its image under the reflection about ℓ.

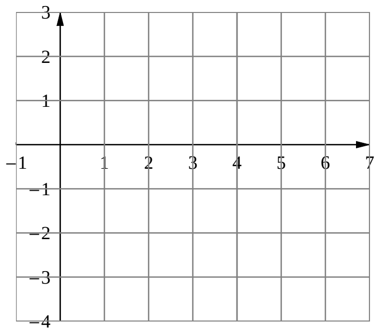


Figure 6.0.16

2. Denote $A = (-1, -1)$, $B = (0, -1)$, $C = (0, 2)$ and $D = (1, 0)$ and denote ℓ to be the line given by $y = x$.
   - (a) In the rectangular coordinate plane shown in Figure 6.0.17, locate the points A, B, C and D and sketch the line ℓ.
   - (b) For each of the points A, B, C and D, find its image under the reflection about ℓ and locate the points in the rectangular coordinate plane shown in Figure 6.0.17.

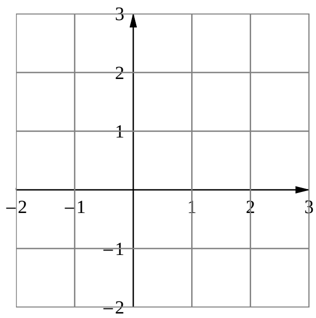


Figure 6.0.17

3. Denote $\mathcal{S}$ to be the square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. Sketch all the axes of symmetry of $\mathcal{S}$ in the rectangular coordinate plane shown in Figure 6.0.18 and write down an equation of each of the axes of symmetry.

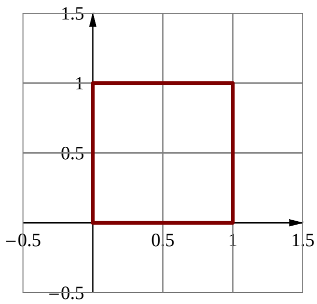


Figure 6.0.18

4. Denote $\mathcal{L}$ to be line segment with endpoints $(-1, 1)$ and $(2, 0)$. Sketch the axis of symmetry of $\mathcal{L}$ in the rectangular coordinate plane shown in Figure 6.0.19 and write down an equation of the axis of symmetry.

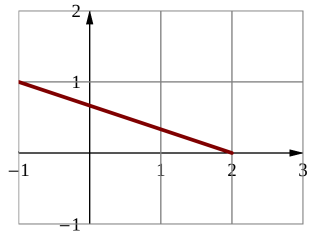


Figure 6.0.19

5. Denote ℓ to be the line given by $2x + 3y - 4 = 0$. Described all the axes of symmetry of ℓ.

6. Denote $\mathcal{S}$ to be the upper half of the circle with center at the point $(1, 1)$ and radius equal to 1 together with the diameter with endpoints $(0, 1)$ and $(2, 1)$. In the rectangular coordinate plane shown in Figure 6.0.20,
   - (a) sketch the axis of symmetry of $\mathcal{S}$;
   - (b) sketch the set such that $\mathcal{S}$ and that set are symmetric about the x-axis;
   - (c) sketch the translate of $\mathcal{S}$ by 1-unit right;
   - (d) sketch the translate of $\mathcal{S}$ by 2-unit down.

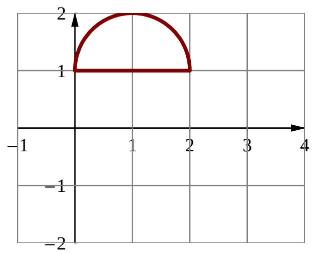


Figure 6.0.20

6.1 Graphs of Quadratic Functions

Definition 6.1.1 We call ***a quadratic function*** to mean a function, denoted by f, satisfying the following conditions:

- the domain and codomain of f are $\mathbb{R}$;
- there exist $a, b, c \in \mathbb{R}$ with $a \neq 0$ such that

$$\text{for every } x \in \mathbb{R}, \quad f(x) = ax^2 + bx + c \tag{6.1.1}$$

Remark In many practical problems, we will consider functions that can be described in the form $x \mapsto ax^2 + bx + c$, but having domains different from $\mathbb{R}$ (because of restrictions on x). For example, in the problem considered at the beginning of this chapter, the domain of the area function is $\{x \in \mathbb{R} : 0 < x < 40\}$. For convenience, such functions are also called *quadratic functions*.

Suppose f is a quadratic function given by (6.1.1). Putting $y = f(x)$, we get

$$y = ax^2 + bx + c$$

which can be considered as an equation with two unknowns x and y. The graph of this equation is called the graph of the function f. In Section 6.1.1, we will consider the graph of f for the special case where b and c are both 0. In Section 6.1.3, we will consider the graph of f in general.

6.1.1 Graph of $y = ax^2$

First we consider the simplest case: $y = x^2$.

- Table 6.1.1 is obtained by putting $x = -3, -2, -1, 0, 1, 2, 3$ respectively, and finding the corresponding values of y.
- The points shown in Figure 6.1.2 are obtained using Table 6.1.1. They are points belonging to the graph of the equation $y = x^2$.

$y = x^2$

x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

Table 6.1.1

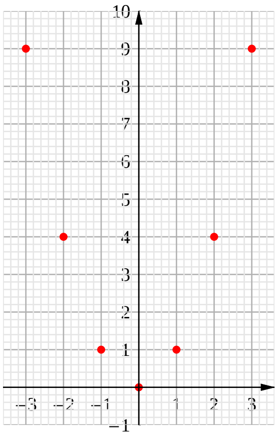


Figure 6.1.2

- Using the same method, we obtain Figure 6.1.3 which shows more points on the graph of the equation $y = x^2$.
- The graph of the equation $y = x^2$ for $-3 \le x \le 3$ is shown in Figure 6.1.4.

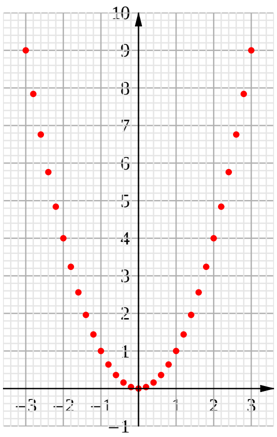


Figure 6.1.3

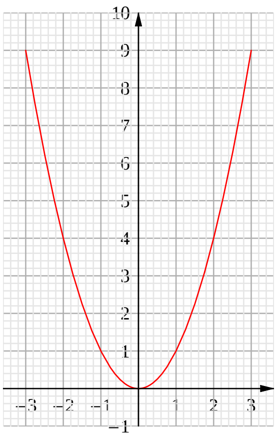


Figure 6.1.4

Properties of the graph of $y = x^2$

- The origin $(0, 0)$ is the lowest point on the graph of $y = x^2$.
- The graph of $y = x^2$ is symmetric about the y-axis. In other words, the y-axis is an axis of symmetry of the graph of $y = x^2$.

Next, we consider the graph of $y = ax^2$ for $a > 0$. As illustrations, we take $a = 2$ and $a = \frac{1}{2}$.

- Using Table 6.1.1, we can obtain Table 6.1.5 and locate a few points on the graphs of the equations $y = 2x^2$ and $y = \frac{1}{2}x^2$.
- The graphs of the equations $y = x^2$, $y = 2x^2$ and $y = \frac{1}{2}x^2$ (for $-3 \le x \le 3$ and $0 \le y \le 10$) are shown in Figure 6.1.6.

x	x^2	$2x^2$	$\frac{1}{2}x^2$
-3	9		$\frac{9}{2}$
-2	4	8	2
-1	1	2	$\frac{1}{2}$
0	0	0	0
1	1	2	$\frac{1}{2}$
2	4	8	2
3	9		$\frac{9}{2}$

Table 6.1.5

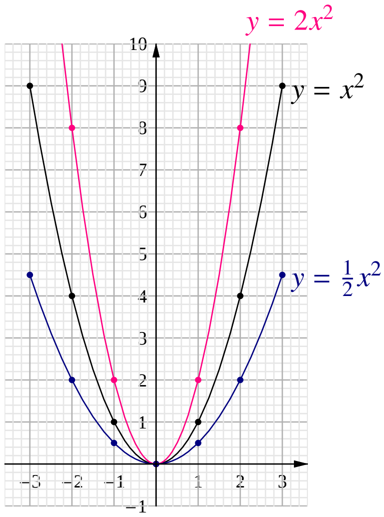


Figure 6.1.6

Note that the graphs of $y = 2x^2$ and $y = \frac{1}{2}x^2$ look like that of $y = x^2$. The graph of $y = 2x^2$ is "narrower" whereas that of $y = \frac{1}{2}x^2$ is "wider". In general, the graph of the equation $y = ax^2$ where $a > 0$ looks like that of $y = x^2$.

Properties of the graph of $y = ax^2$ where $a > 0$

- The graph has a lowest point at the origin and it opens upward (narrow if the value of a is large and wide if the value of a is small).
- The graph is symmetric about the y-axis.

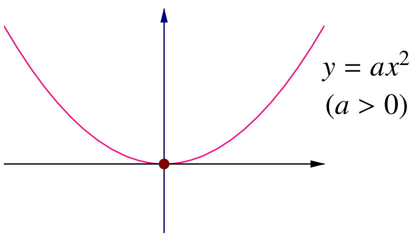


Figure 6.1.7

Finally, we consider the graph of $y = ax^2$ for $a < 0$. As illustrations, we take $a = -1$, $a = -2$ and $a = -\frac{1}{2}$.

- Figure 6.1.8 shows the graphs of the equations $y = -x^2$, $y = -2x^2$ and $y = -\frac{1}{2}x^2$ (for appropriate ranges of values of x and y).

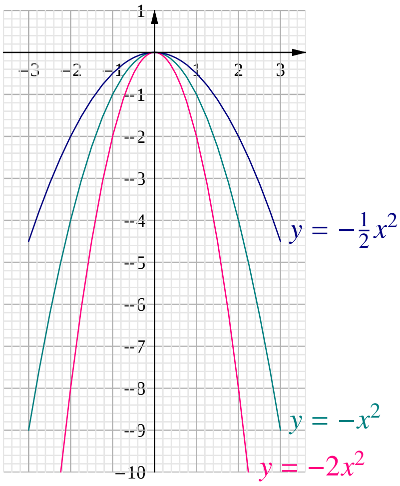


Figure 6.1.8

Note that the graphs of $y = -x^2$, $y = -2x^2$ and $y = -\frac{1}{2}x^2$ looks like that of $y = x^2$ inverted. In general, the graph of the equation $y = ax^2$ where $a < 0$ looks like that of $y = x^2$ inverted.

Properties of the graph of $y = ax^2$ where $a < 0$

- The graph has a highest point at the origin and it opens downward (narrow if the magnitude of a is large and wide if the magnitude of a is small).

  Explain The magnitude of a positive real number is the real number itself and the magnitude of a negative number is the number obtained by omitting the negative sign of the negative number. For example, the magnitude of -3 is 3.

- The graph is symmetric about the y-axis.

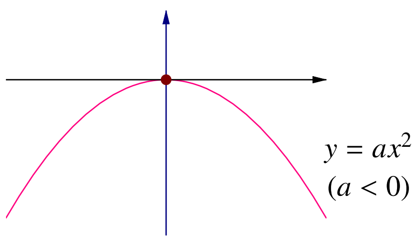


Figure 6.1.9

Observant readers may notice that the graphs in Figure 6.1.8 can be obtained from that in Figure 6.1.6 by inverting the graphs about the x-axis.

- Figure 6.1.10 shows the graph of $y = -x^2$ together with that of $y = x^2$. The graphs of $y = -x^2$ and $y = x^2$ are symmetric about the x-axis (*if we consider the x-axis as a mirror and the graph of $y = x^2$ an object in front of the mirror, then the graph of $y = -x^2$ is the image behind the mirror*).
- Figure 6.1.11 shows the graph of $y = -2x^2$ together with that of $y = 2x^2$. The graphs of $y = -2x^2$ and $y = 2x^2$ are symmetric about the x-axis.

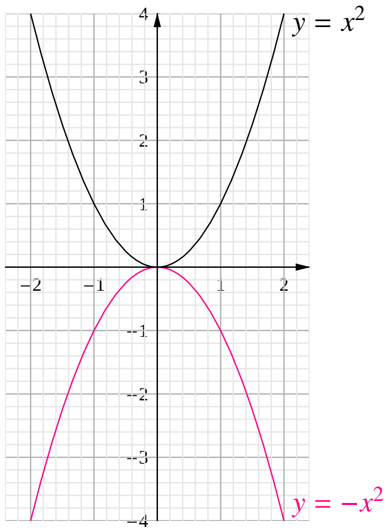


Figure 6.1.10

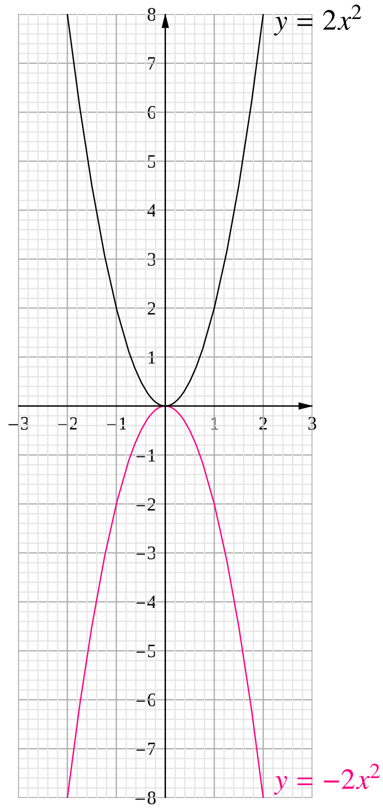


Figure 6.1.11

In general, we have the following:

- The graphs of $y = \alpha x^2$ and $y = -\alpha x^2$, where $\alpha \neq 0$, are symmetric about the x-axis.

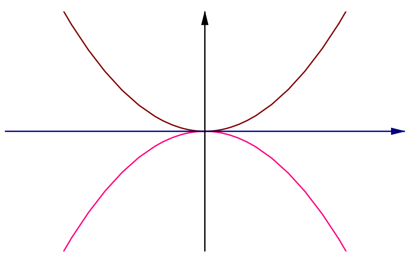


Figure 6.1.12

Example 6.1.1 Suppose a is a non-zero real number and (part of) the graph of $y = ax^2$ is as shown in Figure 6.1.13. Find the value of a.

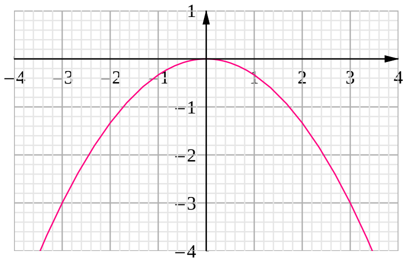


Figure 6.1.13

Explanation Note that the graph opens downward. Thus we can tell that $a < 0$. The value of a can be found using a point on the graph.

Solution Note that the point $(3, -3)$ belongs to the graph of $y = ax^2$. Thus

$$-3 = a \cdot 3^2$$

Solving, we get $a = -\frac{1}{3}$. □

The graph of the equation $y = ax^2$ (where $a \neq 0$) is a curve that we call a *parabola*. Before giving the definition of a parabola, we consider an example.

If you find the solution to Example 6.1.2 too tedious, you may skip the solution. But you should try to understand the meaning of (a) and (b).

If you find Definition 6.1.2 too abstract, read Example 6.1.2 again. If you still find it difficult to understand, don't worry, all we need is the summary for graph of $y = ax^2$ (where $a \neq 0$) on Page 22.

Example 6.1.2 Denote $\mathcal{C}$ to be the graph of the equation $y = x^2$. Denote ℓ to be the horizontal line given by $y = -\frac{1}{4}$ and denote F to be the point $(0, \frac{1}{4})$.

(a) Show that if $P(s, t)$ is a point belonging to $\mathcal{C}$, then the distance from P to ℓ is equal to the distance from P to F.

(b) Show that if $P(s, t)$ is a point in the rectangular coordinate plane such that the distance from P to ℓ is equal to the distance from P to F, then P belongs to $\mathcal{C}$.

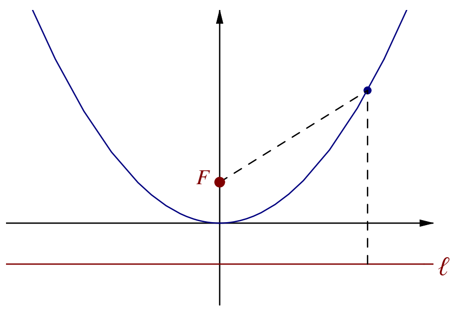


Figure 6.1.14

Solution Let $P(s,t)$ be a point in the rectangular coordinate plane. Denote

$$\begin{aligned} d_1 &= \text{distance from } P \text{ to } F \\ d_2 &= \text{distance from } P \text{ to } \ell \end{aligned}$$

By the distance formula,

$$d_1 = \sqrt{(s-0)^2 + (t - \tfrac{1}{4})^2} \tag{6.1.2}$$

For the case where $t \geq 0$, since the horizontal line ℓ is $\frac{1}{4}$ unit below the x-axis, it follows that

$$d_2 = t + \frac{1}{4} \tag{6.1.3}$$

(*a*) Suppose $P(s,t)$ belongs to $\mathcal{C}$. We want to show that $d_1 = d_2$. Since both d_1 and d_2 are positive real numbers, it suffices to show that $d_1^2 = d_2^2$.

