Contents

Chapter 5

Functions and Graphs

5.1 Introduction

The concept of a *function* is fundamental to many areas of mathematics. The formal definition of a function is rather abstract and requires more set concepts and notation. We will only give an informal definition of a function (see Section 5.2). In this section, we give a few examples to illustrate the idea of a function.

Example 5.1.1 In a small class, there were five students: Chan Tai Man, Li Siu Man, Cheung Wai Yi, Wong Lap Yan and Ho Kwok Yan (represented by Chan, Li, Cheung, Wong and Ho respectively). In a Math test, each student received a letter grade. The test result of the class is represented by the *mapping diagram* shown in Figure 5.1.1. The arrow that maps Chan to A means that Chan Tai Man got an A in the test.

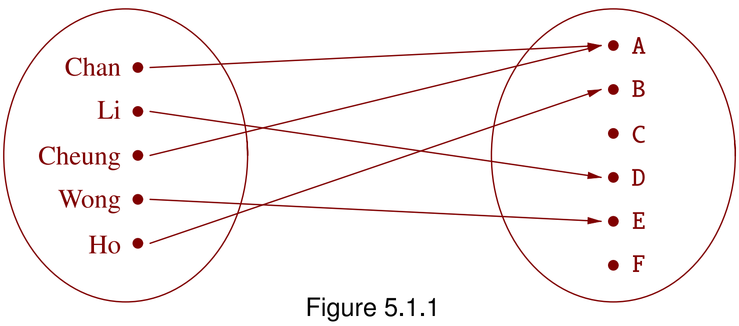


Figure 5.1.1

Note that given any student in the class, we can use the diagram to find out the student's grade in the test.

Example 5.1.2 The *table* below shows the total mid-year population for the world between 1997 and 2005.

Year	Population
1997	5,852,360,768
1998	5,929,735,977
1999	6,006,163,019
2000	6,081,527,896
2001	6,155,942,526
2002	6,229,629,168
2003	6,303,112,453
2004	6,376,863,118
2005	6,451,058,790

Table 5.1.2

Note that given any year between 1997 and 2005, we can use the table to find out the mid-year population of that year.

Example 5.1.3 Consider a ladder with length 1 meter leaning against a wall.

If the angle between the ladder and the ground is x° and the distance from the top of the ladder to the ground is y meter, the relation between the values x and y can be represented by the *graph* shown in Figure 5.1.4.

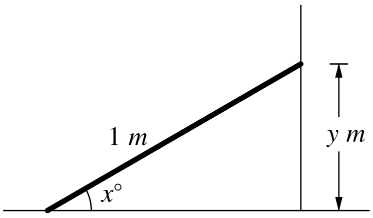


Figure 5.1.3

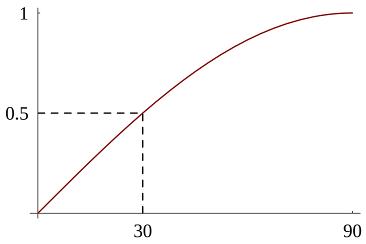


Figure 5.1.4

The point $(30, 0.5)$ lies on the graph means that when $x = 30$, the corresponding value of y is 0.5. That is, when the angle between the ladder and the ground is 30°, the distance from the top of the ladder to the ground is 0.5 meter.

Note that given any value of x with $0 \le x \le 90$ (angle), we can use the graph to find the corresponding value of y (distance).

Remark The graph consists of all the points (x, y) in the rectangular coordinate plane satisfying $y = \sin x^\circ$ $(0 \le x \le 90)$. Thus, instead of using a graph to represent the relation between the angle and the distance, we may use an equation (a *formula*).

Example 5.1.4 For a circular disk with radius r, its area A is given by the following *formula*:

$$A = \pi r^2, \quad r > 0$$

Note that given any positive real number r (radius), we can use the formula to find the corresponding value of A (area).

In each of the above 4 examples, there is a "rule" (a mapping diagram, a table, a graph or a formula) having the following properties:

- there are two collections of objects (people, alphabets, numbers);
- every object in the first collection is mapped to a unique (one and only one) object in the second collection.

Below we give more detail explanation.

1. In Example 5.1.1, the rule is represented by the mapping diagram.
   - The first collection consists of all the students in the class and the second collection consists of all the possible letter grades. In terms of set notations, the first collection is

     $$\{\text{Chan Tai Man},\ \text{Li Siu Man},\ \text{Cheung Wai Yi},\ \text{Wong Lap Yan},\ \text{Ho Kwok Yan}\}$$

     and the second collection is $\{\mathtt{A},\ \mathtt{B},\ \mathtt{C},\ \mathtt{D},\ \mathtt{E},\ \mathtt{F}\}$.
   - Note that for every student in the class, her/his letter grade is unique.

   Remark Although nobody got the grades C and F, they are included in the second collection. Imagine the situation where the teacher changed the letter grade of Li Siu Man from D to C because of a mistake in marking her test. To change the diagram, the teacher only need to change an arrow. However, if the grades C and F are not included in the second collection, the teacher has to make a lot of changes in the diagram.

2. In Example 5.1.2, the rule is represented by the table.
   - The first collection consists of all the years from 1997 to 2005 and the second collection consists of all possible population.
   - Note that for every year between 1997 and 2005, the mid-year population is unique.

   Remark For the second collection, there are many possible choices. One may take the second collection to be the set of all positive integers, or the set of all positive integers between 5 billions and 7 billions etc.

3. In Example 5.1.3, the rule is represented by the graph.

   - The first collection consists of all non-negative real numbers that are less than or equal to 90 (all possible angles) and the second collection consists of all non-negative real numbers (all possible distances). In terms of set notations, the first collection is $\{x \in \mathbb{R} : 0 \leq x \leq 90\}$ and the second collection is $\{y \in \mathbb{R} : y \geq 0\}$.
   - Note that for every real number x satisfying $0 \leq x \leq 90$ (angle, in degrees, between the ladder and the ground), the corresponding value of y (distance, in meter, from the top of the ladder to the ground) is unique.

   Remark The second collection can be taken to be the set of all non-negative real numbers that are less than or equal to 1 (since the length of the ladder is 1 meter). However, it has to be changed if a different ladder is considered.

4. In Example 5.1.4, the rule is represented by the formula.

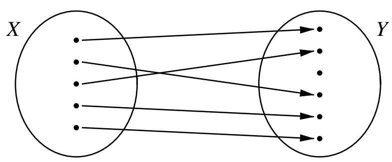
   - The first collection consists of all positive real numbers (radius) and the second collection consists of all positive real numbers (area) too. In other words, both the first and the second collections are $\mathbb{R}^+$.
   - Note that for every positive real number r (radius of the disk), the corresponding value of A (area of the disk) is unique.

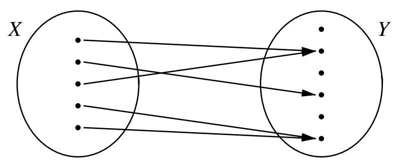
Exercise 5.1

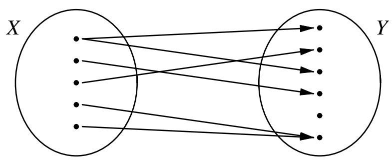
1. For each of the following diagrams, determine whether the following statement is true:

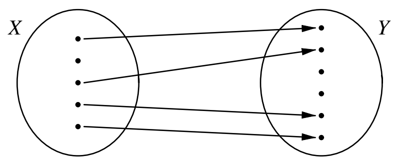
   For every element in the set X, there corresponds one and only one element in the set Y.

   If the answer is 'no', explain why.

   (a)
   

   (b)
   

   (c)
   

   (d)
   

2. For each of the following tables, determine whether the following statement is true:

   For every positive integer x less than 6, there corresponds one and only one positive integer y.

   If the answer is 'no', explain why.

   (*a*)

x	1	2	3	4	5
y	4	5	3	3	1

   (*b*)

x	1	2	3	4	5	2
y	2	5	3	1	10	8

   (*c*)

x	4	2	1	5
y	4	2	3	1

   (*d*)

x	3	2	1	4	5
y	7	2	1	3	1

   (*e*)

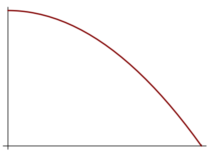
x	4	2	1	3	5
y	1	2	0	1	2

3. For each of the following graphs, determine whether the following statement is true:

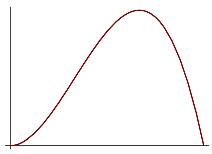
   For every real number x between 0 and 1, there is one and only one real number y such that (x, y) lies on the graph.

   If the answer is 'no', explain why.

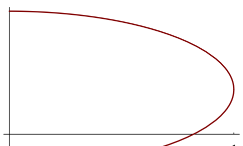
   (*a*)



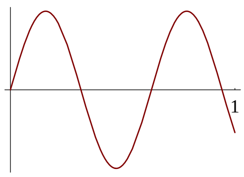
   (*b*)



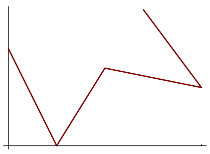
   (*c*)



   (*d*)



   (*e*)



4. For each of the following equations, determine whether the following statement is true:

   For every real number x, there is one and only one real number y such that (x, y) satisfies the equation.

