Contents

Chapter 4

Complex Numbers

In the real number system, we can perform arithmetic operations and we can take (principle) square roots for non-negative real numbers, however we can't take square roots for negative real numbers; this is because the square of every real number is at least 0. It would be nice if we can enlarge the set $\mathbb{R}$ to a set in which we can extend the arithmetic operations and take square root for every number in that set.

First we consider the equation $x^2 = -1$. By introducing a number, denoted by i, such that $\mathsf{i}^2 = -1$, the number -1 has a square root. The set we want to construct should include the number i together with all the real numbers. Since we want to have addition and multiplication,

- for every real number r, the set must contain $r + \mathsf{i}$;
- for every real number s, the set must contain $s \times \mathsf{i}$, which is also written as $s \cdot \mathsf{i}$ or $s\mathsf{i}$,

hence for every real number a and every number b, the set must contain $a + b\mathsf{i}$.

4.1 Notations and Terminologies

Informal Definition An expression in the form $a + b\mathsf{i}$, where a and b are real numbers, is called a ***complex number***.

Remark Inquisitive readers may ask what are the meaning of + and · (note that $b\mathsf{i}$ means $b \cdot \mathsf{i}$). To be vigorous, we should first define complex numbers without using the notations '+' and '·' and then define addition and multiplication for complex numbers and obtain, under some conventions, the result that every complex number can be written in the form $a + b\mathsf{i}$. Details of such constructions are tedious and are thus omitted.

Example 4.1.1 Each of the The following is a complex numbers:

$$1 + 2\mathrm{i}, \quad -3 + 7\mathrm{i}, \quad \pi + (-5)\mathrm{i}, \quad \frac{1}{2} + 0.34\mathrm{i}, \quad 0 + 3\mathrm{i}, \quad \sqrt{2} + 0\mathrm{i}$$

Equality of Complex Numbers Complex numbers $a + b\mathrm{i}$ and $c + d\mathrm{i}$ (where $a, b, c, d \in \mathbb{R}$) are defined to be ***equal***, written $a + b\mathrm{i} = c + d\mathrm{i}$, if and only if $a = c$ and $b = d$.

Remark We also say "$a + b\mathrm{i}$ *is equal to* $c + d\mathrm{i}$" or "$a + b\mathrm{i}$ *equals* $c + d\mathrm{i}$".

Example 4.1.2 Suppose a and b are real numbers such that $a + 4\mathrm{i} = b + 2a\mathrm{i}$. Find the values of a and b.

Solution Since $a + 4\mathrm{i} = b + 2a\mathrm{i}$, it follows from Equality of Complex Numbers that

$$a = b \qquad \text{and} \qquad 4 = 2a$$

Solving, we get $a = b = 2$. □

Notation 4.1.1 We denote $\mathbb{C}$ to be the set of all complex numbers.

Convention For every real number a, we identify a as the complex number $a + 0\mathrm{i}$, that is,

$$a = a + 0\mathrm{i}$$

For every real number b, we write $b\mathrm{i}$ to denote the complex number $0 + b\mathrm{i}$, that is,

$$b\mathrm{i} = 0 + b\mathrm{i}$$

For the case where $b = 1$, we write i to denote the complex number $1\mathrm{i}$. The number i is called the *imaginary unit*.

Remark Every real number is a complex number, that is, $\mathbb{R} \subseteq \mathbb{C}$.

Example 4.1.3 Each of the following is a complex number:

$$0, \quad 1.234, \quad \pi, \quad -\sqrt{3}, \quad 2\mathrm{i}, \quad -\frac{4}{3}\mathrm{i}, \quad \frac{\pi}{2}\mathrm{i}, \quad 0\mathrm{i}$$

Remark By convention, $0\mathrm{i} = 0 + 0\mathrm{i} = 0$ is a real number.

A complex number, as an element of the set $\mathbb{C}$, can also be denoted by a single letter. Every complex number z can be written as

$$z = a + b\mathrm{i}$$

where a and b are real numbers. The real number a is called the *real part* of z and the real number b is called the *imaginary part* of z.

Example 4.1.4

$$\begin{aligned} \text{The real part of } (2 + 3\mathrm{i}) &= 2 \\ \text{The imaginary part of } (2 + 3\mathrm{i}) &= 3 \end{aligned}$$

Example 4.1.5

$$\begin{aligned} \text{The real part of } -5 &= -5 \\ \text{The imaginary part of } -5 &= 0 \qquad \text{since } -5 = -5 + 0\mathrm{i} \end{aligned}$$

Example 4.1.6

$$\begin{aligned} \text{The real part of } \frac{2}{3}\mathrm{i} &= 0 \qquad \text{since } \frac{2}{3}\mathrm{i} = 0 + \frac{2}{3}\mathrm{i} \\ \text{The imaginary part of } \frac{2}{3}\mathrm{i} &= \frac{2}{3} \end{aligned}$$

In Section 4.2 and Section 4.3, we will introduce arithmetic operations $+$, $\times$, $-$ and $\div$ in $\mathbb{C}$. By convention, the priority rules for the arithmetic operations in $\mathbb{C}$ is the same as that in $\mathbb{R}$, for example, *multiplication has a higher priority than addition*, thus $u + v \times w$ means $u + (v \times w)$. Similarly, terminologies, like *sum* and *product* etc., for complex numbers are the same as that for real numbers.