Since $P(s,t)$ belongs to $\mathcal{C}$, it follows that

$$t = s^2 \tag{6.1.4}$$

Thus
$$\begin{aligned} d_1^2 - d_2^2 &= \left(s^2 + (t - \tfrac{1}{4})^2\right) - \left(t + \tfrac{1}{4}\right)^2 && \text{By (6.1.2) and (6.1.3) since } t \geq 0 \\ &= \left(s^2 + t^2 - \frac{1}{2}t + \frac{1}{16}\right) - \left(t^2 + \frac{1}{2}t + \frac{1}{16}\right) \\ &= s^2 - t \\ &= 0 && \text{By (6.1.4)} \end{aligned}$$

Therefore, $d_1^2 = d_2^2$.

(*b*) Suppose that $d_1 = d_2$. We want to show that $P(s,t)$ belongs to $\mathcal{C}$.

Since $d_1 = d_2$, it follows that the point $P(s,t)$ cannot be below the x-axis and so $t \geq 0$. Hence

$$\begin{aligned} \sqrt{s^2 + (t - \tfrac{1}{4})^2} &= t + \frac{1}{4} && \text{By (6.1.2) and (6.1.3) since } t \geq 0 \\ s^2 + (t - \tfrac{1}{4})^2 &= \left(t + \tfrac{1}{4}\right)^2 \\ s^2 + t^2 - \frac{1}{2}t + \frac{1}{16} &= t^2 + \frac{1}{2}t + \frac{1}{16} \\ s^2 &= t \end{aligned}$$

Therefore, the point $P(s,t)$ belongs to $\mathcal{C}$.

□

Example 6.1.2 shows that the graph of $y = x^2$ is set consisting of all points in the rectangular coordinate plane that are equidistant to a certain point and a certain line. Such a set is called a *parabola*.

Definition 6.1.2 We call ***a parabola*** to mean a subset (denoted by $\mathcal{C}$) of a plane (denoted by Π) satisfying the following condition:

- There exists a line ℓ in the plane Π and there exists a point F belonging to Π but not belonging to ℓ such that

$$\mathcal{C} \;=\; \{P \in \Pi : \text{distance from } P \text{ to } \ell = \text{distance from } P \text{ to } F\}$$

Remark $\mathcal{P}$ is called *the parabola with directrix ℓ and focus F.*

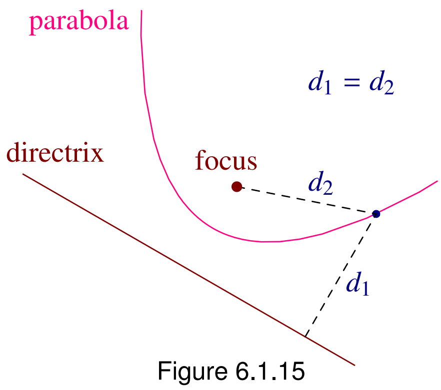


Figure 6.1.15

Example 6.1.2 shows that the graph of the equation $y = x^2$ is a parabola. In fact, it is the parabola with the horizontal line $y = -\frac{1}{4}$ as the directrix and the point $(0, \frac{1}{4})$ as the focus. More generally, we have the following

Theorem 6.1.1 Let a be a non-zero real number. Then the graph of the equation $y = ax^2$ is a parabola.

Idea of Proof The idea of proof of the theorem can be found in Example 6.1.2. Consider the horizontal line $y = -\frac{a}{4}$ and the point $(0, \frac{a}{4})$.

Remark If $a = 0$, then the graph of $y = 0x^2$ is not a parabola; it is the x-axis.

For the parabola given by the equation $y = x^2$, the y-axis is an axis of symmetry. Note that (see Example 6.1.2) the y-axis is the line passing through the focus $(0, \frac{1}{4})$ and perpendicular to the directrix (the horizontal line $y = -\frac{1}{4}$). In general, we have the following

Theorem 6.1.2 Let ℓ be a line and let F be a point not belonging to the line. Denote $\mathcal{P}$ to be the parabola with directrix ℓ and focus F. Then the line passing through F and perpendicular to ℓ is an axis of symmetry of $\mathcal{P}$.

Idea of Proof Denote $\mathcal{A}$ to be the line passing through F and perpendicular to ℓ.
Suppose P is a point belonging to the parabola and Q is the point on the other side of $\mathcal{A}$ such that

$$\text{Line } PQ \perp \mathcal{A} \quad \text{and} \quad \text{distance from } P \text{ to } \mathcal{A} = \text{distance from } Q \text{ to } \mathcal{A}$$

Then Q belongs to the parabola too. This is because

- distance from P to directrix = distance from P to focus

  Reason P belongs to the parabola.
- distance from P to directrix = distance from Q to directrix

  Reason From the relation between P and Q.
- distance from P to focus = distance from Q to focus

  Reason From the relation between P and Q.

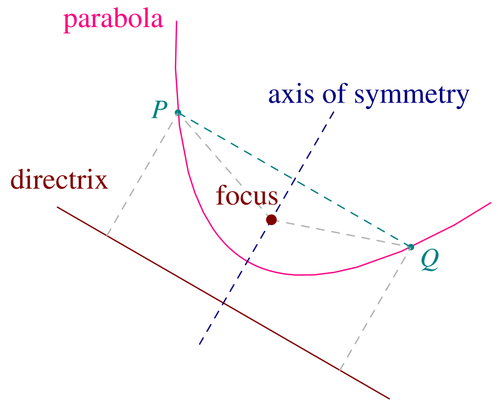


Figure 6.1.16

Remark $\mathcal{P}$ has only one axis of symmetry and so the line passing through F and perpendicular to ℓ is called *the axis of symmetry of* $\mathcal{P}$.

Definition 6.1.3 Let $\mathcal{P}$ be a parabola. We call ***the vertex of*** $\mathcal{P}$ to mean the point of intersection of $\mathcal{P}$ and its axis of symmetry.

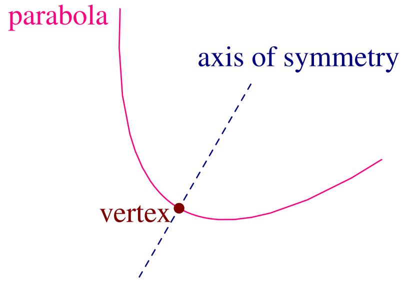


Figure 6.1.17

Summary for graph of $y = ax^2$ where $a \neq 0$

- The graph is a parabola.
- The y-axis is the axis of symmetry of the parabola.
- The origin is the vertex of the parabola.
  - If $a > 0$, the parabola opens upward and the vertex is the lowest point of the parabola.
  - If $a < 0$, the parabola opens downward and the vertex is the highest point of the parabola.

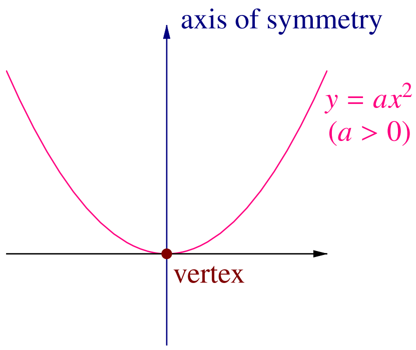


Figure 6.1.18

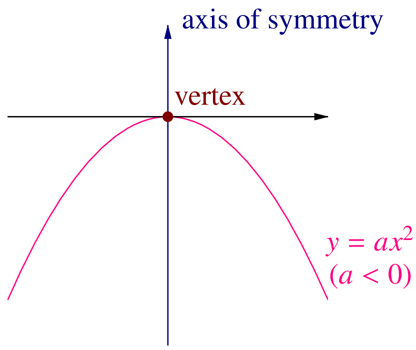


Figure 6.1.19

Interesting Results Related to Parabolas

- If an object is thrown (with a forward component), the trajectory of the object is (part of) a parabola, assuming that gravity is the only force acting on the object.

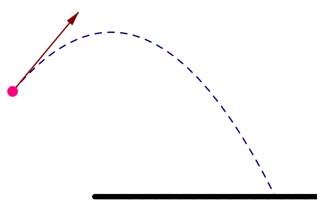


Figure 6.1.20

- If a parabola is considered as a mirror, then rays parallel to the axis of symmetry will be reflected towards the focus.

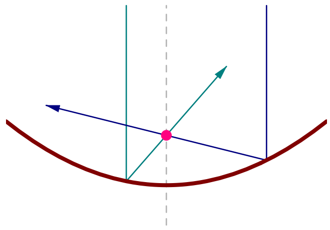


Figure 6.1.21

Exercise 6.1.1

1. For each of the following, use a computer software to sketch the graphs of the given equations on the same rectangular coordinate plane (choose appropriate values of x and y).

   (a) $y = x^2$, $y = 2x^2$, $y = 3x^2$, $y = 4x^2$, $y = 5x^2$

   (b) $y = -x^2$, $y = -2x^2$, $y = -3x^2$, $y = -4x^2$, $y = -5x^2$

   (c) $y = x^2$, $y = \frac{3}{2}x^2$, $y = \frac{3}{5}x^2$

   (d) $y = -x^2$, $y = -\frac{5}{3}x^2$, $y = -\frac{2}{5}x^2$

2. For each of the following, use a computer software to sketch the graphs of the given equations on the same rectangular coordinate plane (choose appropriate values of x and y).

   (a) $y = 3x^2$, $y = -3x^2$

   (b) $y = -4x^2$, $y = 4x^2$

   (c) $y = \frac{1}{2}x^2$, $y = -\frac{1}{2}x^2$

   (d) $y = -\frac{4}{3}x^2$, $y = \frac{4}{3}x^2$

3. For each of the following, determine whether the graph of the given equation opens upward or downward.

   (a) $y = -7x^2$ (b) $y = 2.3x^2$ (c) $y = \frac{2}{-3}x^2$

4. For each of the following, determine whether the origin is the highest or lowest point on the graph of the given equation.

   (a) $y = 4.5x^2$ (b) $y = -\frac{11}{12}x^2$ (c) $y = \frac{-6}{5}x^2$

5. Figure 6.1.22 shows the graphs of $y = ax^2$, $y = bx^2$, $y = cx^2$ and $y = dx^2$. Arrange the numbers a, b, c and d in ascending order.

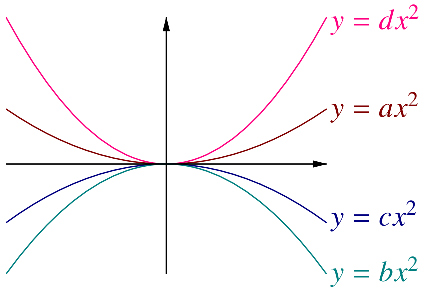


Figure 6.1.22

6. Figure 6.1.23 shows the graph of $y = ax^2$ together with that of $y = x^2$. What can you tell about the value of a?

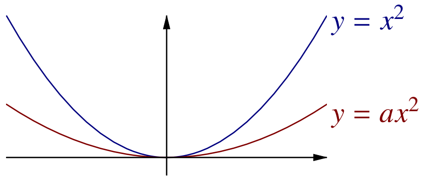


Figure 6.1.23

7. For each of the following, find the value of a if the graph of $y = ax^2$ passes through the given point P.

   (a) $P = (5, 100)$, (b) $P = (-4, 24)$, (c) $P = (-7, 0)$

8. Show that for every point on the graph of $y = 4x^2$, its distance to the point $(0, 1)$ is equal to that to the horizontal line $y = -1$.

6.1.2 Graph of $y = a(x - h)^2 + k$

Consider the graph of the equation $y = x^2 - 2x + 3$.

- Table 6.1.24 is obtained by putting $x = -2, -1, 0, 1, 2, 3$ respectively, and finding the corresponding values of y.
- The points shown in Figure 6.1.25 are obtained using Table 6.1.24. They are points belonging to the graph of $y = x^2 - 2x + 3$.

$y = x^2 - 2x + 3$

x	y
-2	11
-1	6
0	3
1	2
2	3
3	6

Table 6.1.24

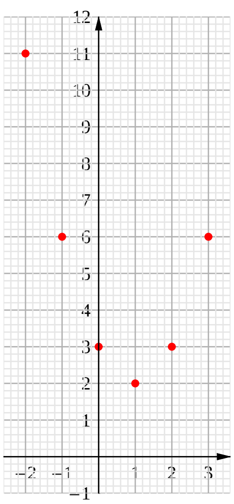


Figure 6.1.25

- The graph of $y = x^2 - 2x + 3$ for $-2 \le x \le 3$ is shown in Figure 6.1.26. Note that the part of the graph for $-1 \le x \le 3$ is symmetrical about the vertical line with equation $x = 1$.
- The graph of $y = x^2 - 2x + 3$ for $-2 \le x \le 4$, together with that of $y = x^2$ for $-3 \le x \le 3$, is shown in Figure 6.1.27.

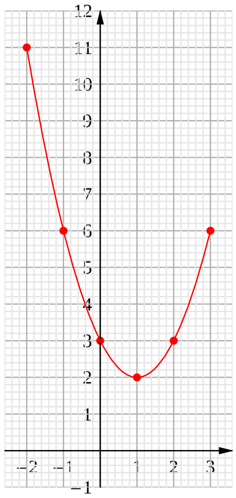


Figure 6.1.26

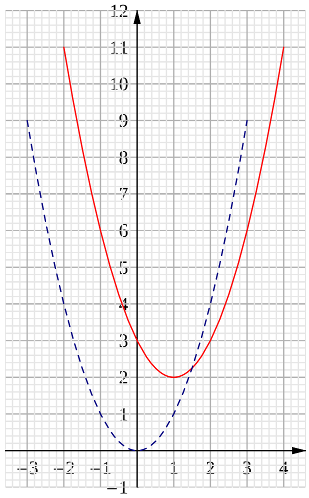


Figure 6.1.27

Figure 6.1.27 shows that the graph of $y = x^2 - 2x + 3$ has the same shape as that of $y = x^2$. More precisely, the graph of $y = x^2 - 2x + 3$ is the translate of the graph of $y = x^2$ by 1-unit right and 2-unit up. Below we consider how to obtain translates

- to the left or right;
- up or down.

Translate to left or right Figure 6.1.28 shows the graph of $y = (x - 1)^2$. Note that the point $(1, 0)$ belongs to the graph and it is the lowest point on the graph. This is because for every real number x, the corresponding value of y is at least 0. The graph of $y = (x - 1)^2$ is the translate of the graph of $y = x^2$ by 1-unit right (see Figure 6.1.29).

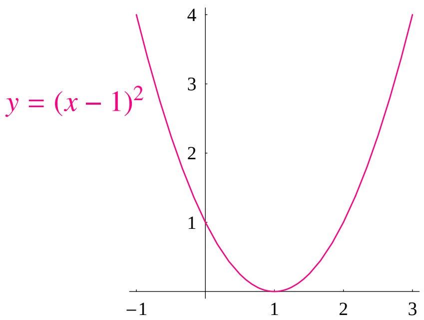


Figure 6.1.28

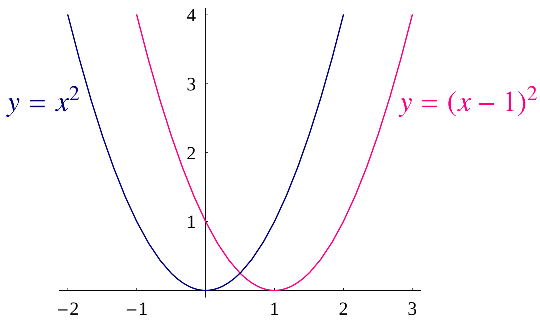


Figure 6.1.29

The equation $y = (x-1)^2$ is obtained by changing x to $(x-1)$ in the equation $y = x^2$. In general, in the equation $y = ax^2$, if we change x to $(x-h)$, we get the equation $y = a(x-h)^2$. The graph of $y = a(x-h)^2$ is a translate of that of $y = ax^2$ to the right or left depending on whether h is positive or negative:

$$\text{Graph of } y = a(x-h)^2 = \begin{cases} \text{Translate of graph of } y = ax^2 \text{ by } m\text{-unit right} & \text{if } h > 0 \\ \text{Translate of graph of } y = ax^2 \text{ by } m\text{-unit left} & \text{if } h < 0 \\ \text{Graph of } y = ax^2 & \text{if } h = 0 \end{cases}$$

where m is the magnitude of h (if $h > 0$, then $m = h$ and if $h < 0$, then $m = -h$).