   If the answer is 'no', explain why.

   (*a*) $x^2 + 2y = 3$ (*b*) $x - 3 = 2y^2$

   (*c*) $2y^2 - 3xy - 4 = 0$ (*d*) $x^2 + 2xy = y$

   (*e*) $\dfrac{x+1}{3y-2} = 5$ (*f*) $x - \sqrt{y} = 2$

5.2 Functions and Basic Concepts

Informal Definition By a ***function***, we mean a "rule" from a set X to a set Y such that every element of X is mapped to a unique element of Y.

To specify a function, we have to tell it is from which set to which set and to tell how every element in the first set is mapped to an element in the second set. The following example illustrates how to do this (the notation $\mapsto$ is read '*maps to*').

Example 5.2.1 The function from $\mathbb{R}^+$ to $\mathbb{R}^+$ given by

$$x \mapsto \pi x^2 \tag{5.2.1}$$

is the function considered in Example 5.1.4. In specifying the function, we state that it is

- from $\mathbb{R}^+$ to itself;
- given by (5.2.1), that is, every $x \in \mathbb{R}^+$ is mapped to $\pi x^2 \in \mathbb{R}^+$.

Remark In (5.2.1), the symbol x is a dummy variable; it can be replaced by any other symbol. For example, we may write $t \mapsto \pi t^2$.

Notation Usually, we use a single letter to denote a function. For example, we may write f to denote the function described in Example 5.2.1.

For some functions that are used very often, we use more than one letters to denote them. For example, we write 'sin' to denote the sine function, 'log' to denote the common logarithmic function etc. Details about these functions (and other related functions) can be found in later chapters. In Example 5.2.6, we will discuss the notation for the square root function.

Terminology Given a function f, according to the (informal) definition, we know that f is a "rule" from a set certain X to a certain set Y such that some conditions are satisfied. Sometimes, we want to specify the sets X and Y. For this, we say that f is a *function from X to Y*.

Example 5.2.2 Denote f to be the function described in Example 5.1.1. Denote

$$X = \{\text{Chan Tai Man, Li Siu Man, Cheung Wai Yi, Wong Lap Yan, Ho Kwok Yan}\}$$

and denote $Y = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}\}$.

The function f is a function from X to Y

Terminology 5.2.1 Let f be a function from X to Y.

- The set X is called the ***domain of f***.
- The set Y is called the ***codomain of f***.

Example 5.2.3 Denote g to be the function described in Example 5.1.3. Note that g is a function from $\{x \in \mathbb{R} : 0 \leq x \leq 90\}$ to $\{y \in \mathbb{R} : y \geq 0\}$.

- The domain of g is $\{x \in \mathbb{R} : 0 \leq x \leq 90\}$.
- The codomain of g is $\{y \in \mathbb{R} : y \geq 0\}$.

Remark We may also consider the function described in Example 5.1.3 as a function from $\{x \in \mathbb{R} : 0 \leq x \leq 90\}$ to $\{y \in \mathbb{R} : 0 \leq y \leq 1\}$. However, the function thus considered is different from the function g discussed above. This is because the their codomains are different.

To state the domain and codomain of a function, we introduce the following

Abbreviation We write *$f : X \to Y$ is a function* to mean that f is a function from X to Y, that is, f is a function with domain X and codomain Y.

Example 5.2.4 Consider the following:

(1) $f : \mathbb{R} \to \mathbb{R}$ is a function

(2) Let $g : \mathbb{R}^+ \to \mathbb{R}$ be a function.

(1) means that f is a function from $\mathbb{R}$ to itself;

(2) means the sentence "Let g be a function from $\mathbb{R}^+$ to $\mathbb{R}$.".

Notation 5.2.2 Let $f : X \to Y$ be a function. According to the (informal) definition, every element x of X is mapped to a unique element y of Y. This unique element is denoted by $f(x)$ and is called the *image of x under f*.

Remark $f(x)$ is read 'f of x'.

Caution Some authors write $f(x)$ to denote a function. This notation may be misleading because it also means an image (an element in the codomain). Readers can determine whether $f(x)$ means a function or an image from the context.

To specify a function, besides stating its domain and codomain, we have to state the image of every element in the domain. For this, we can use the notation '$\mapsto$' as in Example 5.2.1. The following example describes an alternative way to do this.

Example 5.2.5 Denote $a : \mathbb{R}^+ \to \mathbb{R}^+$ to be the function given by

$$a(r) = \pi r^2 \tag{5.2.2}$$

Then a is the function discussed in Example 5.2.1.

- The domain and the codomain of a are $\mathbb{R}^+$.
- For every r belonging to the domain, the image of r under a is πr^2.

  For example, 3 belongs to the domain and we have $a(3) = \pi \cdot 3^2 = 9\pi$.

Remark In (5.2.2), the symbol r is a dummy variable; it can be replaced by any other symbol. For example, we may write $a(t) = \pi t^2$.

Example 5.2.6 Denote f to be the function from $\{x \in \mathbb{R} : x \geq 0\}$ to $\mathbb{R}$ given by

$$f(x) = \sqrt{x} \tag{5.2.3}$$

Then the domain of f is $\{x \in \mathbb{R} : x \geq 0\}$ and the codomain of f is $\mathbb{R}$.

- Note that the number 4 belongs to the domain of f. Thus we can use (5.2.3) to find the image of 4 under f.

  $$\begin{aligned} f(4) &= \sqrt{4} && \text{In (5.2.3), substitute } x = 4 \\ &= 2 \end{aligned}$$

  We say that $f(4)$ *is defined* or f *is defined at* 4.

- Note that the number -3 does not belong to the domain of f. It is meaningless to talk about the image of -3 under f; we say that $f(-3)$ *is undefined* or f *is undefined at* -3.

Remark The function given in the example is called the *square root function*. Sometimes we write 'sqrt' to denote the square root function:

$$\text{sqrt}(x) = \sqrt{x} \quad \text{for } x \geq 0$$

Sometimes for convenience, we simply write $\sqrt{x}$ to denote the square root function.

Example 5.2.7 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = x^2 + 1$$

For each of the following, find its value.

(a) $f(2)$ (b) $f(3)$ (c) $f(5)$

Explanation To find $f(2)$, $f(3)$ and $f(5)$, substitute $x = 2$, $x = 3$ and $x = 5$ respectively into the expression defining f.

Solution

(a)
$$\begin{aligned} f(2) &= 2^2 + 1 \\ &= 5 \end{aligned}$$

(b)
$$\begin{aligned} f(3) &= 3^2 + 1 \\ &= 10 \end{aligned}$$

(c)
$$\begin{aligned} f(5) &= 5^2 + 1 \\ &= 26 \end{aligned}$$

□

Remark The number $f(2)$, for example, is also called the *value* of f when $x = 2$.

Example 5.2.8 Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by

$$g(x) = 2x - x^2$$

For each of the following, find its value.

(a) $g(3)$ (b) $g(-3)$

Explanation To find $g(3)$, substitute $x = 3$ in *every occurrence* of x in the expression defining g.

Solution

(a)
$$\begin{aligned} g(3) &= 2 \cdot 3 - 3^2 \\ &= -3 \end{aligned}$$

(b)
$$\begin{aligned} g(-3) &= 2 \cdot (-3) - (-3)^2 \\ &= -6 - 9 \\ &= -15 \end{aligned}$$

□

Given a function F from $\mathbb{R}$ to itself and given real numbers a and b, one may wonder whether we have $F(a + b) = F(a) + F(b)$ and $F(-a) = -F(a)$.

- Example 5.2.7 illustrates that $F(a+b)$ and $F(a)+F(b)$ may be unequal (take $a = 2$ and $b = 3$).
- Example 5.2.8 illustrates that $F(-a)$ and $-F(a)$ may be unequal (take $a = 3$).

In the exercises, readers are asked to give examples to illustrate that $F(ab)$ and $F(a)(F(b)$ may be unequal etc.

Example 5.2.9 Let h be the function from $\{x \in \mathbb{R} : x^2 - 3x - 4 \neq 0\}$ to $\mathbb{R}$ given by

$$h(x) = \frac{2x+1}{x^2 - 3x - 4}$$

For each of the following, determine whether it is defined or undefined; find its value if it is defined.

(*a*) $h(4)$ (*b*) $h(3)$

Solution (*a*) Since $4^2 - 3(4) - 4 = 16 - 12 - 4 = 0$, it follows that the number 4 does not belong to the domain of h. Hence $h(4)$ is undefined.

(*b*) Since $3^2 - 3(3) - 4 = 9 - 9 - 4 \neq 0$, it follows that the number 3 belongs to the domain of h. Hence $h(3)$ is defined.

$$\begin{aligned} h(3) &= \frac{2 \cdot 3 + 1}{3^2 - 3 \cdot 3 - 4} \\ &= \frac{6+1}{9-9-4} \\ &= -\frac{7}{4} \end{aligned}$$

□

In the rest of the examples in this section, the domain and codomain of the functions are taken to be $\mathbb{R}$. More discussion on domains and codomains can be found in the next section.