Exercise 4.1

1. In each of the following, a and b are real numbers. Find the values of a and b.

   (*a*) $a + 2\mathrm{i} = 4 + b\mathrm{i}$ (*b*) $(2a + 1) + b\mathrm{i} = -5 + (a + 5)\mathrm{i}$

   (*c*) $(2 - a) + (3 - b)\mathrm{i} = 4$ (*d*) $-3\mathrm{i} = (2a + b) + (2a - b)\mathrm{i}$

2. For each of the following complex numbers, write down its real part and imaginary part.

   (*a*) $-5 + 32\mathrm{i}$ (*b*) $7 + (-11)\mathrm{i}$ (*c*) $0.23\mathrm{i}$

   (*d*) $-\sqrt{5}$ (*e*) i (*f*) 0

3. For each of the following statements, determine whether it is true or false.

   (*a*) All rational numbers are complex numbers.

   (*b*) Some complex numbers are not integers.

   (*c*) All complex numbers are irrational numbers.

   (*d*) All complex numbers that are not rational numbers are irrational numbers.

   (*e*) Some complex numbers that are not rational numbers are irrational numbers.

4.2 Addition and Multiplication of Complex Numbers

Addition in $\mathbb{C}$ Let a, b, c, d be real numbers. We define

$$(a + b\mathrm{i}) + (c + d\mathrm{i}) = (a + c) + (b + d)\mathrm{i}$$

Explanation To find the sum of two complex numbers, we can simply add their real parts and add their imaginary parts.

Remark In the expression

$$(a + c) + (b + d)\mathrm{i}$$

there are three '+' symbols. The first and the third ones are the symbols for addition of real numbers, whereas the second one is the symbol used in an expression for a complex number.

In the expression

$$(a + b\mathrm{i}) + (c + d\mathrm{i})$$

there are also three '+' symbols. The second one is the symbol for addition of complex numbers, whereas the first and third ones are the symbols used in expressions for complex numbers. Later, we will see that the symbol $+$ in $a + b\mathrm{i}$ can also be considered as the symbol for addition of complex numbers.

Example 4.2.1

$$\begin{aligned} (2 + 5\mathrm{i}) + (3 + 4\mathrm{i}) &= (2 + 3) + (5 + 4)\mathrm{i} && \text{Definition of } + \text{ in } \mathbb{C} \\ &= 5 + 9\mathrm{i} && \text{Arithmetic in } \mathbb{R} \end{aligned}$$

Example 4.2.2

$$\begin{aligned} (-5 + 6\mathrm{i}) + 1 &= (-5 + 6\mathrm{i}) + (1 + 0\mathrm{i}) && \text{Rewrite 1 in the form } a + b\mathrm{i} \\ &= (-5 + 1) + (6 + 0)\mathrm{i} && \text{Definition of } + \text{ in } \mathbb{C} \\ &= -4 + 6\mathrm{i} && \text{Arithmetic in } \mathbb{R} \end{aligned}$$

Example 4.2.3

$$\begin{aligned} 7\mathrm{i} + (2 + (-3)\mathrm{i}) &= (0 + 7\mathrm{i}) + (2 + (-3)\mathrm{i}) && \text{Rewrite } 7\mathrm{i} \text{ in the form } a + b\mathrm{i} \\ &= (0 + 2) + (7 + (-3))\mathrm{i} && \text{Definition of } + \text{ in } \mathbb{C} \\ &= 2 + 4\mathrm{i} && \text{Arithmetic in } \mathbb{R} \end{aligned}$$

Example 4.2.4

$$\begin{aligned} 1 + 2 &= (1 + 0\mathrm{i}) + (2 + 0\mathrm{i}) && \text{Rewrite 1 and 2 in the form } a + b\mathrm{i} \\ &= (1 + 2) + (0 + 0)\mathrm{i} && \text{Definition of } + \text{ in } \mathbb{C} \\ &= 3 + 0\mathrm{i} && \text{Arithmetic in } \mathbb{R} \\ &= 3 \end{aligned}$$

Remark To find $1 + 2$, we may consider 1 and 2 as complex numbers or as real numbers. In general, to find $s + t$, where s and t are real numbers, we may consider s and t as complex numbers or as real numbers. In other words, addition in $\mathbb{C}$ is an extension of that in $\mathbb{R}$.

Example 4.2.5 Suppose a and b are real numbers satisfying $(a + b\mathrm{i}) + (2 + 3\mathrm{i}) = -5 + 6\mathrm{i}$. Find the values of a and b.

Solution
$$\begin{aligned} (a + b\mathrm{i}) + (2 + 3\mathrm{i}) &= -5 + 6\mathrm{i} \\ (a + 2) + (b + 3)\mathrm{i} &= -5 + 6\mathrm{i} \qquad \text{Addition in } \mathbb{C} \end{aligned}$$

By Equality of Complex Numbers,

$$a + 2 = -5 \qquad \text{and} \qquad b + 3 = 6$$

Solving we get $a = -7$ and $b = 3$.

Remark Later, we will discuss subtraction of complex numbers. Using subtraction notation, we have $a + b\mathrm{i} = (-5 + 6\mathrm{i}) - (2 + 3\mathrm{i})$.