Example 6.1.3 The equation $y = \frac{1}{2}(x-3)^2$ is in the form $y = a(x-h)^2$ where $a = \frac{1}{2}$ and $h = 3$. Note that $h > 0$ and the magnitude of h is 3.

The graph of $y = \frac{1}{2}(x-3)^2$ is the translate of the graph of $y = \frac{1}{2}x^2$ by 3-unit right.

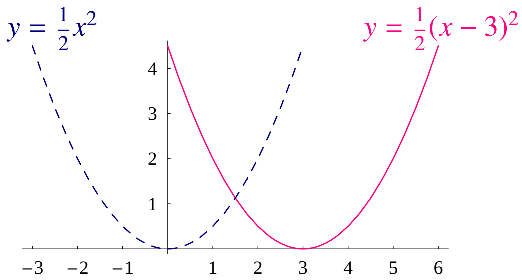


Figure 6.1.30

Example 6.1.4 The equation $y = -\frac{3}{2}(x+4)^2$ is in the form $y = a(x-h)^2$ where $a = -\frac{3}{2}$ and $h = -4$. Note that $h < 0$ and the magnitude of h is 4.

The graph of $y = -\frac{3}{2}(x+4)^2$ is the translate of the graph of $y = -\frac{3}{2}x^2$ by 4-unit left.

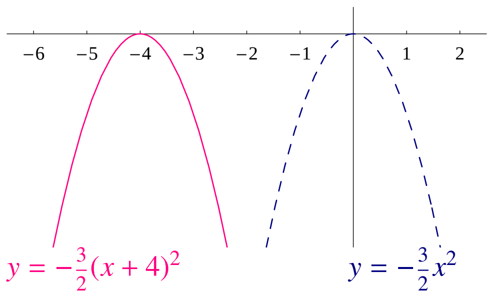


Figure 6.1.31

To determine whether the graph of $y = a(x-h)^2$ is a translate of the graph of $y = ax^2$ to the left or right, we may consider the highest or lowest point on the graph.

[Example 6.1.3] For the graph of $y = \frac{1}{2}(x-3)^2$, the lowest point is $(3, 0)$. The graph is 3 units to the right of the graph of $y = \frac{1}{2}x^2$.

[Example 6.1.4] For the graph of $y = -\frac{3}{2}(x+4)^2$, the highest point is $(-4, 0)$. The graph is 4 units to the left of the graph of $y = -\frac{3}{2}x^2$.

Translate up or down Figure 6.1.32 shows the graph of $y = -\frac{1}{3}x^2 + 2$. Note that the point $(0, 2)$ belongs to the graph and it is the highest point on the graph. This is because for every real number x, the corresponding value of y is at most 2. The graph of $y = -\frac{1}{3}x^2 + 2$ is the translate of the graph of $y = -\frac{1}{3}x^2$ by 2-unit up (see Figure 6.1.33).

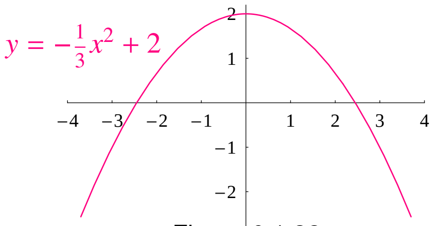


Figure 6.1.32

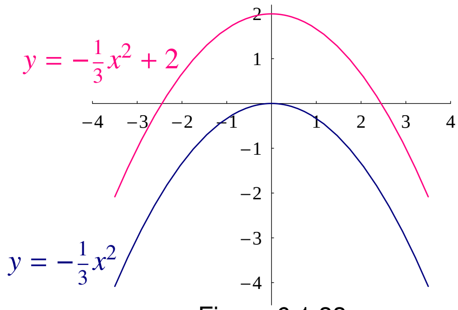


Figure 6.1.33

The equation $y = -\frac{1}{3}x^2 + 2$ is the same as the equation $y - 2 = -\frac{1}{3}x^2$ which can be obtained by changing y to $(y - 2)$ in the equation $y = -\frac{1}{3}x^2$. In general, in the equation $y = ax^2$, if we change y to $(y - k)$, we get the equation $y - k = ax^2$ which is the same as the equation $y = ax^2 + k$. The graph of $y = ax^2 + k$ is a translate of that of $y = ax^2$ up or down depending on whether k is positive or negative:

$$\text{Graph of } y = ax^2 + k = \begin{cases} \text{Translate of graph of } y = ax^2 \text{ by } m\text{-unit up} & \text{if } k > 0 \\ \text{Translate of graph of } y = ax^2 \text{ by } m\text{-unit down} & \text{if } k < 0 \\ \text{Graph of } y = ax^2 & \text{if } k = 0 \end{cases}$$

where m is the magnitude of k.

Example 6.1.5 The equation $y = \frac{1}{4}x^2 - 3$ is in the form $y = ax^2 + k$ where $a = \frac{1}{4}$ and $k = -3$. Note that $k < 0$ and the magnitude of k is 3.

The graph of $y = \frac{1}{4}x^2 - 3$ is the translate of the graph of $y = \frac{1}{4}x^2$ by 3-unit down.

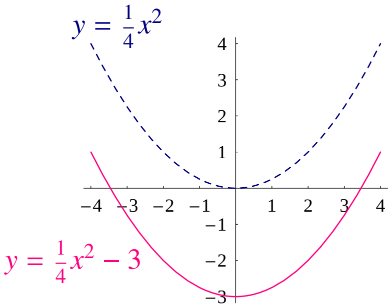


Figure 6.1.34

Example 6.1.6 The equation $y = 2 - x^2$ is in the form $y = ax^2 + k$ where $a = -1$ and $k = 2$. Note that $k > 0$ and the magnitude of k is 2.

The graph of $y = 2 - x^2$ is the translate of the graph of $y = -x^2$ by 2-unit up.

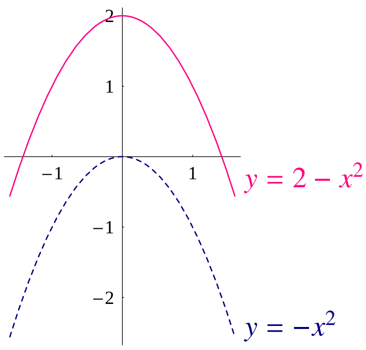


Figure 6.1.35

The idea of moving the graph of $y = ax^2$ up or down applies to the graph of any function (from a subset of $\mathbb{R}$ to $\mathbb{R}$). The following is a general result on the effect of adding a constant to a function.

Theorem 6.1.3 Let $f : X \to \mathbb{R}$ be a function where X is a subset of $\mathbb{R}$. Let k be a non-zero real number. Denote g to be the function from X to $\mathbb{R}$ given by

$$g(x) = f(x) + k \quad \text{for } x \in X$$

Then the graph of g is a translate of that of f:

$$\text{Graph of } g = \begin{cases} \text{Translate of graph of } f \text{ by } m\text{-unit up} & \text{if } k > 0 \\ \text{Translate of graph of } f \text{ by } m\text{-unit down} & \text{if } k < 0 \end{cases}$$

where m is the magnitude of k.

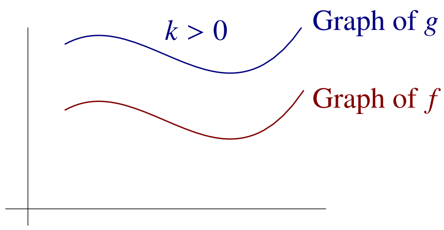


Figure 6.1.36

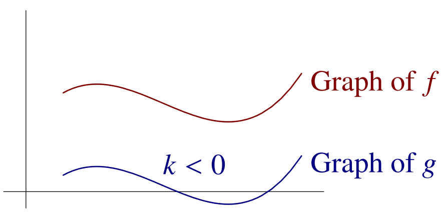


Figure 6.1.37

Remark If $k = 0$, then $g = f$.

Combining the two process of translating to the left or right and that up or down, we can translate subsets of the rectangular coordinate plane to any position. For translating a parabola, instead of specifying the directions (left or right, up or down) we can simply tell where the vertex is.

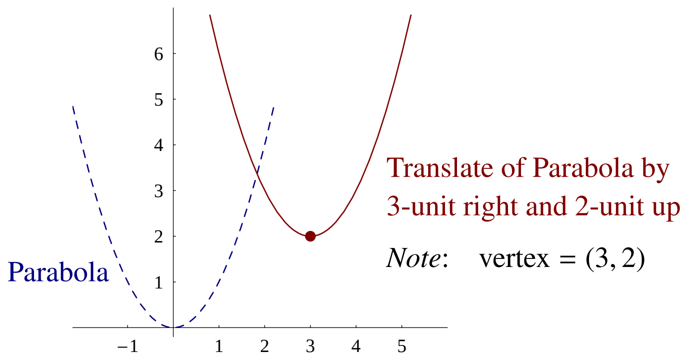


Figure 6.1.39

Theorem 6.1.4 Let a be a non-zero real number. Let h and k be real numbers not both zero. Then the graph of the equation

$$y = a(x - h)^2 + k$$

is a parabola. More precisely, it is the translate of the graph of $y = ax^2$ such that the vertex is at the point (h, k).

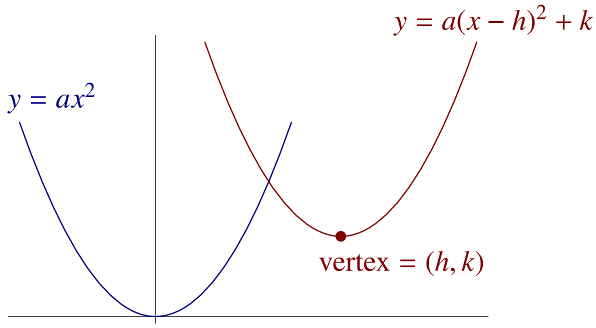


Figure 6.1.40

Remark The vertical line $x = h$ is the axis of symmetry of the graph of $y = a(x - h)^2 + k$. It is the vertical line passing through the vertex (h, k).

Remark If $h = 0$ and $k = 0$, then the equation $y = a(x - h)^2 + k$ reduces to $y = ax^2$ and so its graph is a parabola with vertex at the origin (the graph of $y = ax^2$ without translation).

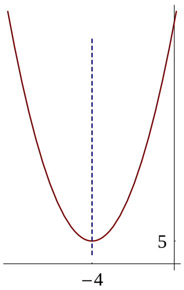
Example 6.1.7 Consider the parabola given by the equation

$$y = 3(x+4)^2 + 5$$

(a) Determine whether the parabola opens upward or downward.

(b) Write down the vertex of the parabola.

(c) Write down an equation of the axis of symmetry of the parabola.



Solution The given equation takes the form $y = a(x-h)^2 + k$, where $a = 3$, $h = -4$ and $k = 5$.

(a) Since $a > 0$, it follows that the the parabola opens upward.

(b) The vertex is $(-4, 5)$.

(c) An equation of the axis of symmetry is: $x = -4$. □

Remark The given parabola is the translate of the graph of $y = 3x^2$ by 4-unit left and 5-unit up.

Example 6.1.8 Figure 6.1.41 shows the graph of $y = a(x-h)^2 + k$, where a, h and k are real numbers. Find the values of a, h and k.

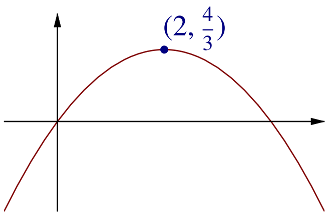


Figure 6.1.41

Solution Since the vertex of the parabola is $(2, \frac{4}{3})$, it follows that $h = 2$ and $k = \frac{4}{3}$. With the above values of h and k, the given equation is

$$y = a(x-2)^2 + \frac{4}{3}$$

Since the origin belongs to the parabola, it follows that

$$0 = a \cdot (0-2)^2 + \frac{4}{3}$$

Hence $a = -\frac{1}{3}$. □

Exercise 6.1.2

1. For each of the following, find the coordinates of the vertex of the graph of the given equation and determine whether the vertex is the lowest or highest point of the graph.

   (a) $y = 3(x-5)^2 - 7$ (b) $y = -7(x+6)^2 - 5$ (c) $y = x^2 + 3$

   (d) $y = -2(x+6)^2$ (e) $y = 7 - 3(x+4)^2$ (f) $y = 2(5 - 6x^2)$

   (g) $y = (2x-4)^2$ (h) $y = 1 - (2x-3)^2$

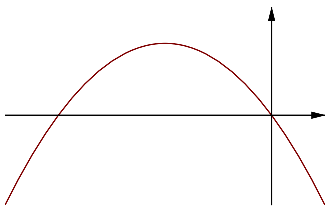
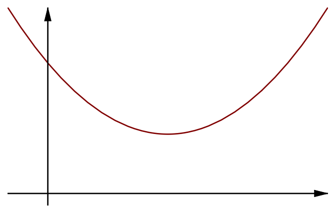
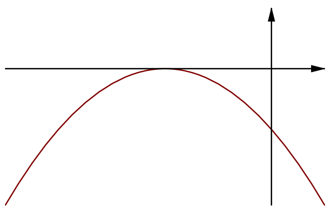
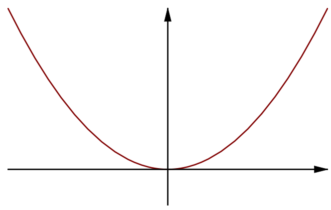
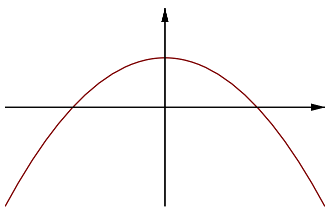
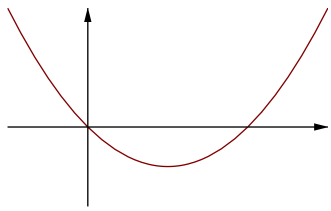
2. For each of the following, determine whether the graph of the given equation opens upward or downward.

   (a) $y = 3(x+4)^2 - 5$ (b) $y = 3 - (x-2)^2$ (c) $y = 2 - (3-4x)^2$

3. For each of the following, find an equation of the axis of symmetry of the graph of the given equation.

   (a) $y = (x+2)^2 + 3$ (b) $y = 1 - (x-2)^2$ (c) $y = (2-3x)^2 + 4$

4. In each of the following figures, the parabola is the graph of $y = a(x-h)^2 + k$. Determine whether the values of a, h and k are positive, negative or 0.

   (a)
   
   (b)
   
   (c)
   
   (d)
   
   (e)
   
   (f)
   

5. Figure 6.1.42 shows that graph of $y = \frac{1}{3}(x-2)^2 + k$. Find the value of k.

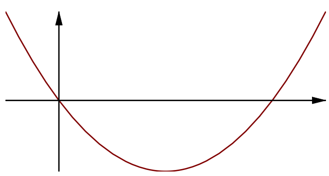


Figure 6.1.42

6. Figure 6.1.43 shows that graph of $y = a(x+3)^2 + k$. Find the values of a and k.

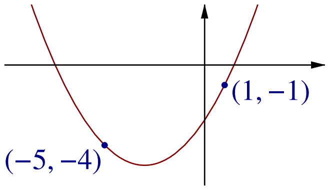


Figure 6.1.43

7. Figure 6.1.44 shows that graph of $y = (x-h)^2 + k$. Find the values of h and k.

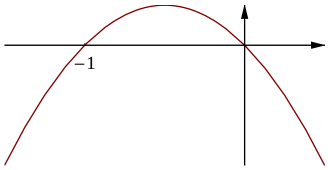


Figure 6.1.44

8. Figure 6.1.45 shows that graph of $y = a(x-h)^2 + k$. Find the values of a, h and k.

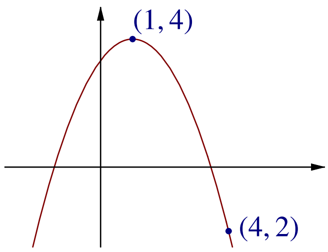


Figure 6.1.45

9. Suppose the point $(1, 1)$ belongs to the graph of the equation $y = a(x+3)^2 - 7$. Find the value of a.
10. Suppose the y-intercept of the graph of the equation $y = -2(x+1)^2 + k$ is 7. Find the value of k.
11. Suppose the vertical line with equation $x + 2 = 0$ is the axis of symmetry of the graph of the equation $y = \frac{1}{2}(x-h)^2 + k$ and $(2, 3)$ is a point belonging to the graph. Find the value of h and k.
12. Suppose $(-1, 3)$ is the vertex of the parabola given by $y = a(x-h)^2 + k$ and -2 is an x-intercept of the parabola. Find the values of a, h and k.