Example 5.2.10 Let $f(x) = 5x$ and $g(x) = x^2 + 1$. Find

(*a*) $f(3) + g(3)$ (*b*) $f(2)\,g(1)$ (*c*) $\dfrac{f(4)}{g(2)}$

Explanation When we write "*Let* $f(x) = 5x$.", we mean "*Denote* f *to be the function (from* $\mathbb{R}$ *to itself) given by* $f(x) = 5x$.".

Note that $f(3) + g(3)$ is the sum of two real numbers $f(3)$ and $g(3)$. Similarly, the expression in (2) is the product of two real numbers and that in (3) is the quotient of two real numbers.

Solution (*a*)

$$\begin{aligned} f(3) + g(3) &= 5 \cdot 3 + (3^2 + 1) && \text{The first term is } f(3) \text{ and the second term is } g(3) \\ &= 15 + 10 \\ &= 25 \end{aligned}$$

(*b*)

$$\begin{aligned} f(2)\,g(1) &= (5 \cdot 2)(1^2 + 1) && \text{The first factor is } f(2) \text{ and the second factor is } g(1) \\ &= 10 \cdot 2 \\ &= 20 \end{aligned}$$

(soln cont'd) (c)

$$\frac{f(4)}{g(2)} = \frac{5 \cdot 4}{2^2 + 1} \quad \text{The numerator is } f(4) \text{ and the denominator is } g(2)$$
$$= \frac{20}{5}$$
$$= 4$$

□

Example 5.2.11 Let $f(x) = x^2 - x$. Express each of the following in terms of n.

(a) $f(n)$ (b) $f(3n)$

(c) $f(n^2)$ (d) $f(n+1)$

Explanation In the expressions $f(n)$, $f(3n)$ etc., the symbol n is considered as a variable (more detail discussion on variables can be found in the next section).

To find $f(n)$, in the expression defining f, substitute $x = n$; to find $f(3n)$ substitute $x = 3n$ etc.

Solution

(a) $f(n) = n^2 - n$ Substitute $x = n$

(b) $f(3n) = (3n)^2 - 3n$ Substitute $x = 3n$
$= 9n^2 - 3n$

(c) $f(n^2) = (n^2)^2 - n^2$ Substitute $x = n^2$
$= n^4 - n^2$

(d) $f(n+1) = (n+1)^2 - (n+1)$ Substitute $x = n + 1$
$= (n^2 + 2n + 1) - (n + 1)$ Perfect Square Identity
$= n^2 + 2n + 1 - n - 1$
$= n^2 + n$

□

Example 5.2.12 Let $f(x) = 5 + 3x$. Suppose that $f(a) = a + 1$. Find the value of a.

Explanation The equality $f(a) = a + 1$ is an equation with unknown a.

Solution

$f(a) = a + 1$
$5 + 3a = a + 1$ Substitute $x = a$ in the expression defining $f(x)$
$3a - a = 1 - 5$ Rearrange terms
$2a = -4$

Hence $a = -2$.

□

Example 5.2.13 Let $f(x) = ax - 5$, where a is an unknown constant. Suppose that $f(2) = 3$. Find the value of a.

Explanation $f(2) = 3$ gives an equation with unknown a because $f(2) = a \cdot 2 - 5$.

Solution

$$\begin{aligned} f(2) &= 3 \\ a \cdot 2 - 5 &= 3 && \text{Substitute } x = a \text{ in the expression defining } f(x) \\ 2a &= 8 \end{aligned}$$

Hence $a = 4$. □

Exercise 5.2

1. Let $f(x) = 2x + 3$. For each of the following, find its value.

   (a) $f(1)$ (b) $f(-2)$ (c) $f(4)$

   (d) $f(1) + f(-2)$ (e) $f(1 + 3)$ (f) $f(1 + 2)$

   (g) $f(-2) \times f(4)$ (h) $f(-2 \times 4)$ (i) $f(-2 \times 3)$

2. Let $g(x) = x^2 - x + 2$. For each of the following, find its value.

   (a) $g(1)$ (b) $g(2)$ (c) $g(1 + 2)$ (d) $g(1) + g(2)$

3. Let $h(x) = \dfrac{x}{x^2 + 1}$. For each of the following, find its value.

   (a) $h(2)$ (b) $h(3)$ (c) $h(2 \times 3)$ (d) $h(2) \times h(3)$

4. Let $f(x) = 4 - x$ and $g(x) = x^2 - 3$. For each of the following, find its value.

   (a) $f(1) + g(1)$ (b) $f(2) + g(3)$ (c) $f(2) - g(2)$

   (d) $f(3)\, g(4)$ (e) $f(2)\, f(3)\, f(4)$ (f) $\dfrac{f(-1)\, g(1)}{f(1)}$

5. For each of the following, find the value of f when $x = 3$.

   (a) $f(x) = 3 - 7x$ $f(x) = 2x - 3x^2$ $f(x) = \sqrt{16 + x^2}$

6. Let $f(x) = x(x + 1)$ and let a be a non-zero real number. Express each of the following in terms of a.

   (a) $f(2a)$ (b) $f(a + 1)$ (c) $f(a^2)$ (d) $f\left(\frac{1}{a}\right)$

   (e) $2 f(a)$ (f) $f(a) + 1$ (g) $\big(f(a)\big)^2$

7. Let $f(x) = 4 + 3x$. Suppose that $f(2k) = k - 1$. Find the value of k.

8. Let $g(x) = x^2 + x$. Suppose that $g(k + 1) = k + 2$. Find the value(s) of k.

9. Suppose that $f(x) = ax + 3$ and $f(5) = 2$. Find the value of a.

10. Suppose that $g(x) = kx^2 + k^2$ and $g(2) = 5$. Find the values(s) of k.

11. Let $h(x) = ax + b$. Suppose that $h(1) = 3$ and $h(4) = -1$. Find the values of a and b.
12. Let $f(x) = x^2 + kx$. Suppose that $f(2) = f(3)$. Find $f(4)$.
13. Let $f(x) = 2x^2 + 3x - 4$. Solve the equation $f(x + 2) = f(x - 2)$.
14. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$ such that for every real number a, $f(a + 1) = a^2 + a$. Find
    (a) $f(3)$ (b) $f(-5)$ (c) $f(x)$
15. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers a and b such that $f(ab) \neq f(a)f(b)$.
16. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and real numbers a and b such that $f\left(\frac{a}{b}\right) \neq \frac{f(a)}{f(b)}$.
17. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ and a real number a such that $f(a^2) \neq (f(a))^2$.

5.3 Domains and Codomains

For a function whose domain and codomain are subsets of $\mathbb{R}$,

- elements in the domain are called "*input numbers*";
- elements in the codomains are called "*output numbers*".

Such a function can be considered as a "*number machine*". Given any input number, the machine gives one and only one output number.

- A variable that represents the input numbers is called an *independent variable*.
- A variable that represents the output numbers is called a *dependent variable* because its value depends on the value of the independent variable.

If we denote the independent variable by x and denote the dependent variable by y, we say that '*y is a function of x*'.

Example 5.3.1 Denote $f : \mathbb{R} \to \mathbb{R}$ to be the function given by $f(x) = x^2$.
Putting $y = f(x)$, we get

$$y = x^2 \tag{5.3.1}$$

For each input number x, using (5.3.1), we can find the corresponding output number y. For example, if $x = 2$, then $y = 4$;
if $x = -3$, then $y = 9$.

The independent variable is x and the dependent variable is y.

The equality in (5.3.1) tells that y is a function of x. However, to be more specific, we have to know what are the domain and codomain.

Convention Unless otherwise stated, codomains of functions are taken to be $\mathbb{R}$, that is,

- '*a function*' means '*a function whose codomain is* $\mathbb{R}$'.

For a function whose domain is a subset of $\mathbb{R}$, in many cases, the domain can be specified by stating a condition for the independent variable. The following illustrates how to do this.

Example 5.3.2 Consider the function given by

$$y = \sqrt{x}, \quad x \geq 0 \tag{5.3.2}$$

The condition $x \geq 0$ determines the domain of the function— the domain is $\{x \in \mathbb{R} : x \geq 0\}$.

The function described by (5.3.2) is in fact the square root function given in Example 5.2.6.

In the above example, the condition $x \geq 0$ can be omitted. This is because the expression $\sqrt{x}$ gives a real number only when x is a non-negative real number; it does not give a real number if x is a negative number.

Convention Unless otherwise stated, for a function that is given by an equality in the form

$$f(x) = \text{an expression in } x \qquad \text{or} \qquad y = \text{an expression in } x$$

- the domain of the function is taken to be the set of all real numbers r such that when we substitute $x = r$, the expression gives a real number. The domain taken in this way is called the *natural domain*.

Example 5.3.3 Find the (natural) domain of the function f given by $f(x) = \sqrt{1-2x}$.

Explanation By convention, the codomain of f is $\mathbb{R}$ and the domain of f is the set of all real numbers r such that $\sqrt{1-2r}$ gives a real number, or equivalently, that $1 - 2r \geq 0$.

Solution The domain of f is $\{x \in \mathbb{R} : 1 - 2x \geq 0\}$.