Remark Let a and b be real numbers. By convention, both a and $b\mathrm{i}$ are complex numbers:

$$a = a + 0\mathrm{i}, \qquad b\mathrm{i} = 0 + b\mathrm{i}$$

Using definition, the sum of the complex numbers a and $b\mathrm{i}$ is found as follows:

$$\begin{aligned} a + b\mathrm{i} &= (a + 0\mathrm{i}) + (0 + b\mathrm{i}) && \text{By convention} \\ &= (a + 0) + (0 + b)\mathrm{i} && \text{Definition of addition in } \mathbb{C} \\ &= a + b\mathrm{i} && \text{Arithmetic in } \mathbb{R} \end{aligned}$$

Thus the notation $a + b\mathrm{i}$ can be considered as the sum of the complex numbers a and $b\mathrm{i}$ or an expression that represents a complex number.

Below we list some important properties of addition in $\mathbb{C}$.

Commutative Property of Addition In $\mathbb{C}$, $u + v \equiv v + u$

Associative Property of Addition In $\mathbb{C}$, $(u + v) + w \equiv u + (v + w)$

Zero Element of Addition In $\mathbb{C}$, $0 + u \equiv u$

Cancellation Principle for Addition Let $u, v, w \in \mathbb{C}$.
Then $u + v = u + w$ if and only if $v = w$.

Explanation The above properties can be proved easily using the corresponding properties in $\mathbb{R}$. This is because the sum of two complex numbers is defined using sum of real parts and sum of imaginary parts of the complex numbers.

Multiplication in $\mathbb{C}$ Let a, b, c, d be real numbers. We define

$$(a + b\mathrm{i}) \times (c + d\mathrm{i}) = (ac - bd) + (ad + bc)\mathrm{i}$$

Explanation The expression on the right-side of the above equality seems quite complicated. After discussing properties of addition and multiplication, we will have a simple way to find the product of two complex numbers.

Remark $(a + b\mathrm{i}) \times (c + d\mathrm{i})$ is also written as $(a + b\mathrm{i}) \cdot (c + d\mathrm{i})$ or $(a + b\mathrm{i})(c + d\mathrm{i})$.

Example 4.2.6

$$\begin{aligned} (2 + 3\mathrm{i}) \times (4 + 5\mathrm{i}) &= (2 \cdot 4 - 3 \cdot 5) + (2 \cdot 5 + 3 \cdot 4)\mathrm{i} && \text{Definition of } \times \text{ in } \mathbb{C} \\ &= -7 + 22\mathrm{i} && \text{Arithmetic in } \mathbb{R} \end{aligned}$$

Example 4.2.7

$$\begin{aligned} 2 \cdot (7 + 5\mathrm{i}) &= (2 + 0\mathrm{i}) \cdot (7 + 5\mathrm{i}) && \text{Rewrite 2 in the form } a + b\mathrm{i} \\ &= (2 \cdot 7 - 0 \cdot 5) + (2 \cdot 5 + 0 \cdot 7)\mathrm{i} && \text{Definition of } \times \text{ in } \mathbb{C} \\ &= 14 + 10\mathrm{i} && \text{Arithmetic in } \mathbb{R} \end{aligned}$$

Example 4.2.8

$$\begin{aligned} 2 \times 3 &= (2 + 0\mathrm{i}) \cdot (3 + 0\mathrm{i}) && \text{Rewrite 2 and 3 in the form } a + b\mathrm{i} \\ &= (2 \cdot 3 - 0 \cdot 0) + (2 \cdot 0 + 0 \cdot 3)\mathrm{i} && \text{Definition of } \times \text{ in } \mathbb{C} \\ &= 6 + 0\mathrm{i} && \text{Arithmetic in } \mathbb{R} \\ &= 6 \end{aligned}$$

Remark To find 2×3, we may consider 2 and 3 as complex numbers or as real numbers. In general, to find $s \times t$, where s and t are real numbers, we may consider s and t as complex numbers or as real numbers. In other words, multiplication in $\mathbb{C}$ is an extension of that in $\mathbb{R}$.

Remark Let b be a real number. By convention, b is a complex number:

$$b = b + 0\mathrm{i}$$

Hence

$$\begin{aligned} b \times \mathrm{i} &= (b + 0\mathrm{i}) \cdot (0 + 1\mathrm{i}) && \text{By convention} \\ &= (b \cdot 0 - 0 \cdot 1) + (b \cdot 1 + 0 \cdot 0)\mathrm{i} && \text{By definition of multiplication in } \mathbb{C} \\ &= 0 + b\mathrm{i} && \text{Arithmetic in } \mathbb{R} \\ &= b\mathrm{i} && \text{By convention} \end{aligned}$$

Thus the notation $b\mathrm{i}$ can be considered as the product of the complex numbers b and i or an expression that represents a complex number. For the case where $b = 0$, the expression $0\mathrm{i}$ is a real number (because $0\mathrm{i} = 0$); for the case where $b \neq 0$, the expression $b\mathrm{i}$ is not a real number, it is a non-zero multiple of the pure imaginary unit i and is called a *purely imaginary number*.