6.1.3 Graph of $y = ax^2 + bx + c$

In Section 6.1.2, we have seen that the graph of $y = x^2 - 2x + 3$ has the same shape as that of $y = x^2$ (see Figure 6.1.27). In fact, it is the translate of the graph of $y = x^2$ by 1-unit right and 2-unit up. This can be seen by rewriting the equation $y = x^2 - 2x + 3$ in the form $y = a(x-h)^2 + k$ using the completing square method:

$$y = x^2 - 2x + 3$$

$$y = (x^2 - 2x + 1) - 1 + 3$$

Add 1 to make a complete square (the number 1 is the square of $\frac{1}{2} \cdot 2$); minus 1 to make the value of the right-side unchanged

$$y = (x-1)^2 + 2$$

In general, the graph of $y = ax^2 + bx + c$ (where $a \neq 0$) has the same shape as that of $y = ax^2$. It is a translate of the graph of $y = ax^2$ (for the case where at least one of b and c is not 0). The vertex of the parabola $y = ax^2 + bx + c$ can be found by rewriting the equation in the form $y = a(x - h)^2 + k$ using the complete square method.

Example 6.1.9 Consider the parabola given by the equation

$$y = 2x^2 + 12x$$

(a) Find the coordinates of the vertex of the parabola.

(b) Find an equation of the axis of symmetry of the parabola.

Solution Rewrite the given equation in the form $y = a(x - h)^2 + k$ using the complete square method:

$$\begin{aligned} y &= 2x^2 + 12x \\ y &= 2(x^2 + 6x) && \text{Extract factor } a = 2 \text{ in } x^2\text{-term and } x\text{-term} \\ y &= 2(x^2 + 6x + 9 - 9) && \text{Complete square, value of right-side unchanged} \\ y &= 2(x^2 + 6x + 9) - 18 \\ y &= 2(x + 3)^2 - 18 \end{aligned}$$

Note that $a = 2$, $h = -3$ and $k = -18$.

(a) The vertex is $(-3, -18)$.

(b) An equation of the axis of symmetry is: $x = -3$.

□

Remark Alternatively, we can find the values of a, h and k as follows:

Expand and collect terms:
$$\begin{aligned} y &= a(x - h)^2 + k \\ y &= a(x^2 - 2xh + h^2) + k \\ y &= ax^2 - (2ah)x + (ah^2 + k) \end{aligned}$$

Compare coefficients:
$$\begin{aligned} a &= 2 \\ -2ah &= 12 \\ ah^2 + k &= 0 \end{aligned}$$

With $a = 2$, the second equation gives $h = -3$ and hence the third equation gives $k = -18$.

Next we give a result which is useful in finding vertices of graphs of quadratic functions. It is also useful in finding where quadratic functions attain their extrema (see Theorem 6.3.1).

Theorem 6.1.5 Let a, b and c be real numbers with $a \neq 0$. Then the graph of the equation

$$y = ax^2 + bx + c$$

is a parabola. Moreover, the x-coordinate of the vertex of the parabola is $-\frac{b}{2a}$.

Proof The graph of the given equation is a parabola. This is because the equation can be written in the form $y = a(x-h)^2 + k$:

$$\begin{aligned}
y &= ax^2 + bx + c \\
y &= a\Big(x^2 + \frac{b}{a}x\Big) + c && \text{Extract factor } a \text{ in } x^2\text{-term and } x\text{-term} \\
y &= a\Big(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 - (\frac{b}{2a})^2\Big) + c && \text{Complete square, value of right-side unchanged} \\
y &= a\Big(x^2 + \frac{b}{a}x + (\frac{b}{2a})^2\Big) - a \cdot (\frac{b}{2a})^2 + c \\
y &= a\Big(x + \frac{b}{2a}\Big)^2 - a \cdot \frac{b^2}{4a^2} + c \\
y &= a\Big(x - \frac{-b}{2a}\Big)^2 + \Big(c - \frac{b^2}{4a}\Big)
\end{aligned}$$

By Theorem 6.1.4, the x-coordinate of the vertex is $\frac{-b}{2a}$. □

Remark The y-coordinate of the vertex is $c - \frac{b^2}{4a} = -\frac{b^2 - 4ac}{4a}$. The value can be found by substituting $x = \frac{-b}{2a}$ directly into the equation of the parabola (see Solution 2 to Example 6.1.10).

By Theorem 6.1.4, the graph of $y = ax^2 + bx + c$ is a translate of that of $y = ax^2$. Thus

- If $a > 0$, the parabola opens upward and the vertex is the lowest point of the parabola.
- If $a < 0$, the parabola opens downward and the vertex is the highest point of the parabola.

Example 6.1.10 Find the vertex of the parabola given by

$$y = 5 + 6x - 3x^2$$

Solution 1 Rewrite the given equation in the form $y = a(x-h)^2 + k$.

$$\begin{aligned}
y &= 5 + 6x - 3x^2 \\
y &= (-3x^2 + 6x) + 5 && \text{Collect } x^2\text{-term and } x\text{-term} \\
y &= -3(x^2 - 2x) + 5 && \text{Extract factor } a = -3 \text{ in } x^2\text{-term and } x\text{-term} \\
y &= -3(x^2 - 2x + 1 - 1) + 5 && \text{Value of right-side unchanged} \\
y &= -3(x^2 - 2x + 1) + 3 + 5 && (-3) \cdot (-1) = 3 \\
y &= -3(x-1)^2 + 8
\end{aligned}$$

The vertex of the parabola is $(1, 8)$. Note that $a = -3$, $h = 1$ and $k = 8$. □

Solution 2 Note that the given equation can be written as

$$y = -3x^2 + 6x + 5$$

With $a = -3$ and $b = 6$,

$$-\frac{b}{2a} = -\frac{6}{2 \cdot (-3)} = 1 \qquad x\text{-coordinate of vertex (by Theorem 6.1.5)}$$

Substituting $x = 1$ into the given equation, we get

$$y = 5 + 6 \cdot 1 - 3 \cdot 1^2 = 8$$

The vertex of the parabola is $(1, 8)$. □

Next, we consider intercepts of the graph of $y = ax^2 + bx + c$ (where $a \neq 0$).

- To find x-intercept(s), if there is any, put $y = 0$ and solve for the corresponding values of x. Note that there may be two, one or no x-intercepts.
  - If there are two x-intercepts, then the axis of symmetry is given by $x = h$, where h is the average of the two x-intercepts (see Example 6.1.11).
  - If there is only one x-intercept, then the vertex is the point $(h, 0)$, where h is the x-intercept (see Example 6.1.14).
- To find y-intercept, put $x = 0$ and find the corresponding value of y. Note that there is exactly one y-intercept.

Example 6.1.11 Denote $\mathcal{P}$ to be the graph of the equation $y = \frac{1}{2}x^2 - x - 4$.

(*a*) Find the x-intercept(s), if any, of $\mathcal{P}$.

(*b*) Find the y-intercept of $\mathcal{P}$.

(*c*) Use the result in (a) to find an equation of the axis of symmetry of $\mathcal{P}$.

(*d*) Use the result in (c) to find the vertex of $\mathcal{P}$.

Solution (*a*) Put $y = 0$ into the given equation and solve for x

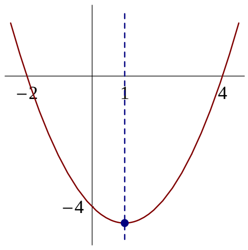
$$\begin{aligned} 0 &= \frac{1}{2}x^2 - x - 4 && \text{Put } y = 0 \\ 0 &= x^2 - 2x - 8 && \text{Multiply both sides by 2} \\ 0 &= (x-4)(x+2) \\ x - 4 = 0 &\quad \text{or} \quad x + 2 = 0 \\ x = 4 &\quad \text{or} \quad x = -2 \end{aligned}$$

The x-intercepts of $\mathcal{P}$ are 4 and -2.

(soln cont'd) (*b*) Put $x = 0$ into the given equation:

$$y = \frac{1}{2} \cdot 0^2 - 0 - 4 = -4$$

The y-intercept of the graph is -4.



(*c*) By (a), the points $A(4, 0)$ and $B(-2, 0)$ belong to $\mathcal{P}$. The midpoint of A and B is $M(1, 0)$. Since the axis of symmetry of $\mathcal{P}$ is the vertical line that passes through M, it follows that $x = 1$ is an equation of the axis of symmetry.

(*d*) By (c), the x-coordinate of the vertex is 1.

Put $x = 1$ into the given equation:

$$y = \frac{1}{2} \cdot 1^2 - 1 - 4 = -\frac{9}{2}$$

The vertex is $(1, \frac{9}{2})$. □

Remark Suppose that the parabola given by $y = ax^2 + bx + c$ has two x-intercepts. Then the average of the two x-intercepts is the x-coordinate of the vertex of the parabola.

Proof The two x-intercepts of the parabola are the solutions to the quadratic equation $ax^2+bx+c = 0$. By the quadratic formula, the solutions are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The average of these two solution is

$$\frac{\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}}{2} = \frac{\frac{-b - b}{2a}}{2} = \frac{-b}{2a}$$

which, by Theorem 6.1.5, is the x-coordinate of the vertex of the parabola.

Example 6.1.12 Denote f to be the function given by

$$f(x) = 2x^2 + 4x + 3$$

(*a*) Find the x-intercept(s), if any, of the graph of f.

(*b*) Find the y-intercept of the graph of f.

Solution The graph of f is the graph of the equation $y = f(x)$, that is,

$$y = 2x^2 + 4x + 3$$

(*a*) Put $y = 0$ into the equation $y = f(x)$

$$0 = 2x^2 + 4x + 3$$

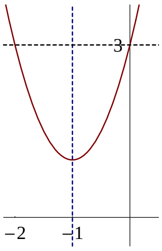
Note that the discriminant $\Delta = 4^2 - 4 \cdot 2 \cdot 3 = -8 < 0$. Thus the above quadratic equation has no solution.

The graph of f does not have any x-intercept.

(soln cont'd) (*b*) Put $x = 0$ into the equation $y = f(x)$

$$y = 2 \cdot 0^2 + 4 \cdot 0 + 3 = 3$$

The y-intercept of the graph of f is 3. □



Remark Although the parabola does not have any x-intercept, we can still use the idea in Example 6.1.11 to find the axis of symmetry.
Putting $y = 3$ into the equation of the parabola, we get $3 = 2x^2 + 4x + 3$. Solving for x, we get $x = 0$ or $x = -2$. The axis of symmetry is the vertical line $x = -1$ (the number -1 is the average of 0 and -2).

Example 6.1.13 Consider the parabola given by the equation

$$y = ax^2 + bx + c$$

where a, b, c are real numbers with $a \neq 0$. Suppose that the x-intercepts of the parabola are 0 and 5 and the point $(1, 4)$ belongs to the parabola. Find the values of a, b and c.

Solution Since the points $(0, 0)$, $(5, 0)$ and $(1, 4)$ belong to the parabola, it follows that

$$\begin{aligned} 0 &= a \cdot 0^2 + b \cdot 0 + c \\ 0 &= a \cdot 5^2 + b \cdot 5 + c \\ 4 &= a \cdot 1^2 + b \cdot 1 + c \end{aligned}$$

From the first equation, we get $c = 0$.
With $c = 0$, the second and third equations reduce to

$$\begin{aligned} 25a + 5b &= 0 \\ a + b &= 4 \end{aligned}$$

Solving (using elimination or substitution), we get $a = -1$ and $b = 5$. □

Example 6.1.14 Consider the graph of the function f given by

$$f(x) = ax^2 + bx + c$$

where a, b, c are real numbers with $a \neq 0$. Suppose that the y-intercept of the graph is 12 and that the graph has only one x-intercept, namely, 2. Find the values of a, b and c.

Explanation Since the graph is a parabola and it has only one x-intercept, it follows that the vertex of the parabola is on the x-axis.

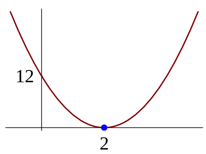


Figure 6.1.46

Solution 1 The graph of the function is the parabola given by the equation

$$y = ax^2 + bx + c$$

Since the y-intercept of the parabola is 12, it follows that

$$12 = a \cdot 0^2 + b \cdot 0 + c$$

which yields $c = 12$.

Since the number 2 is the only x-intercept of the parabola, it follows that the vertex of the parabola is the point $(2, 0)$. Hence

$$0 = a \cdot 2^2 + b \cdot 2 + c$$

which, with $c = 12$, reduces to

$$4a + 2b + 12 = 0 \tag{6.1.5}$$

Moreover, by the formula for x-coordinate of vertex, we get

$$2 = -\frac{b}{2a}$$

which can be written as

$$4a + b = 0 \tag{6.1.6}$$

Solving Equations (6.1.5) and (6.1.6), we get $b = -12$ and $a = 3$. □

Solution 2 The graph of the function is the parabola given by the equation $y = ax^2+bx+c$ which can be written in the form

$$y = a(x - h)^2 + k \tag{6.1.7}$$

Since the number 2 is the only x-intercept of the parabola, it follows that the vertex of the parabola is the point $(2, 0)$. Thus $h = 2$ and $k = 0$ and so Equation (6.1.7) reduces to

$$y = a(x - 2)^2$$

Since the y-intercept of the parabola is 12, it follows that

$$12 = a \cdot (0 - 2)^2$$

which yields $a = 3$. Thus Equation (6.1.7) can be written as

$$\begin{aligned} y &= 3(x - 2)^2 \\ y &= 3(x^2 - 4x + 4) \\ y &= 3x^2 - 12x + 12 \end{aligned}$$

Therefore, $a = 3$, $b = -12$ and $c = 12$. □

Remark Suppose a parabola $\mathcal{P}$ intersects a line ℓ at exactly one point A. Then we say that $\mathcal{P}$ *touches* ℓ at A. For the parabola consider in the example, it touches the x-axis at the point $(2, 0)$.

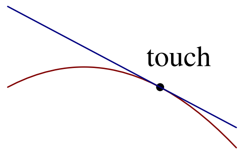


Figure 6.1.47

Remark Suppose that the parabola given by $y = ax^2 + bx + c$ has one and only one x-intercept. Then the x-coordinate of the vertex is the x-intercept and the y-coordinate of the vertex is 0.

Exercise 6.1.3

1. For each of the following, use a computer software to sketch the graphs of the given equations on the same rectangular coordinate plane (choose appropriate values of x and y).

   (a) $y = x^2$, $y = x^2 - 4$, $y = x^2 + 3x$, $y = x^2 + 3x - 4$

   (b) $y = 3x^2$, $y = 3x^2 + 5$, $y = 3x^2 - 2x$, $y = 3x^2 - 2x + 5$

   (c) $y = -2x^2$, $y = -2x^2 - 3$, $y = -2x^2 + x$, $y = -2x^2 + x - 3$

   (d) $y = -x^2$, $y = 2 - x^2$, $y = 3x - x^2$, $y = 2 + 3x - x^2$

2. For each of the following, determine whether the graph of the given equation opens upward or downward.

   (a) $y = 3x^2 - 4x + 5$ (b) $y = -x^2 + x - 3$

   (c) $y = 7x + 2x^2$ (d) $y = 3x - 4 - 2x^2$

3. For each of the following, rewrite the give equation in the form $y = a(x - h)^2 + k$.

   (a) $y = x^2 - 6x$ (b) $y = x^2 - 6x + 13$

   (c) $y = x^2 + 4x$ (d) $x^2 + 4x - 7$

   (e) $y = x^2 + 2x - 5$ (f) $y = x^2 - 10x + 3$

   (g) $y = -x^2 + 2x$ (h) $y = -x^2 + 2x + 3$

   (i) $y = -x^2 - 4x + 3$ (j) $y = 7 - 4x - x^2$

   (k) $y = 2x^2 + 4x$ (l) $y = 2x^2 + 4x + 3$

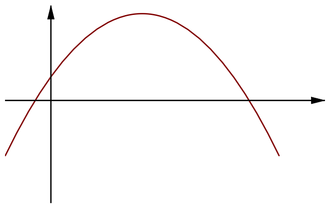
   (m) $y = 3x^2 - 6x + 1$ (n) $y = 12x - 3x^2$

4. For each of the following, find the vertex of the graph of the given equation and determine whether the vertex is the highest or lowest point on the graph.

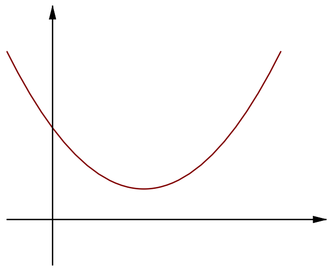
   (a) $y = 3x^2 + 4$
   (b) $y = 5 - 2x^2$
   (c) $y = x^2 - 12x + 13$
   (d) $y = -x^2 + 4x$
   (e) $y = 3x - x^2$
   (f) $y = 2x^2 + 8x + 7$
   (g) $y = 2x^2 - 6x + 7$
   (h) $y = (2 + x)(3 - x)$

5. In each of the following figures, the parabola is the graph of $y = ax^2 + bx + c$. Determine whether the values of a, b and c are positive, negative or 0.