Solving
$$\begin{aligned} 1 - 2x &\geq 0 \\ 1 &\geq 2x \\ \frac{1}{2} &\geq x \end{aligned}$$

The domain is $\left\{x \in \mathbb{R} : \frac{1}{2} \geq x\right\}$. □

Remark The domain can also be written as $\{x \in \mathbb{R} : x \leq \frac{1}{2}\}$.

Example 5.3.4 Find the domain of the function given by $y = \dfrac{2}{3x+4}$

Explanation In Example 5.3.3, we write f to denote the function under consideration. However, in this example, we don't give a notation for the function (it's not important).

The domain of the function under consideration is the set of all real numbers r such that $\dfrac{2}{3r+4}$ gives a real number, or equivalently, that $3r + 4 \neq 0$.

Solution The domain of the function is $\{x \in \mathbb{R} : 3x + 4 \neq 0\}$.

$$\text{Solving} \quad \begin{aligned} 3x + 4 &\neq 0 \\ 3x &\neq -4 \\ x &\neq \frac{-4}{3} \end{aligned}$$

The domain is $\left\{x \in \mathbb{R} : x \neq -\frac{4}{3}\right\}$. □

Remark The domain can also be written as $\mathbb{R} \setminus \{ -\frac{4}{3} \}$.

For the next example, we need the following result which, in fact, is the Product Zero Principle stated in an alternatively way.

Product Non-zero Principle Let u and v be real numbers. Then $u \cdot v \neq 0$ if and only if $u \neq 0$ and $v \neq 0$.

Example 5.3.5 Find the domain of the function $y = \dfrac{3x^2}{x^2 - 5x - 14}$.

Explanation We write "*the function* $y = \frac{3x^2}{x^2-5x-14}$" to mean "*the function given by* $y = \frac{3x^2}{x^2-5x-14}$". The method to find the domain is similar to that in Example 5.3.4.

Solution The domain of the function is $\{x \in \mathbb{R} : x^2 - 5x - 14 \neq 0\}$.

$$\text{Solving} \quad \begin{aligned} x^2 - 5x - 14 &\neq 0 \\ (x-7)(x+2) &\neq 0 \\ x - 7 \neq 0 \quad \text{and} &\quad x + 2 \neq 0 \qquad \text{Product Non-zero Principle} \\ x \neq 7 \quad \text{and} &\quad x \neq -2 \end{aligned}$$

The domain is $\{x \in \mathbb{R} : x \neq 7 \text{ and } x \neq -2\}$. □

Remark The domain can also be written as $\mathbb{R} \setminus \{7, -2\}$.

Example 5.3.6 Suppose f is a function with domain a subset of $\mathbb{R}$ and codomain $\mathbb{R}$ and is given by

$$f(x) = \sqrt{x-1} + \sqrt{2-x}$$

Find the largest possible domain of f.

Explanation In Example 5.3.5 (and the two examples preceding it), under the adopted convention, the function given is unique (hence we write '*the function*'). However, in this example, there are infinitely many functions satisfying the given conditions (hence we write '*f is a function with ...*').

The question is the same as the following:

Find the (natural) domain of the function $y = \sqrt{x-1} + \sqrt{2-x}$.

Solution $\sqrt{x-1} + \sqrt{2-x}$ gives a real number if and only if $x-1 \geq 0$ and $2-x \geq 0$.

$$\begin{array}{lrclcrcl} \text{Solving} & x-1 & \geq & 0 & \text{and} & 2-x & \geq & 0 \\ & x & \geq & 1 & \text{and} & 2 & \geq & x \end{array}$$

Hence $1 \leq x \leq 2$.

The largest possible domain is $\{x \in \mathbb{R} : 1 \leq x \leq 2\}$. □

In the next two examples, the questions are to find some unknown numbers. Similar questions have been considered in some examples in the last section. The difference is that the functions are described in different ways.

Example 5.3.7 Consider the function $y = x^2 + 3x - 4$. Suppose that $y = a^2 + 1$ when $x = a$. Find the value of a.

Explanation This question is similar to that in Example 5.2.12.

Solution

$$\begin{array}{rcll} y & = & x^2 + 3x - 4 & \\ a^2 + 1 & = & a^2 + 3a - 4 & \text{Substitute } x = a \text{ and } y = a^2 + 1 \\ 5 & = & 3a & \end{array}$$

Hence $a = \dfrac{5}{3}$. □

Example 5.3.8 Consider the function $y = x(x+k)$, where k is an unknown constant. Suppose that $y = 6$ when $x = 3$. Find the value of k.

Explanation This question is similar to that in Example 5.2.13.

Solution

$$\begin{array}{rcll} y & = & x(x+k) & \\ 6 & = & 3 \cdot (3+k) & \text{Substitute } x = 3 \text{ and } y = 6 \\ 2 & = & 3+k & \end{array}$$

Hence $k = -1$. □

Example 5.3.9 Let $f(x) = x^2 - x$. Find and simplify the following:

(a) $f(x+2)$ (b) $f(x-2)$ (c) $f(x+2) - f(x-2)$

Explanation By convention, the domain and codomain of f are $\mathbb{R}$. Thus f is the function given in Example 5.2.11.

In the expression $f(x+2)$, the symbol x is considered as a variable that can take values in $\mathbb{R}$. Thus the expression $f(x+2)$ gives a function of x.

Solution

(a)
$$\begin{aligned} f(x+2) &= (x+2)^2 - (x+2) && \text{In the equality defining } f(x)\text{, replace } x \text{ by } x+2 \\ &= (x^2+4x+4) - (x+2) \\ &= x^2+4x+4-x-2 \\ &= x^2+3x+2 \end{aligned}$$

(b)
$$\begin{aligned} f(x-2) &= (x-2)^2 - (x-2) && \text{In the equality defining } f(x)\text{, replace } x \text{ by } x-2 \\ &= (x^2-4x+4) - (x-2) \\ &= x^2-4x+4-x+2 \\ &= x^2-5x+6 \end{aligned}$$

(c)
$$\begin{aligned} f(x+2) - f(x-2) &= (x^2+3x+2) - (x^2-5x+6) && \text{By (1) and (2)} \\ &= x^2+3x+2-x^2+5x-6 \\ &= 8x-4 \end{aligned}$$

□

Remark If we put $y = f(x+2)$ and $z = f(x-2)$, then y and z are functions of x. Part (3) is to find $y - z$ (difference of of two functions of x). In Section 5.5, we will discuss arithmetic operations on functions.

In considering $y = f(x+2)$, if we put $u = x+2$, then u is a function of x and y is a function of u (since $y = f(u)$). In Section 5.6, we will discuss compositions of functions.

Exercise 5.3

1. For each of the following, find the (natural) domain of f.

   (a) $f(x) = 2x + 1$ (b) $f(x) = \frac{1}{x-1}$ (c) $f(x) = \sqrt{2x-3}$

2. For each of the following, the given equation determines y as a function of x. Find the largest possible domain of the function.

   (a) $y = x^2 + 2x + 3$ (b) $y = \frac{3x+4}{x^2-1}$ (c) $y = 7$

   (d) $y = \frac{2x}{\sqrt{3x+4}}$ (e) $y = 3 - \frac{4}{x^2-5x+6}$ (f) $y = \frac{\sqrt{2x-1}}{\sqrt{x+3}}$

5.4 Practical Problems Leading to Functions

Example 5.4.1 Figure 5.4.1 shows a rectangular cardboard of dimensions $20\,cm \times 14\,cm$. Suppose a square of sides $x\,cm$ is cut from the four corners of the cardboard and the remaining part is folded to form an open box (without lid). Denote $V(x)$ to be the volume, in cm^3, of the box.

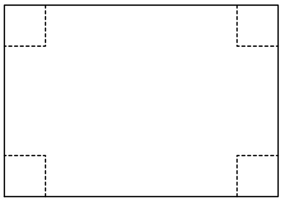


Figure 5.4.1

(a) Express $V(x)$ in terms of x.

(b) Find the domain of V.

Explanation In this problem, x is a variable, it can take the value 2 for example; but it can't take the value 8 for example. Since the volume of the box depends on x, it is denoted by $V(x)$. The symbol V means the volume function.

Solution (a) The length and width of the box are $(20 - 2x)\,cm$ and $(14 - 2x)\,cm$ respectively and the height is $x\,cm$.

Hence $V(x) = (20 - 2x)(14 - 2x)x$.

(b) Note that lengths must be positive.

The domain of the volume function consists of all real numbers x such that $20 - 2x > 0$ and $14 - 2x > 0$ and $x > 0$.

Solving	$20 - 2x > 0$	and $14 - 2x > 0$	and $x > 0$
	$20 > 2x$	$14 > 2x$	
	$10 > x$	$7 > x$	

Hence $0 < x < 7$.

The domain of V is $\{x \in \mathbb{R} : 0 < x < 7\}$. □

Example 5.4.2 An object is released at the top of a 30 m tall building. The height h (in meter) of the object after t seconds is given by $h = 30 - 4.9t^2$.

(a) How far is the object from the ground after 2 seconds?

(b) When will be object hit the ground? Give your answer correct to 1 decimal place.

(c) What is the domain of the height function?

Explanation The equality $h = 30 - 4.9t^2$ tells that h is a function of t. The function is called the height function. Its domain consists of t between 0 and the time when the object hit the ground.

Solution (a) When $t = 2,\ h = 10.4$.