Notation 4.2.1 Let u be a complex number and let n be a positive integer. We denote

$$u^n = \underbrace{u \cdot u \cdots u}_{n \text{ factors}}$$

Theorem 4.2.1 $\mathrm{i}^2 = -1$

Proof
$$\begin{aligned} \mathrm{i}^2 &= \mathrm{i} \cdot \mathrm{i} && \text{By Notation 4.2.1} \\ &= (0 + 1\mathrm{i}) \cdot (0 + 1\mathrm{i}) && \text{By convention} \\ &= (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 1 \cdot 0)\mathrm{i} && \text{By definition of multiplication in } \mathbb{C} \\ &= -1 + 0\mathrm{i} && \text{Arithmetic in } \mathbb{R} \\ &= -1 && \text{By convention} \end{aligned}$$
□

Remark The complex number i is a solution to the equation $x^2 = -1$.

Example 4.2.9 (*a*)
$$\begin{aligned} \mathrm{i}^3 &= \mathrm{i}^2 \cdot \mathrm{i} \\ &= (-1)\mathrm{i} && \text{By Theorem 4.2.1} \end{aligned}$$

(*b*)
$$\begin{aligned} \mathrm{i}^4 &= \mathrm{i}^2 \cdot \mathrm{i}^2 \\ &= (-1) \cdot (-1) && \text{By Theorem 4.2.1} \\ &= 1 \end{aligned}$$

Remark Let n be a positive integer. Then
$$\mathrm{i}^n = \begin{cases} 1 & \text{if } n \text{ is a multiple of 4} \\ \mathrm{i} & \text{if } n \text{ is 1+ a multiple of 4} \\ -1 & \text{if } n \text{ is 2+ a multiple of 4} \\ (-1)\mathrm{i} & \text{if } n \text{ is 3+ a multiple of 4} \end{cases}$$

Remark Using the notation for the negative of a complex number (see Definition 4.3.1), the complex number $(-1)\mathrm{i}$ can also be written as $-\mathrm{i}$.

Commutative Property of Multiplication In $\mathbb{C}$, $u \cdot v \equiv v \cdot u$

Associative Property of Multiplication In $\mathbb{C}$, $(u \cdot v) \cdot w \equiv u \cdot (v \cdot w)$

Distributive Property of Multiplication over Addition In $\mathbb{C}$,
$$u \cdot (v + w) \equiv u \cdot v + u \cdot w$$
$$(u + v) \cdot w \equiv u \cdot w + v \cdot w$$

Identity Element of Multiplication In $\mathbb{C}$, $1 \cdot u \equiv u$

Cancellation Principle for Multiplication Let $u, v, w \in \mathbb{C}$ with $u \neq 0$.
Then $u \cdot v = u \cdot w$ if and only if $v = w$.

Product Zero Principle Let $u, v \in \mathbb{C}$.
Then $u \cdot v = 0$ if and only if $u = 0$ or $v = 0$.

Explanation The proofs of the above properties are straightforward (but some are tedious).

Remark For $a, b \in \mathbb{R}$, the notation $a + b\mathrm{i}$ can be considered as the sum of the complex number a and the complex number that is the product of b and i. Thus we can perform multiplication in $\mathbb{C}$ using properties of addition and multiplication. We use Example 4.2.6 as an illustration.

$$\begin{aligned}
(2+3\mathrm{i})\cdot(4+5\mathrm{i}) &= 2\cdot(4+5\mathrm{i})+3\mathrm{i}\cdot(4+5\mathrm{i}) && \text{Distributive Property of } \times \text{ over } + \\
&= (2\cdot 4+2\cdot 5\mathrm{i})+(3\mathrm{i}\cdot 4+3\mathrm{i}\cdot 5\mathrm{i}) && \text{Distributive Property of } \times \text{ over } + \\
&= 8+2\cdot(5\cdot\mathrm{i})+(3\cdot\mathrm{i})\cdot 4+(3\cdot\mathrm{i})\cdot(5\cdot\mathrm{i}) && \text{Arithmetic in } \mathbb{R} \text{ and Interpret } b\mathrm{i} \text{ as } b\cdot\mathrm{i} \\
&= 8+(2\cdot 5)\cdot\mathrm{i}+(3\cdot 4)\cdot\mathrm{i}+(3\cdot 5)\cdot(\mathrm{i}\cdot\mathrm{i}) && \text{Associative and Commutative Property of } \times \\
&= 8+10\cdot\mathrm{i}+12\cdot\mathrm{i}+15\cdot(-1) && \text{Arithmetic in } \mathbb{R} \text{ and Theorem 4.2.1} \\
&= (8+(-1)\cdot 15)+(10\cdot\mathrm{i}+12\cdot\mathrm{i}) && \text{Associative and Commutative Property of } + \\
&= (8+(-1)\cdot 15)+(10+12)\cdot\mathrm{i} && \text{Distributive Property of } \times \text{ over } + \\
&= -7+22\mathrm{i} && \text{Arithmetic in } \mathbb{R}
\end{aligned}$$

The above steps can be shortened as follows:

$$\begin{aligned}
(2+3\mathrm{i})\cdot(4+5\mathrm{i}) &= 2\cdot 4+2\cdot 5\cdot\mathrm{i}+3\cdot 4\cdot\mathrm{i}+3\cdot 5\cdot\mathrm{i}^2 \\
&= 8+10\mathrm{i}+12\mathrm{i}+15\cdot(-1) \\
&= -7+22\mathrm{i}
\end{aligned}$$