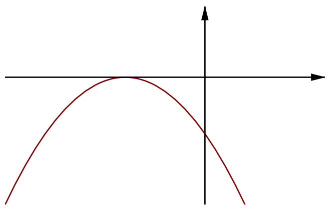
   (a)



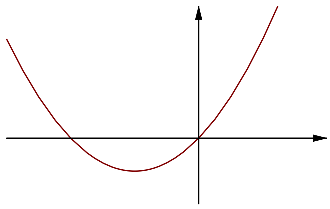
   (b)



   (c)



   (d)



6. Consider the graph of the equation $y = x^2 + 4x - 5$.

   (a) Find the x-intercept(s), if any, of the graph.
   (b) Find the y-intercept of the graph.
   (c) Find the vertex of the graph.
   (d) Sketch the graph for $-5 \leq x \leq 2$.

7. Consider the parabola given by the equation $y = 2x^2 - 3x + 7$.

   (a) Find the x-intercept(s), if any, of the parabola.
   (b) Find the y-intercept of the parabola.
   (c) Denote ℓ to be the horizontal line that intersects the parabola at the y-axis. Find the points of intersection of ℓ and the parabola.
   (d) Use the result in (c) to find an equation of the axis of symmetry of the parabola.
   (e) Find the vertex of the parabola.
   (f) Sketch the parabola for $-1 \leq x \leq 3$.

8. Suppose the point $(-1, 2)$ belongs to the graph of the equation $y = ax^2 - 7x + 8$. Find the value of a.

9. Suppose the x-intercepts of the graph of the equation $y = -x^2 + bx + c$ are -3 and 7. Find the values of b and c.
10. Consider the parabola given by the equation $y = ax^2 + bx$, where a and b are real numbers with $a \neq 0$. Suppose that the point $(1, 3)$ belongs to the parabola and that the vertical line with equation $4x + 1 = 0$ is the axis of symmetry of the parabola. Find the values of a and b.
11. Consider the parabola given by the equation $y = 2x^2 + bx + c$, where b and c are real numbers. Suppose that the point $(-1, 5)$ is the vertex of the parabola. Find the values of b and c.
12. Consider the parabola given by the equation $y = ax^2 + bx + c$, where a, b and c are real numbers with $a \neq 0$. Suppose that the point $(3, -2)$ is the vertex of the parabola and that -2 is an x-intercept of the parabola. Find the values of a, b and c.
13. Consider the parabola given by the equation $y = ax - x^2$, where a is a non-zero real number. Suppose that the axis of symmetry is the vertical line with equation $x = 5$.
    (a) Find the value of a.
    (b) Find the vertex of the graph.

6.2 Graphical Method for Solving Quadratic Equations

Given a quadratic equation

$$ax^2 + bx + c = 0 \tag{6.2.1}$$

where a, b, c are real numbers with $a \neq 0$, the expression ax^2+bx+c can be considered as a function, say denoted by f. To solve Equation (6.2.1) means to find the value(s) of x, if there is any, such that $f(x) = 0$, that is, to find the x-intercept(s), if there is any, of the graph of f.

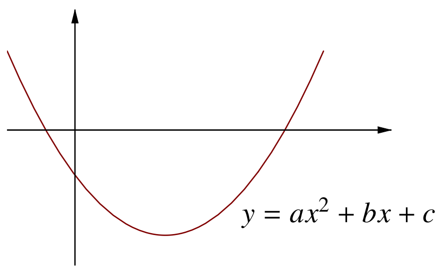


Figure 6.2.1

Thus, to solve Equation (6.2.1), we can sketch the graph of

$$y = ax^2 + bx + c$$

and look for the x-coordinate(s) of the point(s) of intersection of the graph and the x-axis. This method is called the *graphical method*.

Example 6.2.1 Figure 6.2.2 shows the graph of the equation $y = x^2 + 2x - 3$. Note that the x-intercepts of the graph are -3 and 1. Thus the solutions to the quadratic equation $x^2 + 2x - 3 = 0$ are -3 and 1.

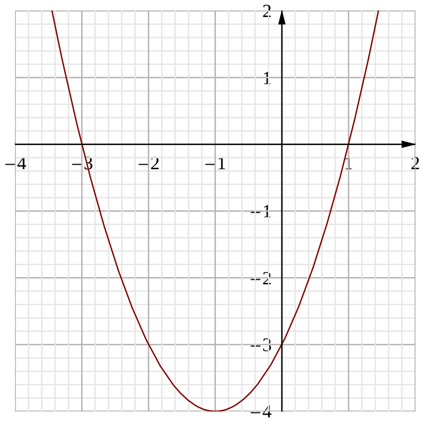


Figure 6.2.2

Remark We can solve the quadratic equation $x^2+2x-3 = 0$ by factorization: $(x+3)(x-1) = 0$.

Example 6.2.2 Figure 6.2.3 shows the graph of the equation $y = x^2 - 6x + 9$. Note that the x-intercept of the graph is 3. Thus the solution to the quadratic equation $x^2 - 6x + 9 = 0$ is 3.

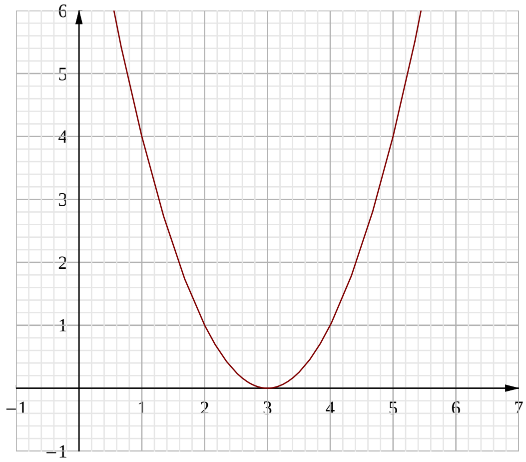


Figure 6.2.3

Remark We can solve the quadratic equation $x^2 - 6x + 9 = 0$ by factorization: $(x - 3)^2 = 0$.

Example 6.2.3 Figure 6.2.4 shows the graph of the equation $y = x^2 - 2x + 3$. Note that the graph lies entirely above the x-axis; it does not have any x-intercept. Thus the quadratic equation $x^2 - 2x + 3 = 0$ has no solution.

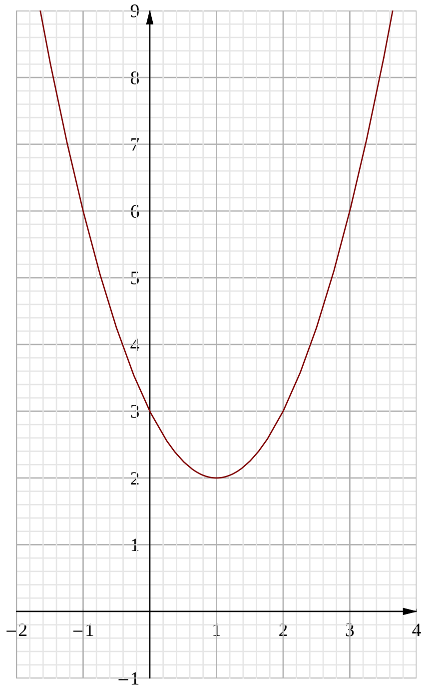


Figure 6.2.4

Remark We can see that the quadratic equation $x^2 - 2x + 3 = 0$ has no solution by considering the discriminant: $\Delta = (-2)^2 - 4 \cdot 1 \cdot 3 = -8 < 0$.

There are several variants of the graphical method for solving quadratic equations. Besides looking for the x-intercepts of a given parabola, we can also look for

- the x-coordinate(s) of the points on the parabola at which the y-coordinate equal to an appropriate value (see Example 6.2.4);
- the x-coordinate(s) of the points of intersection of the parabola and an appropriate straight line (see Example 6.2.5 and Example 6.2.6).

Example 6.2.4 Figure 6.2.5 shows the graph of $y = 4 - 5x - 2x^2$.

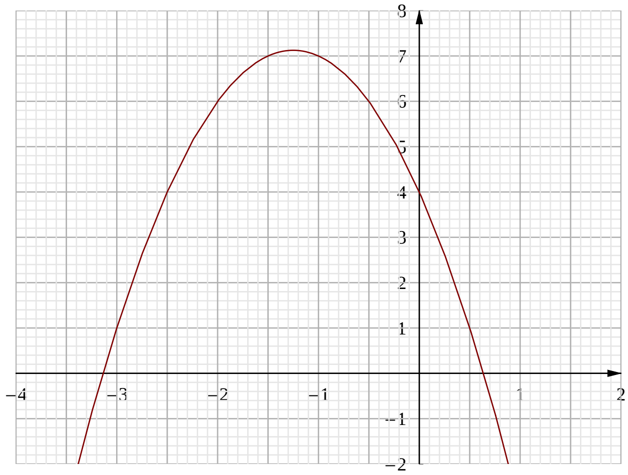


Figure 6.2.5

Solve the quadratic equation $2x^2 + 5x - 1 = 0$ graphically; give your answer correct to 1 decimal place.

Explanation The quadratic equation can be written in the form $4-5x-2x^2 = k$. The solutions can be found by looking for points on the graph whose y-coordinates are equal to k.

Solution The quadratic equation $2x^2 + 5x - 1 = 0$ can be written as

$$\begin{aligned} -1 &= -2x^2 - 5x && \text{Rearrange terms} \\ -1 + 4 &= -2x^2 - 5x + 4 && \text{Add the same number to both sides} \end{aligned}$$

that is, $3 = 4 - 5x - 2x^2$.

From Figure 6.2.5, we see that there are two points on the given graph having y-coordinates equal to 3. The x-coordinates of the two points are −2.7 and 0.2 (correct to 1 decimal place).

The required solutions are −2.7 and 0.2 (correct to 1 decimal place). □

Remark We can solve the given quadratic equation by the quadratic formula:

$$x = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \frac{-5 \pm \sqrt{33}}{4}$$

Example 6.2.5 Figure 6.2.6 shows the graph of $y = x^2$. By adding a suitable line to the figure, find the solutions to the quadratic equation $x^2 + x - 2 = 0$.

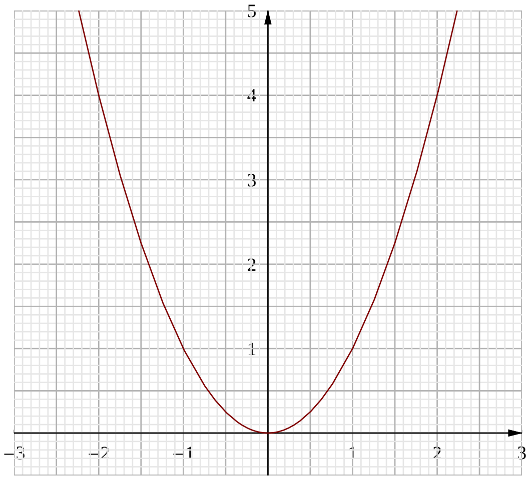
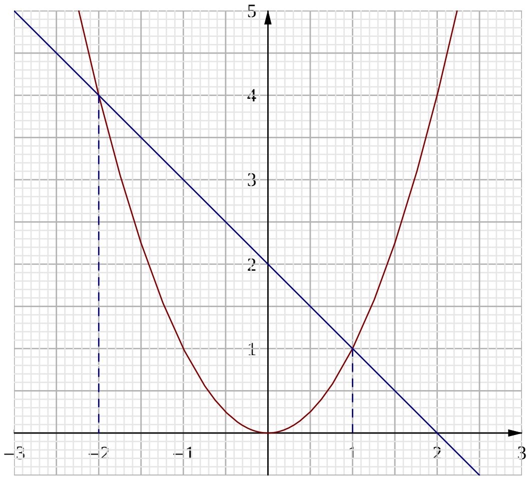


Figure 6.2.6

Explanation The quadratic equation can be written in the form $x^2 = px + q$. The solutions can be found by looking for points of intersection of the given graph and the line with equation $y = px + q$.

Solution The equation $x^2 + x - 2 = 0$ can be written as $x^2 = 2 - x$. In the figure, add the line with equation $y = 2 - x$.



There are two points of intersection of the parabola and the line. The x-coordinates of the two points are −2 and 1.

The required solutions are −2 and 1. □

Example 6.2.6 Figure 6.2.7 shows the graph of $y = x^2$. By adding a suitable line to the figure, find the solutions to the quadratic equation $2x^2 - 3x - 5 = 0$.

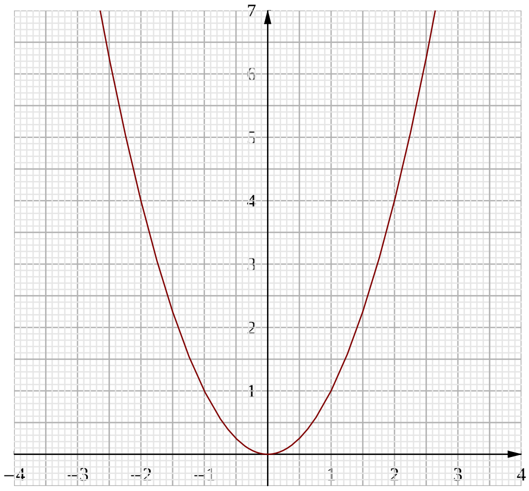
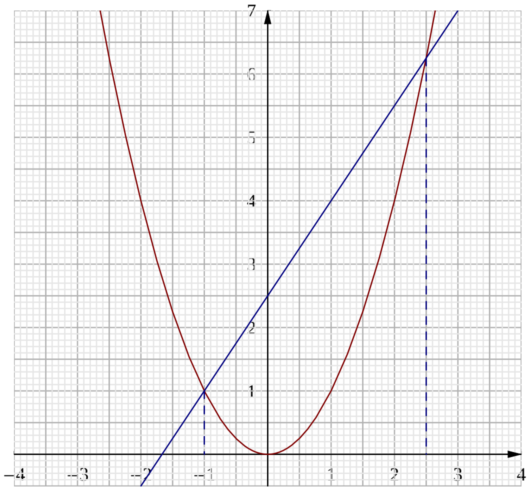


Figure 6.2.7

Solution The quadratic equation $2x^2 - 3x - 5 = 0$ can be written as $2x^2 = 3x + 5$, or equivalently, as $x^2 = \frac{3}{2}x + \frac{5}{2}$. In the figure, add the line with equation $y = \frac{3}{2}x + \frac{5}{2}$.



There are two points of intersection of the parabola and the line. The x-coordinates of the two points are -1 and 2.5.

The required solutions are -1 and 2.5. □

Remark Solutions read from graphs are approximations. Thus we write 2.5 instead of $\frac{5}{2}$.

Remark For quadratic equations, since we can always find the solutions (if there is any) using quadratic formula, the graphical method is not recommended. For other types of equations, there may not be any formula for the solutions. In that case, we may use the graphical method. However, it should be pointed out that solutions thus obtained are approximations only and the precisions are rather low (one or two significant figures). To obtain solutions "quickly" and "accurately", in many branches of mathematics, various methods for different types of equations have been studied.

To close this section, we consider a few examples on the number of x-intercepts of the graph of $y = ax^2 + bx + c$. This is related to the discriminant of the quadratic equation $ax^2 + bx + c = 0$.

Example 6.2.7 Suppose that the graph of the equation $y = 2x^2 + 5x + c$ intersects the x-axis at two points. Find the possible values of c.

Solution The graph of $y = 2x^2 + 5x + c$ intersects the x-axis at two points is equivalent to that the quadratic equation

$$2x^2 + 5x + c = 0$$

has two solutions:

$$\begin{aligned} 5^2 - 4 \cdot 2 \cdot c &> 0 \qquad \text{Discriminant} > 0 \\ 25 &> 8c \end{aligned}$$

Thus $c < \frac{25}{8}$. □

Remark Sometimes, instead of asking for 'possible values of c', the question may ask in the following way: '*Find the range of values of c.*', which implicitly tells that the values of c can be described by inequalities like

$$c > \text{a number} \qquad \text{a number} < c \leq \text{another number}$$

etc. Later, in Section 6.3.2, we will introduce the concept of the *range of a function*. Note that the meanings of the word '*range*' used in the two situations are completely different.

Example 6.2.8 Suppose that the entire graph of the equation $y = ax^2 + 2x + 5$, where a is a non-zero real number, is above the x-axis. Find the possible values of a.