After 2 seconds, the height of the object is 10.4 m.

(b) When the object hits the ground, the height is 0 m.

Solving

$$\begin{aligned} 30 - 4.9t^2 &= 0 \\ 30 &= 4.9t^2 \\ t^2 &= \frac{30}{4.9} \approx 6.12245 \\ t &= \sqrt{6.12245} \qquad t \text{ cannot be negative} \\ &\approx 2.47436 \end{aligned}$$

The object will hit the ground after 2.5 seconds.

(c) The domain of the height function is $\{t \in \mathbb{R} : 0 \le t \le 2.5\}$. □

Exercise 5.4

1. The speed s (in meters per second) of sound in air at t degrees Celsius is approximately given by

$$s(t) = 331.3\sqrt{1 + \frac{t}{273.15}}$$

(a) Find the speed of sound when the temperature is $23°C$.

(b) At what temperature will the speed of sound be $360\, m/s$?

2. Figure 5.4.2 shows a cube of sides $x\, cm$. Denote $A(x)$ to be the total surface area, in cm^2 of the cube.

(a) Express $A(x)$ in terms of x.

(b) Write down the domain of A.

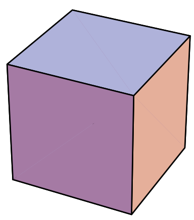


Figure 5.4.2

3. Figure 5.4.3 shows an isosceles triangle where AC and BC are sides of equal lengths. Suppose that the length of AB is 4 units, that the length of BC is x units and that the measure of $\angle ABC$ is less than $60°$. Denote $A(x)$ to be the area of the isosceles triangle.
   - (a) Express $A(x)$ in terms of x.
   - (b) Find the domain of A.

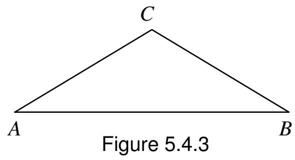


Figure 5.4.3

4. Figure 5.4.4 shows a cylindrical water tank with base diameter $10\,m$ and height $15\,m$. Denote $V(x)$ to be the volume, in cm^3, of water when the depth is $x\,m$.
   - (a) Express $V(x)$ in terms of x.
   - (b) Write down the domain of A.
   - (c) Find the volume of water in the tank when the depth is $10\,m$.
   - (d) Find the depth of water if the volume is $120\pi\,cm^3$.
   - (e) Can the tank hold $1000\,m^3$ of water? Why?

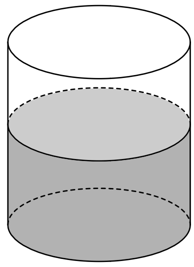


Figure 5.4.4

5. Figure 5.4.5 shows a rectangular piece of land enclosed by a wall and a fence along the other three sides. Suppose the total length of fence is $25\,m$ and the length of the side parallel to the wall is $x\,m$. Denote the area, in m^2, of the land by $A(x)$.
   - (a) Express A in terms of x.
   - (b) Find the domain of A.
   - (c) Find the area of the land if $x = 8$.
   - (d) Find the value of x if the area of the land is $78\,m^2$.
   - (e) Can the area of the land be $80\,m^2$? Why?
   - (f) *Can you find the largest possible area of the land?*


Figure 5.4.5

5.5 Arithmetic Operations on Functions

Suppose that f and g are functions from a set X to $\mathbb{R}$. Then for every $x \in X$, the elements $f(x)$ and $g(x)$ are real numbers. Hence we may consider arithmetic operations on these numbers. For example, for every $x \in X$, we have $f(x) + g(x) \in \mathbb{R}$. In this way, we obtain a function from X to $\mathbb{R}$. This function is called the *sum* of the functions f and g and is denoted by $f + g$. That is, $f + g$ is the function from X to $\mathbb{R}$ such that $(f + g)(x) = f(x) + g(x)$ for $x \in X$.

Using the same idea, we can define the *difference* and *product* of f and g. To define the *quotient* of f and g, we have to assume that for every $x \in X$, the number $g(x)$ is non-zero.

Definition 5.5.1 Let f and g be functions from a set X to $\mathbb{R}$.

- We call *the sum of f and g*, and write $f + g$, to mean the function from X to $\mathbb{R}$ given by

$$(f+g)(x) = f(x) + g(x) \qquad \text{for } x \in X$$

- We call *the difference of f and g*, and write $f - g$, to mean the function from X to $\mathbb{R}$ given by

$$(f-g)(x) = f(x) - g(x) \qquad \text{for } x \in X$$

- We call *the product of f and g*, and write fg, to mean the function from X to $\mathbb{R}$ given by

$$(fg)(x) = f(x)g(x) \qquad \text{for } x \in X$$

Definition 5.5.1a Let f and g be functions from a set X to $\mathbb{R}$ such that for every $x \in X$, $g(x) \neq 0$. We call *the quotient of f and g*, and write $\frac{f}{g}$, to mean the function from X to $\mathbb{R}$ given by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)} \qquad \text{for } x \in X$$

Example 5.5.1 Let f and g be the functions from $\mathbb{R}$ to $\mathbb{R}$ given by

$$f(x) = 5x, \qquad g(x) = x^2 + 1$$

Find $(f+g)(x)$ and $(f+g)(3)$.

Explanation Since f and g are functions with codomain $\mathbb{R}$ and their domains are the same, we may consider $f+g$, which is a function from $\mathbb{R}$ to $\mathbb{R}$. To specify the function $f+g$, we have to know $(f+g)(x)$ for $x \in \mathbb{R}$.

Solution

$$\begin{aligned}
(f+g)(x) &= f(x) + g(x) && \text{By Definition 5.5.1} \\
&= 5x + (x^2 + 1) \\
&= x^2 + 5x + 1 \\
(f+g)(3) &= 3^2 + 5 \cdot 3 + 1 && \text{Substitute } x = 3 \text{ into the above result for } (f+g)(x) \\
&= 25
\end{aligned}$$

□

Remark Note that $(f+g)(3) = f(3) + g(3)$. The value can be found as in Example 5.2.10 (a).

Suppose that f and g are functions with same codomain $\mathbb{R}$ but with different domains X and Y. Then for $x \in X \setminus Y$, the function f is defined at x but the function g is undefined at x; thus $g(x)$ is meaningless. Similarly, for $y \in Y \setminus X$, $f(y)$ is meaningless. However, for $z \in X \cap Y$, both f and g are defined at z and so we may consider $f(z) + g(z)$. In this way, we can defined a function from $X \cap Y$ to $\mathbb{R}$.

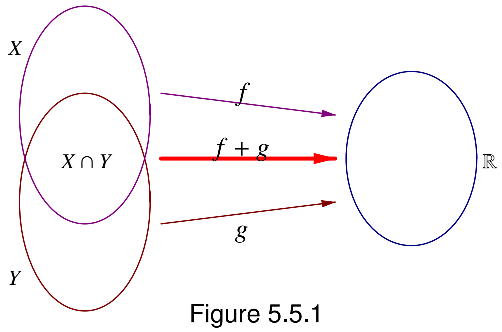


Figure 5.5.1

The following definition is a generalization of Definition 5.5.1.

Definition 5.5.2 Let f be a function from a set X to $\mathbb{R}$ and let g be a function from a set Y to $\mathbb{R}$.

- We call *the sum of f and g*, and write $f + g$, to mean the function from $X \cap Y$ to $\mathbb{R}$ given by
$$(f + g)(z) = f(z) + g(z) \qquad \text{for } x \in X \cap Y$$
- We call *the difference of f and g*, and write $f - g$, to mean the function from $X \cap Y$ to $\mathbb{R}$ given by
$$(f - g)(z) = f(z) - g(z) \qquad \text{for } z \in X \cap Y$$
- We call *the product of f and g*, and write fg, to mean the function from $X \cap Y$ to $\mathbb{R}$ given by
$$(fg)(z) = f(z)g(z) \qquad \text{for } z \in X \cap Y$$
- We call *the quotient of f and g*, and write $\frac{f}{g}$, to mean the function from $X \cap \{y \in Y : g(y) \neq 0\}$ to $\mathbb{R}$ given by
$$\frac{f}{g}(z) = \frac{f(z)}{g(z)} \qquad \text{for } z \in X \cap \{y \in Y : g(y) \neq 0\}$$

Example 5.5.2 Let f and g be the functions given by

$$f(x) = \sqrt{x-1}, \qquad g(x) = \sqrt{2-x}$$

By convention, the codomains of f and g are $\mathbb{R}$. Below we consider the domain of the function $(f+g)$.

Note that

$$\begin{aligned} \text{Domain of } f &= \{x \in \mathbb{R} : x - 1 \geq 0\} \\ &= \{x \in \mathbb{R} : x \geq 1\} \end{aligned} \qquad \begin{aligned} \text{Domain of } g &= \{x \in \mathbb{R} : 2 - x \geq 0\} \\ &= \{x \in \mathbb{R} : x \leq 2\} \end{aligned}$$

Hence

$$\begin{aligned} \text{Domain of } (f+g) &= \{x \in \mathbb{R} : x \geq 1\} \cap \{x \in \mathbb{R} : x \leq 2\} \\ &= \{x \in \mathbb{R} : 1 \leq x \leq 2\} \end{aligned}$$

Remark $(f+g)$ is the function given in Example 5.3.6.