Example 4.2.10

$$\begin{aligned}
(-3+4\mathrm{i})\cdot(5+\mathrm{i}) &= -3\cdot 5+(-3)\cdot\mathrm{i}+4\cdot 5\cdot\mathrm{i}+4\cdot\mathrm{i}^2 \\
&= -15+(-3)\mathrm{i}+20\mathrm{i}+4\cdot(-1) \\
&= -19+17\mathrm{i}
\end{aligned}$$

Example 4.2.11

$$\begin{aligned}
-3\cdot(1+2\mathrm{i}) &= -3\cdot 1+(-3)\cdot 2\cdot\mathrm{i} \\
&= -3+(-6)\mathrm{i}
\end{aligned}$$

Remark The complex number $-3+(-6)\mathrm{i}$ can also be written as $-3-6\mathrm{i}$, where the symbol '$-$' is the notation for subtraction (see Section 4.3).

Example 4.2.12

$$\begin{aligned}
(\pi+\sqrt{2}\mathrm{i})\cdot -3\mathrm{i} &= \pi\cdot(-3)\cdot\mathrm{i}+\sqrt{2}\cdot(-3)\cdot\mathrm{i}^2 \\
&= (-3\pi)\mathrm{i}+\sqrt{2}\cdot(-3)\cdot(-1) \\
&= 3\sqrt{2}+(-3\pi)\mathrm{i}
\end{aligned}$$

Example 4.2.13 Suppose a is a real number and $(1+2\mathrm{i})\cdot(a+\mathrm{i})$ is a purely imaginary number. Find the value of a.

Explanation A purely imaginary number is a complex number that can be written in the form $b\mathrm{i}$ where b is a non-zero real number.

Solution
$$\begin{aligned}(1+2\mathrm{i})\cdot(a+\mathrm{i}) &= a+\mathrm{i}+2a\mathrm{i}+2\mathrm{i}^2\\ &= (a-2)+(1+2a)\mathrm{i}\end{aligned}$$

Since $(1+2\mathrm{i})\cdot(a+\mathrm{i})$ is a purely imaginary number, its real part must be 0, that is,
$$a-2=0$$
Hence $a=2$. □

Remark With $a=2$, the complex number $(1+2\mathrm{i})\cdot(a+\mathrm{i})=5\mathrm{i}$ is a purely imaginary number.

Example 4.2.14 Suppose a and b are real numbers satisfying $(a+b\mathrm{i})\cdot(2+3\mathrm{i})=-4+7\mathrm{i}$. Find the values of a and b.

Solution
$$\begin{aligned}(a+b\mathrm{i})\cdot(2+3\mathrm{i}) &= -4+7\mathrm{i}\\ 2a+3a\mathrm{i}+2b\mathrm{i}+3b\mathrm{i}^2 &= -4+7\mathrm{i}\\ (2a-3b)+(3a+2b)\mathrm{i} &= -4+7\mathrm{i}\end{aligned}$$

Hence by Equality of complex numbers
$$2a-3b=-4 \qquad \text{and} \qquad 3a+2b=7$$

$$\begin{aligned}4a-6b &= -8\\ 9a+6b &= 21\\ \hline 13a &= 13\end{aligned}$$

Solving, we get $a=1$ and $b=2$.

□

Remark Later, we will discuss division of complex numbers. Using division notation, we have $(a+b\mathrm{i})=(-4+7\mathrm{i})\div(2+3\mathrm{i})$.

Exercise 4.2

1. Perform the following addition of complex numbers:

   (a) $(6+5\mathrm{i})+(-2+7\mathrm{i})$ (b) $(2+\mathrm{i})+(-2+3\mathrm{i})$

   (c) $(1+(-2)\mathrm{i})+(-2+3\mathrm{i})$ (d) $(13+7\mathrm{i})+5$

   (e) $-3\mathrm{i}+(11+7\mathrm{i})$ (f) $3\mathrm{i}+\mathrm{i}$

   (g) $-\mathrm{i}+\frac{1}{2}\mathrm{i}$ (h) $(4+\pi\mathrm{i})+(-5+6\mathrm{i})+(7+(-\pi)\mathrm{i})$

2. Suppose a and b are real numbers satisfying $(3+a\mathrm{i})+(b+2\mathrm{i})=a+\mathrm{i}$. Find the values of a and b.
3. Suppose x and y are real numbers satisfying $x+y\mathrm{i}+7\mathrm{i}=y$. Find the values of x and y.
4. Suppose u is a complex number satisfying $u+\mathrm{i}=3+7\mathrm{i}$. Find the value of u.