Solution The entire graph of $y = ax^2 + 2x + 5$ is above the x-axis means that the graph opens upward and does not intersect the x-axis, which is equivalent to that $a > 0$ and the quadratic equation $ax^2 + 2x + 5 = 0$ has no solution:

$$\begin{aligned} a &> 0 \quad \text{and} \quad 2^2 - 4 \cdot a \cdot 5 < 0 \qquad \text{Discriminant} < 0 \\ a &> 0 \quad \text{and} \quad 4 < 20a \end{aligned}$$

Thus $a > \frac{1}{5}$. □

Example 6.2.9 Consider the parabola given by $y = x^2 - x - 3$ and the line given by $y = 2x + k$, where k is a real number. Suppose that the parabola and the line intersect at exactly one point. Find the value of k.

Solution The parabola and the line intersect at exactly one point is equivalent to that the quadratic equation $x^2 - x - 3 = 2x + k$ has only one solution, that is, the quadratic equation

$$x^2 - 3x - 3 - k = 0$$

has only one solution:

$$\begin{aligned} (-3)^2 - 4 \cdot 1 \cdot (-3 - k) &= 0 \qquad \text{Discriminant} = 0 \\ 9 + 12 + 4k &= 0 \end{aligned}$$

Thus, $k = -\frac{21}{4}$ □

Exercise 6.2

1. Figure 6.2.8 shows the graph of the equation $y = x^2 - 3x - 2$. For each of the following, solve the given quadratic equation graphically; give your answer correct to 1 decimal place.

   (*a*) $x^2 - 3x - 2 = 0$ (*b*) $x^2 - 3x - 4 = 0$

   (*c*) $x^2 - 3x + 1 = 0$ (*d*) $x^2 - 3x + 4 = 0$

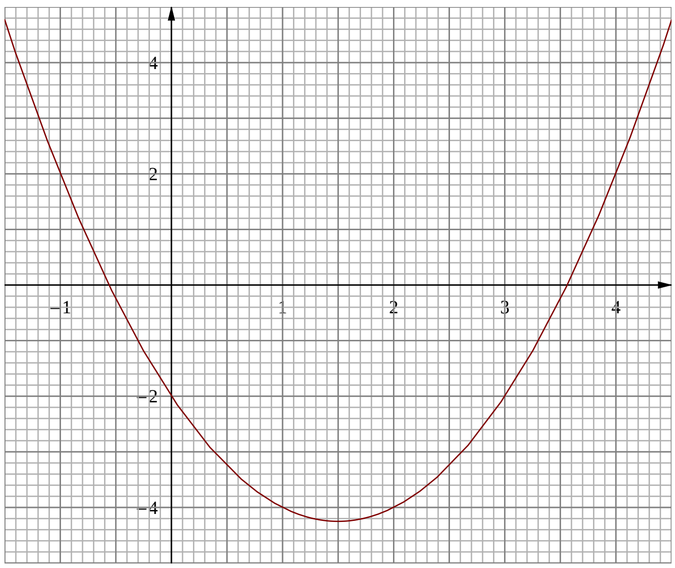


Figure 6.2.8

2. Figure 6.2.9 shows the graph of the equation $y = -4x^2 + 4x - 1$. For each of the following, solve the given quadratic equation graphically; give your answer correct to 1 decimal place.

   (a) $4x^2 - 4x + 1 = 0$ (b) $4x^2 - 4x + 3 = 0$

   (c) $4x^2 - 4x + 5 = 0$ (d) $x^2 - x - 2 = 0$

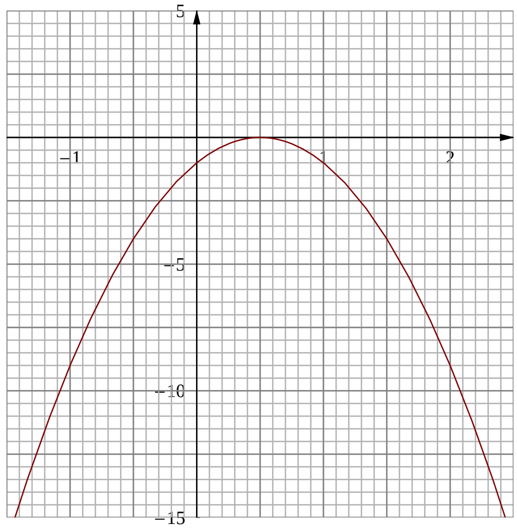


Figure 6.2.9

3. Figure 6.2.10 shows the graph of the equation $y = x^2 + 2x$. For each of the following, solve the given quadratic equation by drawing a suitable line; give your answer correct to 1 decimal place.

   (a) $x^2 + 2x - 1 = 0$ (b) $x^2 + x - 3 = 0$

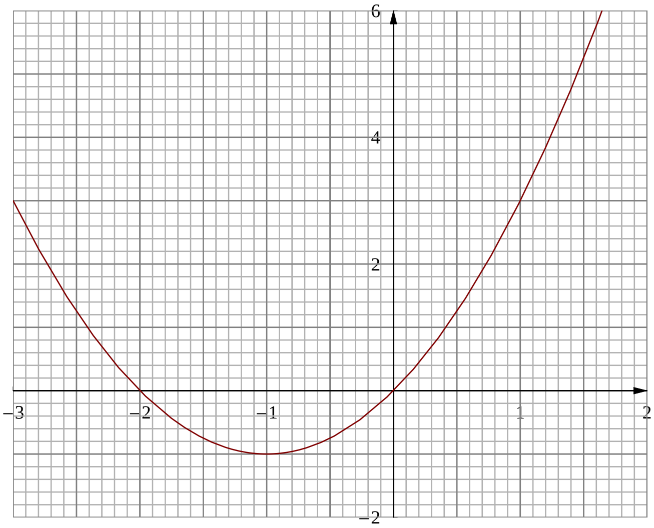


Figure 6.2.10

4. Suppose that the graph of the equation $y = 2x^2 - 3x - 4k$ intersects the x-axis at two points. Find the possible values of k.

5. Suppose that the graph of the parabola with equation $y = 2x^2 + (k - 3)x + 8$ intersects the x-axis at only one point. Find the (possible) value(s) of k.

6. Suppose that the graph of the equation $y = 3x^2 - 2x - (k + 5)$ intersects the horizontal line with equation $x + 3 = 0$ at two points. Find the possible values of k.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = ax^2 + 3x - 5$, where a is a real number.
   (a) Suppose that the graph of f does not intersect the x-axis. Find the possible values of a.
   (b) Suppose that the graph of f intersects the x-axis at exactly one point. Find the possible values of a.
   Note: There are two possible values.

6.3 Maxima and Minima of Quadratic Functions

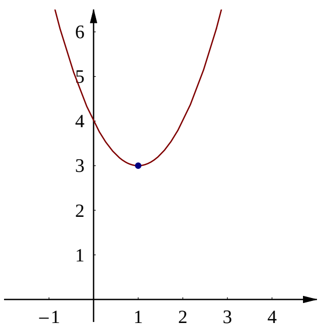
Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = x^2 - 2x + 4$$

The graph of f, which is shown in Figure 6.3.1, is the graph of the equation

$$y = x^2 - 2x + 4$$

Note that the vertex of the graph of f is $(1, 3)$ and it is the lowest point on the graph.



In terms of values under the function f, we have

- $f(1) = 3$
- For every real number x, the value $f(x)$ is at least 3.

We say that 3 is the minimum (value) of f and that the minimum is attained at 1.

There are two concepts concerning maximum and minimum: one is for *sets of numbers* and the other for *functions* (with codomain $\mathbb{R}$).

- In Section 6.3.1, we will consider the concept of maximum and minimum of a subset of $\mathbb{R}$.
- In Section 6.3.2, we will consider the concept of *range* of a function.
- In Section 6.3.3, we will consider the concept of maximum and minimum of a function, using the concepts introduced in Section 6.3.1 and Section 6.3.2.

6.3.1 Maxima and Minima of Sets of Numbers

Definition 6.3.1 Let S be a subset of $\mathbb{R}$. We call

- ***the maximum of S*** to mean the element of S, denoted by $\max S$, such that the following condition is satisfied: for every $s \in S$, $\max S \geq s$;
- ***the minimum of S*** to mean the element of S, denoted by $\min S$, such that the following condition is satisfied: for every $s \in S$, $\min S \leq s$.

Explanation In other words, the maximum (respectively minimum) of a subset of $\mathbb{R}$ means the largest (respectively smallest) number in the set.

Remark The plurals of maximum and minimum are maxima and minima respectively.

Given a subset S of $\mathbb{R}$, if S is a non-empty finite set, then its maximum and minimum always exist; if S is an infinite set, it may happen that its maximum (or minimum) does not exist (see Example 6.3.2 and Example 6.3.3).

Example 6.3.1 Denote $S = \{5, 3, 12, 17, 9\}$. Then

- $\max S = 17$

  Reason $17 \in S$ and for every $s \in S$, $17 \geq s$.
- $\min S = 3$

  Reason $3 \in S$ and for every $s \in S$, $3 \leq s$.

Example 6.3.2 For the set $\mathbb{Z}^+$ (the set of all positive integers), we have the following:

- $\min \mathbb{Z}^+ = 1$

  Reason $1 \in \mathbb{Z}^+$ and for every $n \in \mathbb{Z}^+$, $1 \leq n$.
- $\max \mathbb{Z}^+$ does not exist.

  Meaning There does not exist any element β of $\mathbb{Z}^+$ satisfying the condition: for every $n \in \mathbb{Z}^+$, $\beta \geq n$.

Explanation In other words, 1 is the smallest number in $\mathbb{Z}^+$ but there is no largest number in $\mathbb{Z}^+$.

Example 6.3.3 Denote A to be the set of all negative real numbers, that is, $A = \{x \in \mathbb{R} : x < 0\}$. Then

- $\max A$ does not exist.

  Meaning There does not exist any element β of A satisfying the condition: for every $x \in A$, $\beta \geq x$.

- $\min A$ does not exist.

  Meaning There does not exist any element α of A satisfying the condition: for every $x \in A$, $\alpha \leq x$.

Explanation In other words, there is no largest number and no smallest number in A.

Remark The number 0 is not the maximum of A; it is not an element of A.

Exercise 6.3.1

1. For each of the following subsets of $\mathbb{R}$, find its maximum and minimum (if exist).

   (a) $\{1, 3, 5, 7, 9\}$ (b) $\{2, 4, 6, 8, 10, \ldots\}$
   (c) $\{s \in \mathbb{R} : 1 \leq s \leq 2\}$ (d) $\{s \in \mathbb{R} : -5 < s < 6\}$
   (e) $\{s \in \mathbb{R} : 0 < s \leq 1\}$ (f) $\{s \in \mathbb{R} : 0 \leq s < 1\}$
   (g) $\{s \in \mathbb{R} : s \geq -3\}$ (h) $\{s \in \mathbb{R} : s \leq 4\}$
   (i) $\{s \in \mathbb{R} : s > -3\}$ (j) $\{s \in \mathbb{R} : s < 4\}$
   (k) $\{s \in \mathbb{R} : s - 2 \geq 0\}$ (l) $\{s \in \mathbb{R} : s + 3 \leq 0\}$
   (m) $\{s \in \mathbb{R} : s - 2 > 0\}$ (n) $\{s \in \mathbb{R} : s + 3 < 0\}$

2. For each of the following, find a subset S of $\mathbb{R}$ satisfying the given conditions:

   (a) $\max S = 1$ and $\min S = 0$.
   (b) $\max S = 1$ and $\min S$ does not exist.
   (c) $\max S$ does not exist and $\min S = 0$.
   (d) $1 \in S$ and $0 \notin S$ and $\max S$ and $\min S$ do not exist.

3. Suppose S is a subset of $\mathbb{R}$ and $\max S = \min S$. What can you tell about S?

6.3.2 Ranges of Functions

Definition 6.3.2 Let f be a function from a set X to a set Y. We call ***the range of f*** to mean the following subset of Y:

$$\{b \in Y : \text{there exists } a \in X \text{ such that } f(a) = b\}$$

Explanation The set $\{b \in Y : \text{there exists } a \in X \text{ such that } f(a) = b\}$ consists of elements of Y that are images under f of some elements (belonging to the domain of f). In other words, the range of f is the set of all "outputs".

Example 6.3.4 Denote $X = \{1, 2, 3, 4, 5\}$. Let $f : X \to \mathbb{R}$ be the function given by

$$f(x) = x^2$$

Then we have

$$f(1) = 1, \quad f(2) = 4, \quad f(3) = 9, \quad f(4) = 16, \quad f(5) = 25$$

The range of f is $\{1, 4, 9, 16, 25\}$.

Example 6.3.5 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = x^2 + 1$$

The graph of f is shown in Figure 6.3.2.

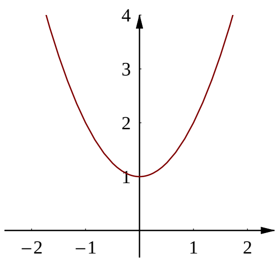


Figure 6.3.2

The range of f is $\{y \in \mathbb{R} : y \geq 1\}$. This is because

- if $t \geq 1$, then there exists (at least) a real number s such that $s^2 + 1 = t$;
- if $t < 1$, then there does not exist any real number s such that $s^2 + 1 = t$.

Remark If we project the graph of f onto the y-axis, we get a half-line (with endpoint included). By considering the y-axis as the real number line, the half-line consists of all real numbers y such that $y \geq 1$; in other words, it is the range of f.

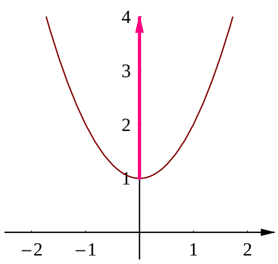


Figure 6.3.3

Example 6.3.6 Denote $X = \{x \in \mathbb{R} : 1 < x < 5\}$. Let $f : X \to \mathbb{R}$ be the function given by

$$f(x) = \frac{1}{2}(x - 1)$$

The graph of f is shown in Figure 6.3.4. It is a line segment with two endpoints excluded (indicated by two little circles).

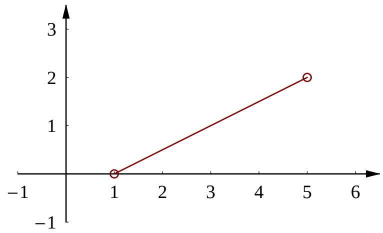


Figure 6.3.4

The range of f is $\{y \in \mathbb{R} : 0 < y < 2\}$. This is because

- if $0 < t < 2$, then there exists a real number s with $1 < s < 5$ such that $\frac{1}{2}(s - 1) = t$;
- if $t \leq 0$ or $t \geq 2$, then there does not exist any real number s with $1 < s < 5$ such that $\frac{1}{2}(s - 1) = t$.

Remark If we project the graph of f onto the y-axis, we get a line segment (with endpoints excluded). By considering the y-axis as the real number line, the line segment consists of all real numbers y such that $0 < y < 2$.

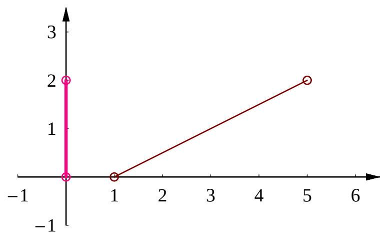


Figure 6.3.5

Exercise 6.3.2

1. For each of the following, find the range of the given function f.

   (a) $f : X \to \mathbb{R}$ given by $f(x) = 2x + 1$, where $X = \{1, 3, 5, 7, 9\}$.

   (b) $f : X \to \mathbb{R}$ given by $f(x) = 3x - 2$, where $X = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$.

   (c) $f : X \to \mathbb{R}$ given by $f(x) = x^2$, where $X = \{x \in \mathbb{R} : 0 < x < 1\}$.

   (d) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 1 - x^2$.

   (e) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 7x + 8$.

6.3.3 Maxima and Minima of Functions

For a function f whose codomain is $\mathbb{R}$, the range of f is a subset of $\mathbb{R}$. We define the maximum (respectively minimum) of f to be the maximum (respectively minimum) of its range.

Definition 6.3.3 Let f be a function with codomain $\mathbb{R}$. We call

- ***the maximum of f*** to mean the maximum of the range of f;
- ***the minimum of f*** to mean the minimum of the range of f.

Remark Sometimes, to emphasize that the maximum and minimum of f are output numbers, we say the *maximum value* and *minimum value* of f.

Given a function $f : X \to \mathbb{R}$, if X is a non-empty finite set, then the maximum and minimum of f always exist; if X is an infinite set, it may happen that maximum (or minimum) of f does not exist (see Example 6.3.8 and Example 6.3.9).

Example 6.3.7 Denote $X = \{1, 2, 3, 4, 5\}$. Let $f : X \to \mathbb{R}$ be the function given by

$$f(x) = x^2$$

By Example 6.3.4, the range of f is $\{1, 4, 9, 16, 25\}$. Thus

- the maximum of f is 25;
- the minimum of f is 1.