Example 5.5.3 Let f and g be the functions given by

$$f(x) = 3x^2, \qquad g(x) = x^2 - 5x - 14$$

Below we consider the domain of the function $\frac{f}{g}$.

Note that the domains of f and g are $\mathbb{R}$.

Hence

$$\begin{aligned} \text{Domain of } \frac{f}{g} &= \mathbb{R} \cap \{x \in \mathbb{R} : x^2 - 5x - 14 \neq 0\} \\ &= \{x \in \mathbb{R} : (x-7)(x+2) \neq 0\} \\ &= \{x \in \mathbb{R} : x \neq 7 \text{ and } x \neq -2\} \end{aligned}$$

Remark $\frac{f}{g}$ is the function given in Example 5.3.5.

By applying arithmetic operations on functions, we can construct complicated functions using the *identity function* and *constant functions* as building blocks.

- By a *constant function*, we mean a function from $\mathbb{R}$ to $\mathbb{R}$ that is given by

  $$x \mapsto c$$

  where c is a real number (called a constant).

  Remark $x \mapsto c$ means that for every $x \in \mathbb{R}$, the image of x under the function is c.

- The *identity function* is the function from $\mathbb{R}$ to $\mathbb{R}$ given by

  $$x \mapsto x$$

  Remark $x \mapsto x$ means that for every $x \in \mathbb{R}$, the image of x under the function is x.

Below we give some examples of functions that can be build from the identity function and constant functions by taking sums and products.

(1) Let f be the function given by $f(x) = x + 3$. Then f is the sum of the identity function and a constant function.

(2) Let g be the function given by $g(x) = -4x$. Then g is the product of the identity function and a constant function.

(3) Let h be the function given by $h(x) = 7 - 5x$. Then h is the sum of a constant function and a function that is the product of the identity function and a constant function.

In Section 5.7, we will discuss graphs of functions. The graphs of the functions f, g and h are the straight lines given by the following equations respectively:

$$y = x + 3, \qquad y = -4x, \qquad y = 7 - 5x$$

For this reasons, the functions f, g and h are called *linear functions*.

(4) Let p be the function given by $p(x) = x^2$. The function is called the *square function*. It is the product of the identity function with itself.

(5) Let q be the function given by $q(x) = 2x^2 + 3x - 4$. Note that q can be build from the identity function and constant functions by taking sums and products.

The functions p and q are called *quadratic functions*. More details on such functions can be found in Chapter **??**.

Exercise 5.5

1. Let $f(x) = 2x + 1$ and let $g(x) = x - 3$.
   - (a) Write down the domains of $f + g$ and fg.
   - (b) Find $(f + g)(x)$ and $(fg)(x)$.
   - (c) Find the domain of $\frac{f}{g}$
   - (d) Find $\frac{f}{g}(x)$.

2. Let $f(x) = \dfrac{1}{x-2}$ and let $g(x) = \sqrt{1-x}$.
   - (a) Find the domain of f.
   - (b) Find the domain of g.
   - (c) Find the domain of fg.
   - (d) For each of the following indicated values of a, determine whether fg is defined at a or not and find the value $(fg)(a)$ if it is defined.

     (i) $a = 3$ (ii) $a = -3$ (iii) $a = 1$

5.6 Compositions of Functions*

Consider the function (from $\mathbb{R}$ into itself) given by

$$y = (x^2 + 1)^3 \tag{5.6.1}$$

Given a real number x, to find the image of x under the function, we can

(1) first calculate $x^2 + 1$; call the resulted value u,

(2) and then calculate u^3.

This is a 2-step process, which can be depicted as follows:

$$x \mapsto x^2 + 1 = u \mapsto u^3 = y$$

The two steps correspond to two functions:

(1) $u = x^2 + 1$

(2) $y = u^3$

Note (5.6.1) tells that y is a function of x and (2) tells that y is a function of u. It is ambiguous to say "*the function* y"; we have to mention the independent variable.

If we denote g and f to be the functions given by

(1) $f(x) = x^2 + 1$

(2) $g(x) = x^3$ (*we can use any symbol for the dummy variable*)

the function obtained by the 2-step process is called the *composition* of g with f.

Definition 5.6.1 Let X, Y and Z be sets. Let $f : X \to Y$ and $g : Y \to Z$ be functions. We call ***the composition of g with f***, and write $g \circ f$, to mean the function from X to Z given by

$$(g \circ f)(x) = g(f(x))$$

Remark $g \circ f$ is read 'g *circle* f' or 'g *composed with* f'.
$(g \circ f)(x)$ is read 'g *circle* f *of* x' and $g(f(x))$ is read 'g *of* f *of* x'.

Given any element x in X, to obtain $g(f(x))$, the element $f(x)$, which is an "output" of the function f, is used as an "input" of the function g; this is legitimate because $f(x)$ belongs to Y which is the domain of g.

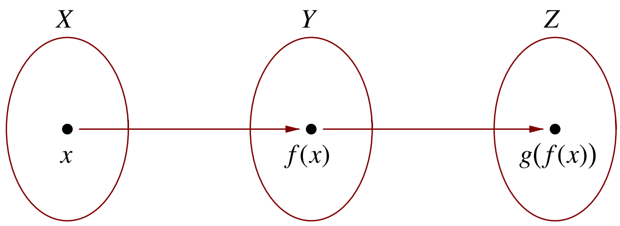


Figure 5.6.1

Example 5.6.1 Let f and g be the functions given by

$$f(x) = 2x + 3, \qquad g(x) = x^2 - 1$$

Find the value of each of the following

(*a*) $(g \circ f)(2)$ (*b*) $(f \circ g)(2)$

Explanation By convention, both f and g are functions from $\mathbb{R}$ to $\mathbb{R}$. Hence the functions $f \circ g$ and $g \circ f$ are functions from $\mathbb{R}$ to $\mathbb{R}$.

Solution (*a*)

$$\begin{aligned}
(g \circ f)(2) &= g(f(2)) && \text{By Definition 5.6.1} \\
&= g(2 \cdot 2 + 3) && \text{Substitute } x = 2 \text{ into the expression defining } f \\
&= g(7) \\
&= 7^2 - 1 && \text{Substitute } x = 7 \text{ into the expression defining } g \\
&= 48
\end{aligned}$$

(*b*)

$$\begin{aligned}
(f \circ g)(2) &= f(g(2)) && \text{By Definition 5.6.1} \\
&= f(2^2 - 1) && \text{Substitute } x = 2 \text{ into the expression defining } g \\
&= f(3) \\
&= 2 \cdot 3 + 3 && \text{Substitute } x = 3 \text{ into the expression defining } f \\
&= 9
\end{aligned}$$

□

Remark Note that $(f \circ g)(2) \neq (g \circ f)(2)$. Hence the functions $f \circ g$ and $g \circ f$ are not equal.

Example 5.6.2 Let f, g and h be functions from $\mathbb{R}$ to itself given by

$$f(x) = x^2 - x, \qquad g(x) = x + 2, \qquad h(x) = x - 2$$

Find and simplify each of the following:

(*a*) $(f \circ g)(x)$ (*b*) $(f \circ h)(x)$ (*c*) $\big((f \circ g) - (f \circ h)\big)(x)$

Explanation $(f \circ g)$ and $(f \circ h)$ are functions from $\mathbb{R}$ to itself. Their difference is the function $\big((f \circ g) - (f \circ h)\big)$.

Solution (*a*)

$$\begin{aligned}
(f \circ g)(x) &= f(g(x)) && \text{By Definition 5.6.1} \\
&= f(x + 2) && \text{Substitute } g(x) = x + 2 \\
&= (x + 2)^2 - (x + 2) \\
&= x^2 + 4x + 4 - x - 2 \\
&= x^2 + 3x + 2
\end{aligned}$$

(soln cont'd) (*b*)

$$\begin{aligned}(f \circ h)(x) &= f(h(x)) && \text{By Definition 5.6.1}\\ &= f(x-2) && \text{Substitute } h(x) = x-2\\ &= (x-2)^2 - (x-2)\\ &= x^2 - 4x + 4 - x + 2\\ &= x^2 - 5x + 6\end{aligned}$$

(*c*)

$$\begin{aligned}\big((f \circ g) - (f \circ h)\big)(x) &= (f \circ g)(x) - (f \circ h)(x) && \text{By Definition 5.5.1}\\ &= (x^2 + 3x + 2) - (x^2 - 5x + 6) && \text{By } (a) \text{ and } (b)\\ &= 8x - 4\end{aligned}$$

□

Remark The function f is the same as that given in Example 5.3.9. This example illustrates how to write the expressions in Example 5.3.9 (*a*), (*b*) and (*c*) using composition and difference of functions.

In defining the composition of g with f, we assume that the codomain of f is equal to the domain of g. This is to guarantee that the following is satisfied:

For every x belonging to the domain of f, the element $f(x)$ belongs to the domain of g. (5.6.2)

Note that if the codomain of f is a subset of the domain of g, Condition (5.6.2) is still satisfied.