5. Perform the following multiplication of complex numbers:

(a)	$4 \cdot (-3 + \mathrm{i})$	(b)	$-2 \cdot (1 + 2\mathrm{i})$
(c)	$2\mathrm{i} \cdot (3 + 4\mathrm{i})$	(d)	$(1 + 2\mathrm{i}) \cdot (1 + 3\mathrm{i})$
(e)	$(-2 + \mathrm{i}) \cdot (3 + (-1)\mathrm{i})$	(f)	$0\mathrm{i} \cdot (2 + \pi\mathrm{i})$
(g)	$(3 + 4\mathrm{i}) \cdot (3 - 4\mathrm{i})$	(h)	$(1 + \mathrm{i}) \cdot (2 + 3\mathrm{i}) \cdot (-4 + 5\mathrm{i})$

6. Simplify the following:

(a)	i^7	(b)	i^{12}	(c)	i^9	(d)	i^6
(e)	$-3\mathrm{i}^4$	(f)	$(-3\mathrm{i})^4$	(g)	$(2\mathrm{i})^5$	(h)	$(-5\mathrm{i})^3$

7. Suppose r and s are real numbers satisfying $2 \cdot (r + s\mathrm{i}) = 4 + r\mathrm{i}$. Find the values of r and s.

8. Suppose a and b are real numbers satisfying $(a + b\mathrm{i}) \cdot (-2 + \mathrm{i}) = \mathrm{i}$. Find the values of a and b.

9. Suppose u is a complex number satisfying $u \cdot 2\mathrm{i} = 4 + 3\mathrm{i}$. Find the value of u.

10. Suppose a is a real number and $(a + 2\mathrm{i}) \cdot (-1 + 3\mathrm{i})$ is a real number. Find the value of a.

4.3 Subtraction and Division of Complex Numbers

To define subtraction of complex numbers, we can use the idea in Example 4.2.5. However, we will adopt a different approach. We will first introduce the concept of the *negative* of a complex number, then we will define subtraction $v - u$ for complex numbers u and v, and finally we will show that $v - u$ is the unique solution to the equation $u + x = v$.

Definition 4.3.1 Let u be a complex number. We call ***the negative of** u*, and write $-u$, to mean the complex number $(-1) \cdot u$.

Example 4.3.1 The negative of $(1 + 3\mathrm{i})$

$$\begin{aligned} \text{The negative of } (1 + 3\mathrm{i}) &= (-1) \cdot (1 + 3\mathrm{i}) && \text{Definition 4.3.1} \\ &= -1 \cdot 1 + (-1) \cdot 3 \cdot \mathrm{i} \\ &= -1 + (-3)\mathrm{i} \end{aligned}$$

Example 4.3.2

(a)
$$\begin{aligned} \text{The negative of } 2\mathrm{i} &= (-1) \cdot 2\mathrm{i} && \text{Definition 4.3.1} \\ &= (-1) \cdot 2 \cdot \mathrm{i} \\ &= -2\mathrm{i} \end{aligned}$$

(b)
$$\begin{aligned} \text{The negative of } -2\mathrm{i} &= (-1) \cdot (-2\mathrm{i}) && \text{Definition 4.3.1} \\ &= (-1) \cdot (-2) \cdot \mathrm{i} \\ &= 2\mathrm{i} \end{aligned}$$

Remark For every complex number u, the negative of (the negative of u) is equal to u, that is, $-(-u) = u$. We say that the complex numbers u and $-u$ are negatives of each other.

Theorem 4.3.1 Let $u \in \mathbb{C}$. Then $u + (-u) = 0$.

Proof

$$\begin{aligned} u + (-u) &= u + (-1) \cdot u && \text{By Definition 4.3.1} \\ &= 1 \cdot u + (-1) \cdot u && \text{Identity Element of } \times \text{ in } \mathbb{C} \\ &= (1 + (-1)) \cdot u && \text{Distributive Property of } \times \text{ over } + \text{ in } \mathbb{C} \\ &= 0 \cdot u && \text{Arithmetic in } \mathbb{R} \\ &= 0 && \text{Product Zero Principle in } \mathbb{C} \end{aligned}$$

□

Subtraction in $\mathbb{C}$ Let u and v be complex numbers. We define $v - u = v + (-u)$.

Explanation Writing $u = a + b\mathrm{i}$ and $v = c + d\mathrm{i}$ (where a, b, c, d are real numbers), by definition

$$v - u = (c - a) + (d - b)\mathrm{i} \tag{4.3.1}$$

Thus, to find the difference of two complex numbers, we can simply find the difference of their real parts as well as that of their imaginary parts.

Remark For real numbers r and s, we can write $r + (-s)\mathrm{i}$ as $r - s\mathrm{i}$. This is because

$$\begin{aligned} r - s\mathrm{i} &= r + (-s\mathrm{i}) && \text{Definition of Subtraction in } \mathbb{C} \\ &= r + (-1) \cdot s\mathrm{i} && \text{Definition 4.3.1} \\ &= r + ((-1) \cdot s) \cdot \mathrm{i} && \text{Associative Property of } \times \text{ in } \mathbb{C} \\ &= r + (-s)\mathrm{i} \end{aligned}$$

Example 4.3.3

$$\begin{aligned} (5 + 2\mathrm{i}) - (1 + 4\mathrm{i}) &= (5 - 1) + (2 - 4)\mathrm{i} && \text{By (4.3.1)} \\ &= 4 + (-2)\mathrm{i} \\ &= 4 - 2\mathrm{i} \end{aligned}$$

Remark The standard notation for a complex number is $a + b\mathrm{i}$ where a and b are real numbers. However $4 + (-2)\mathrm{i}$ is usually written as $4 - 2\mathrm{i}$ which looks simpler.