Example 6.3.8 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = x^2 + 1$$

By Example 6.3.5, the range of f is $\{y \in \mathbb{R} : y \geq 1\}$. Thus

- the maximum of f does not exist;
- the minimum of f is 1.

Meaning In the range of f, 1 is the smallest number but there is no largest number .

Example 6.3.9 Denote $X = \{x \in \mathbb{R} : 1 < x < 5\}$. Let $f : X \to \mathbb{R}$ be the function given by

$$f(x) = \frac{1}{2}(x - 1)$$

By Example 6.3.6, the range of f is $\{y \in \mathbb{R} : 0 < y < 2\}$. Thus

- the maximum and minimum of f do not exist.

Meaning In the range of f, there is no largest number and no smallest number.

Remark Note that the numbers 0 and 2 do not belong to the range of f.

For problems concerning the maximum values (or minimum values) of functions, very often, we are also interested in the elements at which the functions attains their maxima (or minima).

Definition 6.3.4 Let $f : X \to \mathbb{R}$ be a function. Let $a \in X$. We say

- ***f attains its maximum at a*** to mean $f(a)$ = the maximum of f.
- ***f attains its minimum at a*** to mean $f(a)$ = the minimum of f.

To find where a function attains its maximum or minimum (if there is any), one way is to find the maximum or minimum value first and then look for element(s) in the domain whose image(s) is/are equal to that value. Alternatively, we can use the following simple results (see the remarks to Examples 6.3.10 and 6.3.11).

- The condition '$f(a)$ = the maximum of f' is equivalent that

$$\text{for every } x \in X, \ f(a) \geq f(x)$$

- The condition '$f(a)$ = the minimum of f' is equivalent to that

$$\text{for every } x \in X, \ f(a) \leq f(x)$$

Example 6.3.10 Denote $X = \{1, 2, 3, 4, 5\}$. Let $f : X \to \mathbb{R}$ be the function given by

$$f(x) = x^2$$

Then the maximum and minimum values of f are 25 and 1 respectively (see Example 6.3.7). Moreover,

- f attains its maximum at 5;
- f attains its minimum at 1.

Remark *Alternatively*, note that

$$\text{for every } x \in X, \quad f(1) \leq f(x) \leq f(5)$$

Thus f attains its minimum at 1 and attains its maximum at 5.

Example 6.3.11 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = x^2 + 1$$

Then minimum of f is 1 (see Example 6.3.8). Moreover,

- f attains its minimum at 0.

Remark *Alternatively*, note that

$$\text{for every } x \in \mathbb{R}, \quad f(0) \le f(x)$$

Thus f attains its minimum at 0.

Remark Maximum of f does not exist (we also say that f does not attain its maximum).

If the maximum (or minimum) of a function exists, then it is unique. However, there may exist more than one elements in the domain at which the function attains its maximum (or minimum). The following is such an example.

Example 6.3.12 Denote $X = \{1, 3, 5, 7\}$ and let $f : X \to \mathbb{R}$ be the function given by

$$f(x) = (x - 4)^2$$

Then we have

$$f(1) = 9, \quad f(3) = 1, \quad f(5) = 1, \quad f(7) = 9$$

From the above values, we see that

- the maximum of f is 9 and f attains its maximum at 1 and also at 7;
- the minimum of f is 1 and f attains its minimum at 3 and also at 5.

Exercise 6.3.3

1. Let $f : X \to \mathbb{R}$ be the function given by $f(x) = 2x + 3$, where $X = \{1, 2, 3, 4, 5\}$.
   - (a) Write down the range of f.
   - (b) Determine whether the maximum and minimum of f exist or not and write down the value(s) that exist(s).
   - (c) At which element(s) in X, if there is any, does f attains its maximum? its minimum?
2. Let $f : X \to \mathbb{R}$ be the function given by $f(x) = x^2$, where $X = \{-2, -1, 0, 1, 2\}$.
   - (a) Write down the range of f.
   - (b) Determine whether the maximum and minimum of f exist or not and write down the value(s) that exist(s).
   - (c) At which element(s) in X, if there is any, does f attains its maximum? its minimum?

3. Let $f : X \to \mathbb{R}$ be the function given by $f(x) = 2 - x$, where $X = \{x \in \mathbb{R} : 0 \le x < 3\}$. The graph of f is shown in Figure 6.3.6.

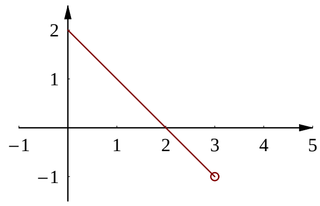


Figure 6.3.6

(*a*) Write down the range of f.

(*b*) Determine whether the maximum and minimum of f exist or not and write down the value(s) that exist(s).

(*c*) At which element(s) in X, if there is any, does f attains its maximum? its minimum?

4. Let $f : X \to \mathbb{R}$ be the function given by $f(x) = x^2 - 2x$, where $X = \{x \in \mathbb{R} : 0 \le x \le 3\}$. The graph of f is shown in Figure 6.3.7.

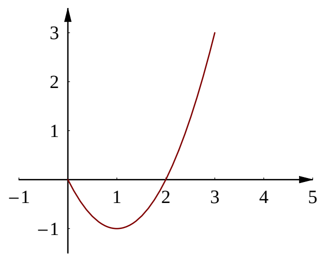


Figure 6.3.7

(*a*) Write down the range of f.

(*b*) Determine whether the maximum and minimum of f exist or not and write down the value(s) that exist(s).

(*c*) At which element(s) in X, if there is any, does f attains its maximum? its minimum?

5. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = x^2 - 2x$. Part of the graph of f is shown in Figure 6.3.8.

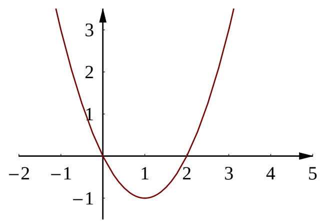


Figure 6.3.8

(*a*) Write down the range of f.

(*b*) Determine whether the maximum and minimum of f exist or not and write down the value(s) that exist(s).

(*c*) At which element(s) in $\mathbb{R}$, if there is any, does f attains its maximum? its minimum?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 3 - 2x - x^2$. Part of the graph of f is shown in Figure 6.3.9.

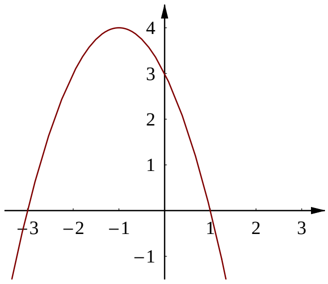


Figure 6.3.9

(a) Write down the range of f.

(b) Determine whether the maximum and minimum of f exist or not and write down the value(s) that exist(s).

(c) At which element(s) in $\mathbb{R}$, if there is any, does f attains its maximum? its minimum?

6.3.4 Maxima and Minima of Quadratic Functions

Let a, b and c be real numbers with $a \neq 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be the quadratic function given by

$$f(x) = ax^2 + bx + c$$

Then we have the following:

- if $a > 0$, then the minimum of f exists but the maximum of f does not exist;
- if $a < 0$, then the maximum of f exists but the minimum of f does not exist.

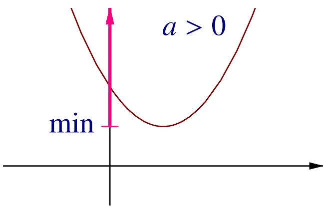


Figure 6.3.10 (a)

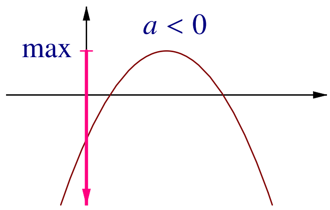


Figure 6.3.10 (b)

In fact, using the complete square method, we can rewrite $f(x)$ in the following form:

$$f(x) = a(x - h)^2 + k$$

Note that $f(h) = a \cdot 0^2 + k = k$ and that

$$\text{for every } x \in \mathbb{R}, \quad f(x) = a \cdot (\text{a non-negative number}) + k \begin{cases} \geq k & \text{if } a > 0 \\ \leq k & \text{if } a < 0 \end{cases}$$

Thus we have the following:

Properties of Quadratic Functions $a(x-h)^2+k$

- If $a>0$, then the function attains its minimum at h,
  the minimum value of the function is k.
- If $a<0$, then the function attains its maximum at h,
  the maximum value of the function is k.

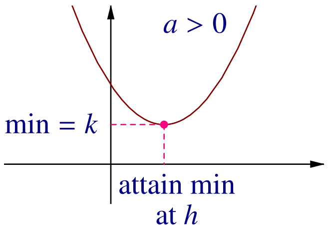


Figure 6.3.11 (a)

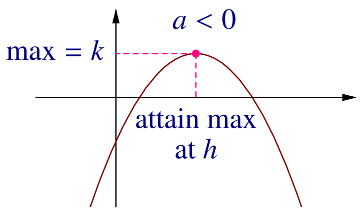


Figure 6.3.11 (b)

Terminology In mathematics, maxima and minima (of functions) are collectively known as *extrema* (singular: *extremum*).

Remark No matter whether $a>0$ or $a<0$, the quadratic function $a(x-h)^2+k$ attains its extremum at h.

The following result is useful in finding where a quadratic function attains its extremum. It is a simple consequence of Theorem 6.1.5.

Theorem 6.3.1 Let a, b and c be real numbers with $a \neq 0$. Let $f:\mathbb{R}\to\mathbb{R}$ be the quadratic function given by

$$f(x)=ax^2+bx+c$$

- If $a>0$, then f attains its minimum at $-\frac{b}{2a}$.
- If $a<0$, then f attains its maximum at $-\frac{b}{2a}$.

Remark The extremum value of f can be found by direct substitution: the value is $f(-\frac{b}{2a})$.

Example 6.3.13 Let f be the function given by

$$f(x) = x^2 + 4x - 9$$

Find the minimum value of f.

Explanation It is understood that f is a function from $\mathbb{R}$ to $\mathbb{R}$.
Since $a = 1 > 0$, it follows that the minimum of f exists.

Solution 1 Rewrite $f(x)$ in the form $a(x-h)^2 + k$:

$$\begin{aligned} f(x) &= x^2 + 4x - 9 \\ &= (x^2 + 4x + 4) - 4 - 9 && \text{Complete square and make } f(x) \text{ unchanged} \\ &= (x+2)^2 - 13 \end{aligned}$$

The minimum value of f is -13. □

Solution 2 With $a = 1$ and $b = 4$, the number $-\dfrac{b}{2a} = -\dfrac{4}{2\cdot 1} = -2$.

The function f attains its minimum at -2. (By Theorem 6.3.1)

Direct substitution gives $f(-2) = (-2)^2 + 4\cdot(-2) - 9 = -13$.

The minimum value of f is -13. □

Example 6.3.14 Let f be the function given by

$$f(x) = 3 + 2x - x^2$$

Find the real number at which f attains its maximum.

Explanation Since $a = -1 < 0$, it follows that the maximum of f exists. The question is to find where the function f attains its maximum.

Solution 1 Rewrite $f(x)$ in the form $a(x-h)^2 + k$:

$$\begin{aligned} f(x) &= 3 + 2x - x^2 \\ &= -(x^2 - 2x) + 3 && \text{Extract factor } a = -1 \\ &= -(x^2 - 2x + 1 - 1) + 3 && \text{Complete square and make } f(x) \text{ unchanged} \\ &= -(x^2 - 2x + 1) + 1 + 3 \\ &= -(x-1)^2 + 4 \end{aligned}$$

The real number at which f attains its maximum is 1. □

Solution 2 Note that $f(x) = -1\cdot x^2 + 2x + 3$.

With $a = -1$ and $b = 2$, the number $-\dfrac{b}{2a} = -\dfrac{2}{2\cdot(-1)} = 1$.

The real number at which f attains its maximum is 1. (By Theorem 6.3.1) □

Example 6.3.15 Consider the function given by

$$y = 2x^2 + 10x + 13$$

Find where the function attains its minimum and find the minimum of the function.

Explanation The function under consideration is described by a relation between the independent variable x and dependent variable y.

Solution 1 Rewrite y in the form $a(x-h)^2 + k$:

$$\begin{aligned} y &= 2x^2 + 10x + 13 \\ &= 2(x^2 + 5x) + 13 && \text{Extract factor } a = 2 \\ &= 2\left(x^2 + 5x + (\tfrac{5}{2})^2 - (\tfrac{5}{2})^2\right) + 13 && \text{Complete square and make } y \text{ unchanged} \\ &= 2\left(x^2 + 5x + (\tfrac{5}{2})^2\right) - 2 \cdot \frac{25}{4} + 13 \\ &= 2\left(x + \frac{5}{2}\right)^2 - \frac{25}{2} + 13 \\ &= 2\left(x - (-\frac{5}{2})\right)^2 + \frac{1}{2} \end{aligned}$$

The function attains its minimum at $-\frac{5}{2}$.

The minimum of the function is $\frac{1}{2}$. □

Solution 2 With $a = 2$ and $b = 10$, the number $-\frac{b}{2a} = -\frac{10}{2 \cdot 2} = -\frac{5}{2}$.

The function attains its minimum at $-\frac{5}{2}$. (By Theorem 6.3.1)

When $x = -\frac{5}{2}$, $y = 2 \cdot \left(-\frac{5}{2}\right)^2 + 10 \cdot \left(-\frac{5}{2}\right) + 13 = \frac{1}{2}$.

The minimum of the function is $\frac{1}{2}$. □

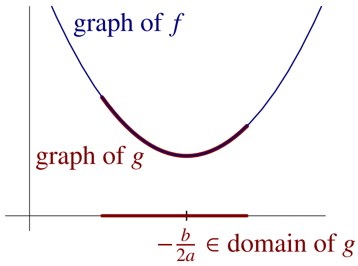
In practical problems, we may have to consider extrema of quadratic functions whose domains are not $\mathbb{R}$. The following variant of Theorem 6.3.1 is useful in this regard (see solution 2 or alternative solution in the examples in Section 6.4).

Theorem 6.3.2 Let a, b and c be real numbers with $a \neq 0$. Let X be a subset of $\mathbb{R}$ such that $-\frac{b}{2a} \in X$. Let $g : X \to \mathbb{R}$ be the quadratic function given by

$$g(x) = ax^2 + bx + c$$

- If $a > 0$, then g attains its minimum at $-\frac{b}{2a}$.
- If $a < 0$, then g attains its maximum at $-\frac{b}{2a}$.

Explanation The results are not difficult to see using the graphs of f and g. The lowest (or highest) point on the graph of f is the lowest (or highest) point on the graph of g.



Below we give a detail proof.

Proof Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = ax^2 + bx + c$$

By Theorem 6.3.1,

(1) if $a > 0$, the function f attains its minimum at $-\frac{b}{2a}$, thus

$$\text{for every } x \in \mathbb{R}, \quad f(x) \geq f(-\tfrac{b}{2a})$$

(2) if $a < 0$, the function f attains its maximum at $-\frac{b}{2a}$, thus

$$\text{for every } x \in \mathbb{R}, \quad f(x) \leq f(-\tfrac{b}{2a})$$

from which we obtain (note that $-\frac{b}{2a}$ belongs to the domain of g)

(1) if $a > 0$, then

$$\text{for every } x \in X, \quad g(x) = f(x) \geq f(-\tfrac{b}{2a}) = g(-\tfrac{b}{2a})$$

and so g attains its minimum at $-\frac{b}{2a}$;

(2) if $a < 0$, then

$$\text{for every } x \in X, \quad g(x) = f(x) \leq f(-\tfrac{b}{2a}) = g(-\tfrac{b}{2a})$$

and so g attains its maximum at $-\frac{b}{2a}$. □

Remark The extremum value of g is $g(-\frac{b}{2a})$.

Exercise 6.3.4

1. For each of the following, find (if there is any) the maximum and minimum of the given function and find where the function attains its maximum and minimum.

   (a) $f(x) = x^2 - 6x - 5$ (b) $f(x) = x^2 + 6x + 7$

   (c) $f(x) = x^2 - 3x + 11$ (d) $f(x) = x^2 + 3x - 3$

   (e) $f(x) = 7 + 4x - x^2$ (f) $f(x) = 8 - 4x - x^2$

   (g) $y = 2x^2 + 8x + 9$ (h) $y = 2x^2 - 10x + 9$

   (i) $y = 3x^2 - 6x - 2$ (j) $y = 3x^2 + 9x$

   (k) $y = 6x - 3x^2$ (l) $y = 4 - 3x - 2x^2$

2. Let f be the function given by $f(x) = -2x^2 + 4x - 5$.

   (a) Express $f(x)$ in the form $a(x - h)^2 + k$.

   (b) Write down the maximum of f. Explain why the value you give is the maximum.

   (c) Write down the real number at which f attains its maximum. Explain why the number you give is the only real number at which f attains its maximum.

3. Consider the function given by $y = x^2 + 6x + c$, where c is a real number. Suppose that the minimum value of the function is 5. Find the value of c.