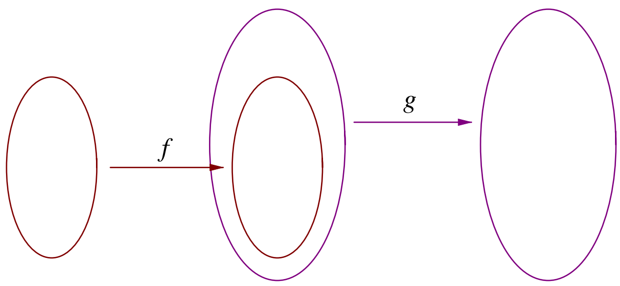


Figure 5.6.2

If the codomain of f is *not* a subset of the domain of g,

- there may exist an element a in the domain of f such that $f(a)$ does not belong to the domain of g; for such an element a, it is meaningless to consider $g(f(a))$.
- there may also exist an element b in the domain of f such that $f(b)$ belongs to the domain of g; for such an element b, we may consider $g(f(b))$.

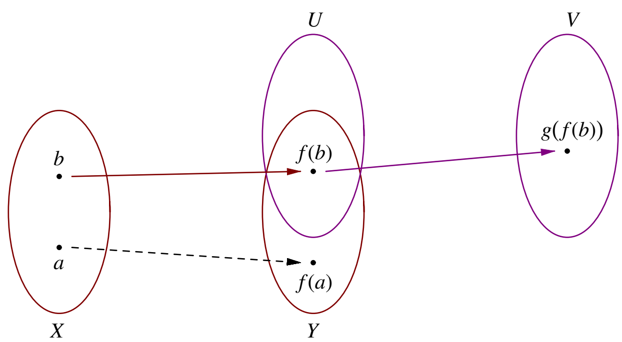


Figure 5.6.3

The following is a generalization of Definition 5.6.1.

Definition 5.6.2 Let X, Y, U and V be sets. Let $f : X \to Y$ and $g : U \to V$ be functions. We call ***the composition of g with f***, and write $g \circ f$, to mean the function from $\{x \in X : f(x) \in U\}$ to V given by

$$(g \circ f)(x) = g(f(x))$$

Explanation For every b belonging to the set $\{x \in X : f(x) \in U\}$, the element $f(b)$ belongs to the domain of g and so we may consider $g(f(b))$.

Example 5.6.3 Let $g(x) = 1 - 2x$ and $h(x) = \sqrt{x}$.

(a) Write down the domain of g.

(b) Find the domain of h.

(c) Find the domain of $h \circ g$.

(d) Find $(h \circ g)(x)$.

Solution (a) The domain of g is $\mathbb{R}$.

(b) $\sqrt{x}$ gives a real number if and only if $x \geq 0$.
The domain of h is $\{x \in \mathbb{R} : x \geq 0\}$.

(c)
$$\begin{aligned} \text{Domain of } h \circ g &= \{x \in \mathbb{R} : g(x) \text{ belongs to the domain of } h\} \\ &= \{x \in \mathbb{R} : g(x) \geq 0\} \\ &= \{x \in \mathbb{R} : 1 - 2x \geq 0\} \\ &= \{x \in \mathbb{R} : \tfrac{1}{2} \geq x\} \end{aligned}$$

(d)
$$\begin{aligned} (h \circ g)(x) &= h(g(x)) \\ &= h(1 - 2x) \\ &= \sqrt{1 - 2x} \end{aligned}$$
□

Remark $h \circ g$ is the function f considered in Example 5.3.3.

Exercise 5.6

1. Let $f(x) = 2x + 3$ and let $g(x) = 4 - x^2$. Find

   (a) $(f \circ g)(1)$ (b) $(g \circ f)(1)$ (c) $(f + g)(1)$ (d) $(fg)(1)$

2. Let $f(x) = x - 5$ and let $g(x) = 1 + 2x$. Find

   (a) $(f \circ g)(-2)$ (b) $(g \circ f)(3)$ (c) $(f \circ g)(x)$
   (d) $(g \circ f)(x)$ (e) $(g \circ f)(x^2)$ (f) $(f \circ g)(x^2)$
   (g) $(g \circ f)(x + 2)$ (h) $(f \circ g)(x - 5)$

5.7 Graphs of Functions

Graphs of Equations Recall that the graph of an equation with two unknowns x and y is the subset of the rectangular coordinate plane consisting of all ordered pairs of real numbers that are solutions to the equation (see Definition **??** in Chapter **??**). For example,

- the graph of the equation $x - 2y + 3 = 0$ is the line having slope equal to $\frac{1}{2}$ and y-intercept equal to $\frac{3}{2}$;

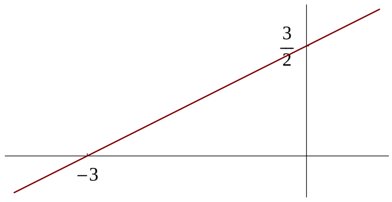


Figure 5.7.1

- the graph of the equation $x^2 + y^2 = 1$ is the circle with center at the origin and radius equal to 1,

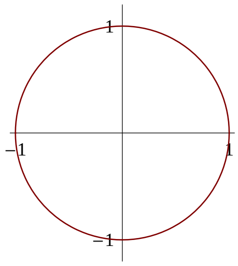


Figure 5.7.2

more details on equations of circles can be found in Chapter **??**.

Before stating the definition for the graph of a function (from a subset of $\mathbb{R}$ to $\mathbb{R}$), we consider an example. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ given by

$$f(x) = x^2 - 2x - 3$$

Putting $y = f(x)$, we get

$$y = x^2 - 2x - 3 \tag{5.7.1}$$

which can be considered as an equation with two unknowns x and y. The graph of Equation (5.7.1) is called the graph of the function f.

The graph of Equation (5.7.1) (that is, the graph of the function f) is shown in Figure 5.7.3. More details on graphs of quadratic functions can be found in Chapter **??**.

Note that an ordered pair (a, b) of real numbers is a solution to Equation (5.7.1) if and only if $b = f(a)$. Thus the graph of f is the subset of the rectangular plane consisting of all points (a, b) such that $b = f(a)$.

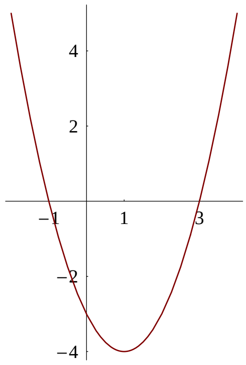


Figure 5.7.3

Definition 5.7.1 Let X be a subset of $\mathbb{R}$. Let $f : X \to \mathbb{R}$ be a function. We call ***the graph of f*** to mean the set $\{(x, y) \in \mathbb{R}^2 : x \in X \text{ and } y = f(x)\}$.

Remark In describing the graph of f, the condition '$x \in X$' can be omitted. This is because in order that an ordered pair (a, b) satisfies $f(a) = b$, the number a must belong to the domain of f.

Example 5.7.1 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = 4$. Then

- The point $(1, 3)$ does not belong to the graph of f.

  Reason $f(1) \neq 3$ since $f(1) = 4$.

- The point $(1, 4)$ belongs to the graph of f.

  Reason $f(1) = 4$.

The graph of f is the set of all $(x, y) \in \mathbb{R}^2$ such that $y = 4$. It is the horizontal line having y-intercept equal to 4 (see Figure 5.7.4).

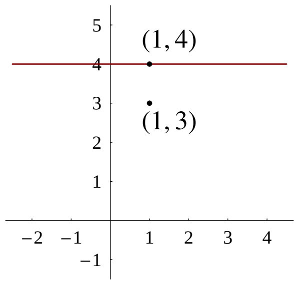


Figure 5.7.4

The function f described in Example 5.7.1 is a constant function. Its graph is a horizontal line. More generally, we have the following result.

Graphs of Constant Functions Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = c$$

where c be a real number (a constant). Then the graph of f is the set $\{(x, y) \in \mathbb{R}^2 : y = c\}$. It is the line given by the equation $y = c$, that is, the horizontal line having y-intercept equal to c.

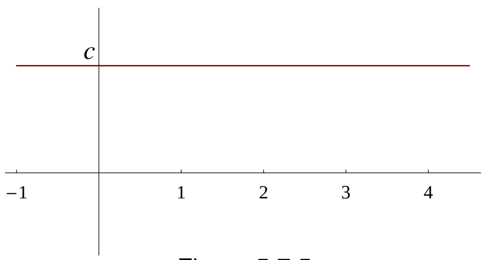


Figure 5.7.5

Example 5.7.2 Let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = 2x + 3$. Then

- The point $(1, 5)$ belongs to the graph of g.

  Reason $g(1) = 5$.

- The point $(2, 6)$ does not belong to the graph of g.

  Reason $g(2) \neq 6$ since $g(2) = 7$.

The graph of g is the set of all $(x, y) \in \mathbb{R}^2$ such that $y = 2x + 3$. It is the line having slope equal to 2 and y-intercept equal to 3 (see Figure 5.7.6).

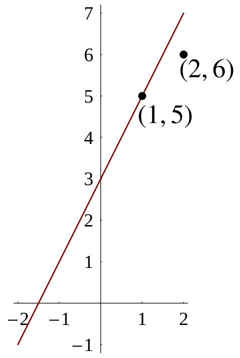


Figure 5.7.6

The function g described in Example 5.7.2 is a linear function (the meaning is given below). Its graph is a line that is not horizontal and not vertical.