Example 4.3.4

$$\begin{aligned} 8\mathrm{i} - 13\mathrm{i} &= (0 + 8\mathrm{i}) - (0 + 13\mathrm{i}) && \text{Rewrite complex numbers in the form } a + b\mathrm{i} \\ &= (0 - 0) + (8 - 13)\mathrm{i} && \text{By (4.3.1)} \\ &= 0 + (-5)\mathrm{i} \\ &= -5\mathrm{i} \end{aligned}$$

Alternative Method Apply the Distributive Property:

$$(u - v) \cdot w = u \cdot w - v \cdot w$$

we get $8\mathrm{i} - 13\mathrm{i} = (8 - 13)\mathrm{i} = -5\mathrm{i}$.

Theorem 4.3.2 Let u and v be complex numbers. Then $v - u$ is the unique solution to the equation $u + x = v$.

Explanation By direct substitution, we see that $v - u$ is a solution to the equation. However, this method doesn't tell whether there is any other solution.
In the proof below, we add a *suitable* complex number to both sides of the equation, applying the Cancellation Property for Addition in $\mathbb{C}$.

Proof

$$\begin{aligned} u + x &= v \\ (-u) + u + x &= -u + v && \text{Add } -u \text{ to both sides} \\ 0 + x &= v - u && \text{By Theorem 4.3.1 and Definition of Subtraction in } \mathbb{C} \\ x &= v - u \end{aligned}$$

□

The following identities are similar to that in $\mathbb{R}$. They can be proved easily using properties of addition and multiplication in $\mathbb{C}$

Identity for Difference of Squares In $\mathbb{C}$, $u^2 - v^2 \equiv (u + v)(u - v)$

Perfect Square Identities In $\mathbb{C}$,

$$(u + v)^2 \equiv u^2 + 2uv + v^2$$
$$(u - v)^2 \equiv u^2 - 2uv + v^2$$

Example 4.3.5

$$\begin{aligned} (9 + 2\mathrm{i}) \cdot (9 - 2\mathrm{i}) &= 9^2 - (2\mathrm{i})^2 && \text{By Identity for Difference of Squares} \\ &= 81 - 4 \cdot \mathrm{i}^2 \\ &= 81 - 4 \cdot (-1) \\ &= 81 + 4 \\ &= 85 \end{aligned}$$

Example 4.3.6

$$\begin{aligned} (4 + 7\mathrm{i})^2 &= 4^2 + 2 \cdot 4 \cdot 7\mathrm{i} + (7\mathrm{i})^2 && \text{By Perfect Square Identity} \\ &= 16 + 56\mathrm{i} + 7^2 \cdot (-1) \\ &= 16 - 49 + 56\mathrm{i} \\ &= -33 + 56\mathrm{i} \end{aligned}$$

The following theorem can be obtained from the Identity for Difference of Squares easily. Example 4.3.5 is a special case of the result.

Theorem 4.3.3 Let a and b be real numbers. Then $(a + b\mathrm{i}) \cdot (a - b\mathrm{i}) = a^2 + b^2$.

Proof

$$\begin{aligned}(a + b\mathrm{i}) \cdot (a - b\mathrm{i}) &= a^2 - (b\mathrm{i})^2 && \text{By Identity for Difference of Squares}\\ &= a^2 - b^2 \cdot \mathrm{i}^2\\ &= a^2 - b^2 \cdot (-1)\\ &= a^2 + b^2\end{aligned}$$

□

To define division of complex numbers, we can use the idea in Example 4.2.14. However, we will adopt a different approach. We will first define the concept of the *inverse* of a non-zero complex number, then we will define subtraction $v \div u$ for complex numbers u and v with $u \neq 0$, and finally we will show that $v \div u$ is the unique solution to the equation $u \cdot x = v$.

To define the concept of the *inverse* of a non-zero complex number, we introduce the concept of the *conjugate* of a complex number. Before that, we re-do Example 4.2.14 using a different method.

Example 4.2.14 Suppose a and b are real numbers satisfying $(a + b\mathrm{i}) \cdot (2 + 3\mathrm{i}) = -4 + 7\mathrm{i}$. Find the values of a and b.

Explanation The idea is to multiply both sides of the equality by a *suitable* non-zero complex number, using the Cancellation Principle for Multiplication in $\mathbb{C}$.

Alternative Solution In view of Theorem 4.3.3, we multiply both sides of the given equality by $(2 - 3\mathrm{i})$ so that the left-side becomes $(a + b\mathrm{i}) \cdot$ a real number.

$$\begin{aligned}(a + b\mathrm{i}) \cdot (2 + 3\mathrm{i}) &= (-4 + 7\mathrm{i})\\ (a + b\mathrm{i}) \cdot (2 + 3\mathrm{i}) \cdot (2 - 3\mathrm{i}) &= (-4 + 7\mathrm{i}) \cdot (2 - 3\mathrm{i}) && \text{Multiply both sides by } (2 - 3\mathrm{i})\\ (a + b\mathrm{i}) \cdot (2^2 + 3^2) &= (-4 + 7\mathrm{i}) \cdot (2 - 3\mathrm{i}) && \text{By Theorem 4.3.3}\\ 13(a + b\mathrm{i}) &= (-4 + 7\mathrm{i}) \cdot (2 - 3\mathrm{i})\\ \frac{1}{13} \cdot 13 \cdot (a + b\mathrm{i}) &= \frac{1}{13} \cdot (-4 + 7\mathrm{i}) \cdot (2 - 3\mathrm{i})\\ a + b\mathrm{i} &= \frac{1}{13} \cdot \big((-4) \cdot 2 + (-4) \cdot (-3)\mathrm{i} + 2 \cdot 7\mathrm{i} + 7 \cdot (-3)\mathrm{i}^2\big)\\ &= \frac{1}{13}\big(-8 + 12\mathrm{i} + 14\mathrm{i} - 21 \cdot (-1)\big)\\ &= \frac{1}{13}(13 + 26\mathrm{i})\\ &= 1 + 2\mathrm{i}\end{aligned}$$