4. Consider the function given by $y = x^2 + bx + 7$, where b is a real number.

   (a) Suppose that the function attains its minimum value at 13. Find the value of b.

   (b) Suppose that the minimum value of the function is 3. Find the possible values of b.

5. Consider the function given by $y = ax^2 + 6x + 7$, where a is a non-zero real number.

   (a) Suppose that the function attains its minimum at $-\frac{3}{2}$. Find the value of a.

   (b) Suppose that the maximum value of the function is 10. Find the value of a.

6. Let f be the function given by $f(x) = x^2 + kx + (k + 2)$, where k is a real number. Suppose that $f(2) = 18$.

   (a) Find the value of k.

   (b) Find the minimum value of f.

7. Let f be the function given by $f(x) = 3x^2 - 6x + 4$. Show that for every real number a, the value $f(a)$ is positive.

6.4 Problems Leading to Quadratic Functions

In this section, we will consider some practical problems that lead to quadratic functions. Since there are restrictions on the independent variables, the domains of the quadratic functions discussed in the examples are not equal to $\mathbb{R}$ (subsets of $\mathbb{R}$ only). There are two methods to find the extrema of the functions (Solution 1 and Solution 2):

(*Method 1*) Rewrite the functions in the form $a(x-h)^2+k$.

(*Method 2*) Apply Theorem 6.3.2.

Example 6.4.1 An object is thrown vertically upward. After t seconds, the height h (in meters) of the object is given by $h = 2 + 8t - 5t^2$. Find the maximum height of the object.

Explanation The equality $h = 2 + 20t - 5t^2$ describes a function with domain $\{t \in \mathbb{R} : 0 \le t \le t_g\}$, where t_g is the time at which the object hits the ground. To solve the problem, there is no need to know the value of t_g.

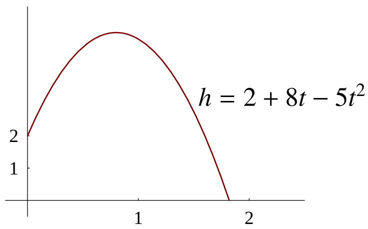


Figure 6.4.1

Solution 1

$$\begin{aligned} h &= 2 + 8t - 5t^2 \\ &= -5(t^2 - 1.6t) + 2 \\ &= -5(t^2 - 1.6t + 0.8^2 - 0.8^2) + 2 \\ &= -5(t^2 - 1.6t + 0.8^2) + 5 \cdot 0.8^2 + 2 \\ &= -5(t - 0.8)^2 + 5.2 \\ &\le 5.2 \qquad \text{for all } t \text{ belonging to the domain of the function} \end{aligned}$$

The maximum height of the object is $5.2\,m$. □

Remark The object reaches its maximum height when $t = 0.8$. To find the time at which the object hits the ground, we can use quadratic formula to solve $2 + 8t - 5t^2 = 0$.

Solution 2 With $a = -5$ and $b = 8$,

$$-\frac{b}{2a} = -\frac{8}{2 \cdot (-5)} = 0.8 \quad \text{(belongs to domain of height function)}$$

The height function attains its maximum at 0.8. (By Theorem 6.3.2)

When $t = 0.8$, $h = 2 + 8 \times 0.8 - 5 \times 0.8^2 = 5.2$.

The maximum height of the object is $5.2\,m$. □

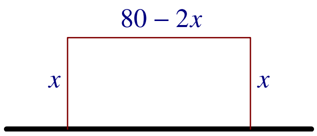
Example 6.4.2 Mr. Chan wants to enclose a rectangular garden in front of a wall, using a fence 80 m long. Denote x to be the length, in m, of a side perpendicular to the wall, and denote A to be the area, in m^2, of the garden.

(a) Express A in terms of x.

(b) What is the domain of the area function?

(c) Find the maximum area of the garden and the corresponding dimensions of the garden.

Solution Since the length of the fence is 80 m, it follows that the length of the side parallel to the wall is $(80 - 2x)\,m$.



(a) $A = (80 - 2x) \cdot x$

(b) Note that $x > 0$ and $80 - 2x > 0$. Solving the second inequality, we get $x < 40$.

The domain of the area function is $\{x \in \mathbb{R} : 0 < x < 40\}$.

(c)
$$\begin{aligned} A &= 80x - 2x^2 \\ &= -2(x^2 - 40x) \\ &= -2(x^2 - 40x + 20^2 - 20^2) \\ &= -2(x^2 - 40x + 20^2) + 2 \cdot 20^2 \\ &= -2(x - 20)^2 + 800 \\ &\leq 800 \qquad \text{for all } x \text{ belonging to the domain} \end{aligned}$$

The maximum area of the garden is 800 m^2 (attained when $x = 20$).

When $x = 20$, the other side $(80 - 2x) = 40$.

The dimensions for maximum area are $20\,m \times 40\,m$, where 40 m is the length of the side parallel to the wall. □

Alternative solution to (c) With $a = -2$ and $b = 80$,

$$-\frac{b}{2a} = -\frac{80}{2 \cdot (-2)} = 20 \quad \text{(belongs to domain of area function)}$$

The area function attains its maximum at 20. (By Theorem 6.3.2)

When $x = 20$, $80 - 2x = 40$ and $A = 40 \cdot 20 = 800$.

The maximum area of the garden is 800 m^2 and the corresponding dimensions are $20\,m \times 40\,m$, where 40 m is the length of the side parallel to the wall. □

Example 6.4.3 The cost C, in dollars, of producing a mobile phone by a certain manufacturing company is given by

$$C = \frac{1}{4}x^2 - 30x + 1800$$

where x is the number of mobile phones produced per day.

In order to minimize the cost of producing a mobile phone how many mobile phones should be produced per day? What is the minimum cost?

Explanation The given equality describes the cost function. The question is to find the number at which the cost function attains its minimum and to find the minimum of the cost function. Note that the domain of the cost function is the set of all positive integers.

Solution 1

$$\begin{aligned} C &= \frac{1}{4}x^2 - 30x + 1800 \\ &= \frac{1}{4}(x^2 - 120x) + 1800 \\ &= \frac{1}{4}(x^2 - 120x + 60^2 - 60^2) + 1800 \\ &= \frac{1}{4}(x^2 - 120x + 60^2) - \frac{1}{4} \cdot 60^2 + 1800 \\ &= \frac{1}{4}(x - 60)^2 + 900 \end{aligned}$$

When $x = 60$, the cost $C = 900$. Moreover, for every x belonging to the domain of the cost function, the cost $C \geq 900$.

To minimize the cost of producing a mobile phone, 60 mobile phones should be produced per day. The minimum cost is 900 dollars. □

Solution 2 With $a = \frac{1}{4}$ and $b = -30$,

$$-\frac{b}{2a} = -\frac{-30}{2 \cdot \frac{1}{4}} = 60 \quad \text{(belongs to domain of cost function)}$$

The cost function attains its minimum at 60. (By Theorem 6.3.2)

When $x = 60$, $C = \frac{1}{4} \cdot 60^2 - 30 \cdot 60 + 1800 = 900.$

To minimize the cost of producing a mobile phone, 60 mobile phones should be produced per day. The minimum cost is 900 dollars. □

For a quadratic function f given by $f(x) = ax^2 + bx + c$, in order to apply Theorem 6.3.2, the domain of f must contains the number $-\frac{b}{2a}$. In Exercise 6.4, there are problems in which the domains of the functions do not satisfy the condition. For such a problem, we have to pay special attention to the elements of the domain. To look for where the function attains its extremum, we may rewrite $f(x)$ in the form $a(x-h)^2 + k$ or consider the graph of the function (which is part of a parabola).

- If $a > 0$, then minimum (if exists) is attained at the element(s) in the domain that is closest to h and maximum (if exists) is attained at the element(s) in the domain that is farthest from h.
- If $a < 0$, then maximum (if exists) is attained at the element(s) in the domain that is closest to h and minimum (if exists) is attained at the element(s) in the domain that is farthest from h.

For the case where $a > 0$ and the domain is $\mathbb{Z}$ (the set of all integers), the function f does not attain its maximum.

(a) If h is an integer, then f attains its minimum at h (we can apply Theorem 6.3.2 in this case).

(b) If $h = n - \frac{1}{2}$ for some integer n, then f attains its minimum at n as well as at $n-1$ [see Figure 6.4.2 (b)].

(c) If $n - \frac{1}{2} < h < n$ for some integer n, then f attains its minimum at n [see Figure 6.4.2 (c)].

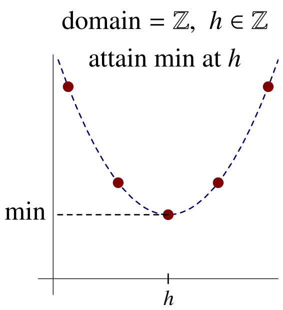


Figure 6.4.2(a)

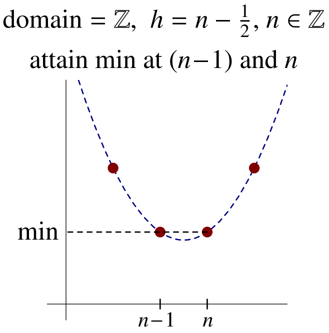


Figure 6.4.2(b)

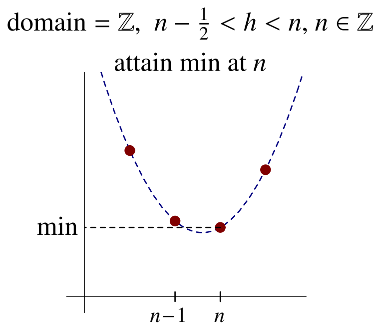


Figure 6.4.2(c)

Exercise 6.4

1. The monthly profit P, in dollars, of a company is given by $P = -2x^2 + 1000x - 30000$, where x is the number of pieces of product produced per day.
   (a) Find the value of x so that the monthly profit is maximum.
   (b) What is the maximum monthly profit?
2. Repeat Question 1 for the case where $P = -2x^2 + 1002x - 30000$.
3. Repeat Question 1 for the case where $P = -2x^2 + 999x - 30000$.

4. An object is projected vertically upward from the ground. After t seconds, the height h, in meters, of the object is given by $h = 80t - 5t^2$.
   (*a*) After how many seconds will the object hit the ground?
   (*b*) What is the maximum height of the object? When will the object reach its maximum height?

   Can you see why the time at which the object hits the ground is twice that at which it reaches its maximum height?

5. An object is projected vertically upward from the ground. After t seconds, the height h, in meters, of the object is given by $h = 10 + 80t - 5t^2$.
   (*a*) What is the initial height of the object?
   (*b*) After how many seconds will the object hit the ground? Give your answer correct to three significant figures.
   (*c*) What is the maximum height of the object? When will the object reach its maximum height?

   Can you see why the time at which the object hits the ground is more than twice that at which it reaches its maximum height?

6. Suppose that the lengths of two adjacent sides of a rectangular region are $x\,cm$ and $(18 - 2x)\,cm$. What is the maximum area of the rectangular region?

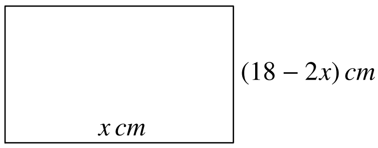


Figure 6.4.3

7. A fence of length $120\,m$ is to enclose a rectangular garden. Denote x to be the length, in m, of one side of the rectangle, and denote A to be the area, in m^2, of the garden.
   (*a*) Express A in terms of x.
   (*b*) Find the maximum area of the garden and the corresponding value of x.

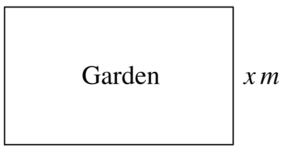


Figure 6.4.4

8. A right-angled triangular region is enclosed by two fences perpendicular to each other, using a wall as the hypotenuse. The total length of the two fences is $30\,m$. Find the maximum area of the triangular region.

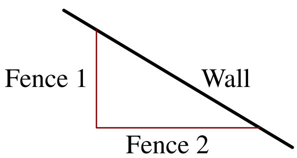


Figure 6.4.5

9. Suppose x and y are real numbers and their sum is 14. Find the largest possible value of $x \cdot y$ and the corresponding values of x and y.

10. Suppose x and y are real numbers and $2x + y = 14$. Find the largest possible value of $x \cdot y$ and the corresponding values of x and y.

11. (a) Suppose x and y are real numbers and $x + y = 6$. Find the smallest possible value of $x^2 + y^2$.
    (b) Use the result in (a) to find the (shortest) distance from the origin to the line given by $x + y = 6$.

12. Suppose x and y are real numbers and the difference between them is 14. Find the smallest possible value of $x \cdot y$.

13. A rectangular lawn of dimensions $x\,m \times y\,m$ is fenced and divided into two rectangular lots as shown in Figure 6.4.6. Suppose that the total length of the fence is $600\,m$.
    (a) Express y in terms of x.
    (b) Denote A to be the area, in m^2, of the lawn. Express A in terms of x.
    (c) Find the maximum area of the lawn.

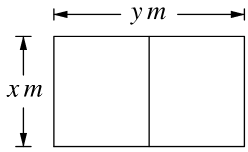


Figure 6.4.6

14. An $80\,cm$ long wire is cut into two pieces and each wire is bent to form a square. Denote x to be the length, in cm, of the side of one of the squares. Denote A to be the total area, in cm^2, of the two square regions.
    (a) Express A in terms of x.
    (b) Whats is the domain of the total area function?
    (c) Find the minimum total area of the two square regions.

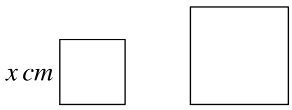


Figure 6.4.7

How should we cut the wire to make the total area as large as possible?

15. An $80\,cm$ long wire is cut into two pieces, one of which is bent to form a square and the other to form a circle. What is the minimum total area of the square region and the circular region?

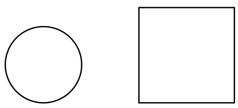


Figure 6.4.8

16. A Norman window has the outline of a semicircular region on top of a rectangular region, as shown in Figure 6.4.9. Suppose there is $6\,m$ of wood trim available. Find the radius of the semicircular region that will maximize the area of the window.

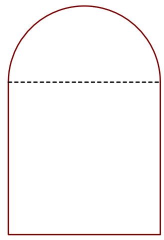


Figure 6.4.9

17. Figure 6.4.10 shows a region consisting of a rectangular region $ABCD$ together with two regions ABP and CDQ each of which is a quarter of a circular region. Suppose that the perimeter of the region is 100 units. What is the maximum area of the region?

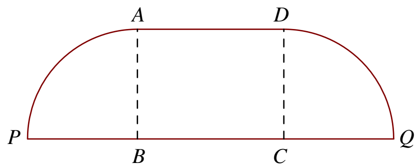


Figure 6.4.10

Perimeter of the region = length of side PQ + length of arc QD + length of side DA + length of arc AP

18. In Figure 6.4.11, $\triangle ABC$ is a right-angled triangle and $\square PQRC$ is a rectangle inscribed in $\triangle ABC$. Suppose the lengths of the sides AC and BC are 10 units and 6 units respectively. Find the maximum area of the rectangular region $PQRC$.

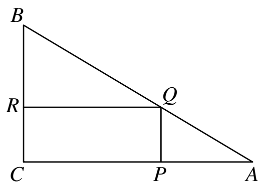


Figure 6.4.11

19. In Figure 6.4.12, $\square ABCD$ is a square of side 12 units and the points P, Q, R and S are on the sides AB, BC, CD and DA respectively such that the lengths of the sides AP, BQ, CR and DS are equal. Find the minimum area of the square region $PQRS$.

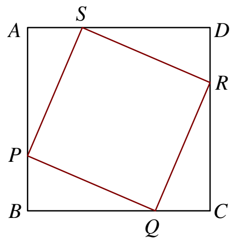


Figure 6.4.12

20. The manager of an 80-unit apartment complex is trying to decide what rent to charge. Experience has shown that at a rent of \$20000 (per month), all the units will be full. On the average, one additional unit will remain vacant for each \$500 increase in rent. Denote n to be the number of \$500 increases and denote R to be the corresponding (monthly) revenue, in dollars, obtained from the rented apartments.
    - (a) Express R in terms of n.
    - (b) What is the domain of the revenue function?
    - (c) What value of n leads to maximum revenue? What is the maximum revenue?
21. A farmer wants to construct a rectangular pen next to a barn $40\,m$ long, using all of the barn as part of one side of the pen. Suppose that $200\,m$ of fencing material is available. Find the dimensions of the pen with the largest area that the farmer can build.

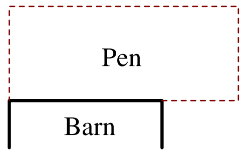


Figure 6.4.13

22. Repeat Question 21 for the case where $100\,m$ of fencing material is available.