- By a *linear function*, we mean a function from $\mathbb{R}$ to $\mathbb{R}$ that is given by

$$x \mapsto ax + b$$

  where a and b are real numbers with $a \neq 0$.

Graphs of Linear Functions Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = ax + b$$

where a and b are real numbers with $a \neq 0$. Then the graph of f is the set $\{(x, y) \in \mathbb{R}^2 : y = ax + b\}$. It is the line given by the equation $y = ax + b$, that is, the line having slope equal to a and y-intercept equal to b.

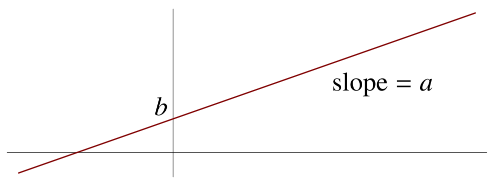


Figure 5.7.7

In Chapter **??**, we will consider quadratic functions and their graphs. Constant functions, linear functions and quadratic functions are special cases of polynomial functions. In Chapter **??**, we will consider polynomial functions. Given a polynomial function, we can use *differentiation* to determine where the graph of the function goes up or down. Figure 5.7.8 and Figure 5.7.9 show the graphs of the functions f and g given by

$$f(x) = x^3 - 2x - 3x + 4, \qquad g(x) = \frac{x^4}{2} - x^2 - 2x - 3$$

respectively. Readers who want to know how to obtain the graphs by "calculation" may take a course on *calculus*.

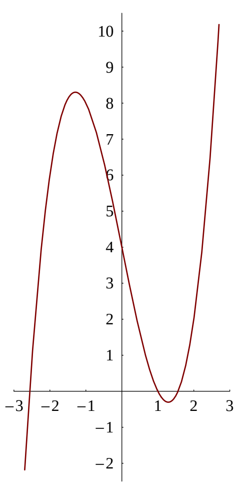


Figure 5.7.8

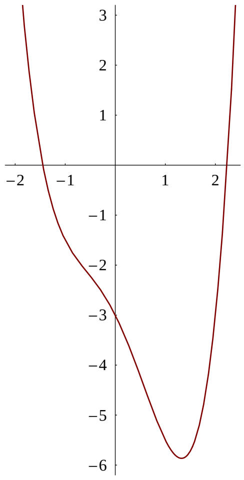


Figure 5.7.9

A property of Graphs of Functions Let $f : X \to \mathbb{R}$ be a function, where X is a subset of $\mathbb{R}$. Then every vertical line in the rectangular coordinate plane intersects the graph of f in at most one point.

Reason Denote $\mathcal{G}$ to be the graph of f. Suppose ℓ is a vertical line in the rectangular coordinate plane. Denote a to be the x-intercept of ℓ.

- If a does not belong to the domain of f, then there does not exist any point on $\mathcal{G}$ whose x-coordinate is equal to a. Hence $\ell \cap \mathcal{G} = \{\}$.
- If a belongs to the domain of f, then there is exactly one point on $\mathcal{G}$, namely $(a, f(a))$, whose x-coordinate is equal to a. Hence the point $(a, f(a))$ is the only element of $\ell \cap \mathcal{G}$.

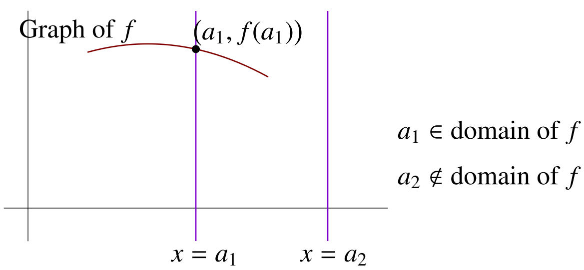


Figure 5.7.10

From the above property of graphs of functions, we obtain the following:

(1) If $\mathcal{C}$ is (equal to) the graph of a function, then every vertical line intersects $\mathcal{C}$ in at most one point.

In fact, the converse of (1) is also true:

(2) If every vertical line intersects $\mathcal{C}$ in at most one point, then $\mathcal{C}$ is (equal to) the graph of a function.

Reason The conclusion means that there exists a function f whose graph is equal to $\mathcal{C}$. To find such an f, which has domain a subset of $\mathbb{R}$ and codomain $\mathbb{R}$, we have to tell what is the domain of f and tell the image under f of every element of the domain. To this end, we denote X to be the set of all real numbers a such that the vertical line $x = a$ intersects the curve $\mathcal{C}$ at exactly one point and denote f to be the function from X to $\mathbb{R}$ given by

for $a \in X$, $\quad f(a) = b \quad$ where b is the y-coordinate of the point belonging to the intersection of the line $x = a$ and $\mathcal{C}$

It is straightforward to check that the graph of f is equal to $\mathcal{C}$.

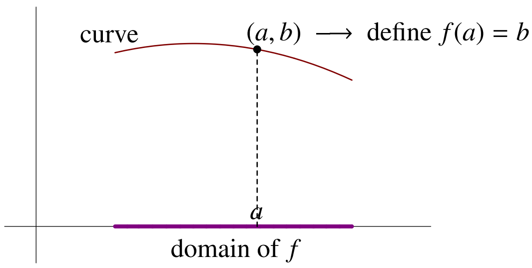


Figure 5.7.11

Combining Results (1) and (2), we get a necessary and sufficient condition for a curve to be the graph of a function.

Vertical Line Test A curve on the rectangular coordinate plane is the graph of a function (from a subset of $\mathbb{R}$ to $\mathbb{R}$) if and only if every vertical line on the rectangular coordinate plane intersects the curve in at most one point.

Explanation The result means two things:

- If there is a vertical line that intersects the curve in two or more points, then the curve can't be the graph of any function.
- If every vertical line doesn't intersect the curve or intersects the curve in only one point, then there is a function f such that the graph of f is the curve.

Example 5.7.3 Consider the curves shown in the following figures.

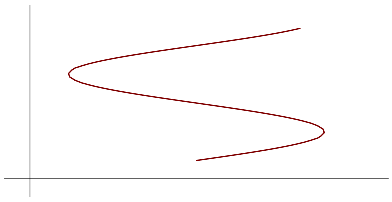


Figure 5.7.12

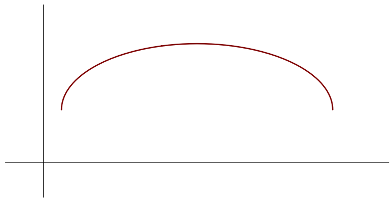


Figure 5.7.13

- The curve shown in Figure 5.7.12 is not the graph of any function. This is because there exists (at least) a vertical line that intersects the curve in more than one points (see Figure 5.7.12 (a)).
- The curve shown in Figure 5.7.13 is the graph of a function. This is because every vertical line does not intersect the curve or intersects the curve in only one point (see Figure 5.7.13 (a)).

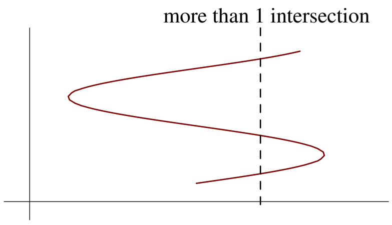


Figure 5.7.12 (a)

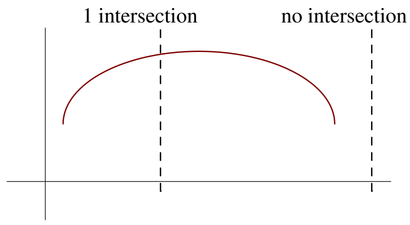
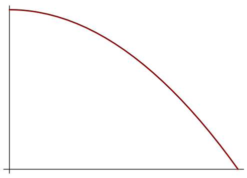


Figure 5.7.13 (a)

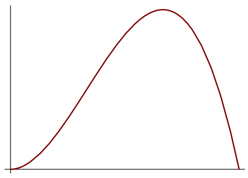
Exercise 5.7

1. For each of the following curves (or polygon), determine whether it is the graph of a function.

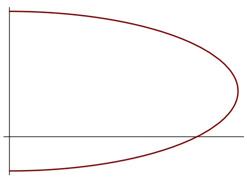
(*a*)



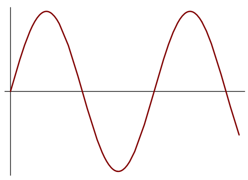
(*b*)



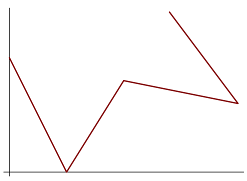
(*c*)



(*d*)



(*e*)



Question 3 in Exercise 5.1 is similar to this question.

2. On the same rectangular coordinate plane, sketch the graphs of the functions f, g and h given as follows:

   (a) $f(x) = 2x$ (b) $g(x) = 2x + 1$ (c) $h(x) = 2x - 3$

   Do you notice any relation between the graphs of f, g and h?

3. On the same rectangular coordinate plane, sketch the graphs of the functions f, g and h given as follows:

   (a) $f(x) = -x$ (b) $g(x) = -(x + 1)$ (c) $h(x) = -(x - 3)$

   Do you notice any relation between the graphs of f, g and h?