Hence $a = 1$ and $b = 2$. □

The complex number $(2 - 3\mathrm{i})$ is called the *conjugate* of $(2 + 3\mathrm{i})$. It is obtained by changing the imaginary part from 3 to -3.

Definition 4.3.2 Let a and b be real numbers. We call ***the conjugate of*** $(a + b\mathrm{i})$, and write $\overline{a + b\mathrm{i}}$, to mean the complex number $(a - b\mathrm{i})$.

Example 4.3.7 The conjugate of $(1 + 2\mathrm{i})$

$$\begin{aligned} &= \overline{1 + 2\mathrm{i}} && \text{Notation for conjugate} \\ &= 1 - 2\mathrm{i} && \text{Definition 4.3.2} \end{aligned}$$

Example 4.3.8 The conjugate of $(3 - 4\mathrm{i})$

$$\begin{aligned} &= \overline{3 - 4\mathrm{i}} && \text{Notation for conjugate} \\ &= \overline{3 + (-4)\mathrm{i}} && \text{Write in the form } a + b\mathrm{i} \\ &= 3 - (-4)\mathrm{i} && \text{Definition 4.3.2} \\ &= 3 + 4\mathrm{i} \end{aligned}$$

Remark Let a and b be real numbers. Then the conjugate of $a - b\mathrm{i}$ is $a + b\mathrm{i}$, that is,

$$\overline{a - b\mathrm{i}} = a + b\mathrm{i}$$

Thus $a + b\mathrm{i}$ and $a - b\mathrm{i}$ are conjugates of each other.
Using the above fact, we can write down the conjugate of $(3 - 4\mathrm{i})$ immediately.

Example 4.3.9 The conjugate of $5\mathrm{i}$

$$\begin{aligned} &= \overline{0 + 5\mathrm{i}} \\ &= 0 - 5\mathrm{i} && \text{Definition 4.3.2} \\ &= -5\mathrm{i} \end{aligned}$$

Remark For every real number b, the conjugate of the complex number $b\mathrm{i}$ is equal to the negative of $b\mathrm{i}$, that is, $\overline{b\mathrm{i}} = -b\mathrm{i}$.

Example 4.3.10 The conjugate of 6

$$\begin{aligned} &= \overline{6 + 0\mathrm{i}} \\ &= 6 - 0\mathrm{i} && \text{Definition 4.3.2} \\ &= 6 \end{aligned}$$

Remark For every real number a, the conjugate of the complex number a is equal to a, that is, $\overline{a} = a$.

Definition 4.3.3 Let u be a complex with $u \neq 0$. We call ***the inverse of*** u, and write u^{-1}, to mean the complex number $\frac{1}{u \cdot \overline{u}} \cdot \overline{u}$.

Explanation By Theorem 4.3.3, the expression $u \cdot \overline{u}$ is a positive real number. The expression $\frac{1}{u \cdot \overline{u}}$ means the real number $1 \div (u \cdot \overline{u})$. Thus $\frac{1}{u \cdot \overline{u}} \cdot \overline{u}$ is the product of the complex numbers $\frac{1}{u \cdot \overline{u}}$ and $\overline{u}$.

Theorem 4.3.4 Let $u \in \mathbb{C}$ with $u \neq 0$. Then $u \cdot u^{-1} = 1$

Proof

$$
\begin{aligned}
u \cdot u^{-1} &= u \cdot \frac{1}{u \cdot \overline{u}} \cdot \overline{u} \\
&= \frac{1}{u \cdot \overline{u}} \cdot (u \cdot \overline{u}) && \text{Commutative and Associative Properties of } \times \\
&= 1 && \text{Product of a real number and its reciprocal}
\end{aligned}
$$

□

Remark The result tells that u^{-1} is a solution to the equation $u \cdot x = 1$. In fact, it is the unique solution (see Corollary 4.3.6).

Example 4.3.11

$$
\begin{aligned}
\text{The inverse of } (2 + 3\mathrm{i}) &= (2 + 3\mathrm{i})^{-1} && \text{Notation for inverse} \\
&= \frac{1}{(2 + 3\mathrm{i}) \cdot (2 - 3\mathrm{i})} \cdot (2 - 3\mathrm{i}) && \text{Definition 4.3.3} \\
&= \frac{1}{2^2 + 3^2} \cdot (2 - 3\mathrm{i}) && \text{By Theorem 4.3.3} \\
&= \frac{1}{13}(2 - 3\mathrm{i}) \\
&= \frac{2}{13} - \frac{3}{13}\mathrm{i}
\end{aligned}
$$

Remark In the alternative solution to Example 4.2.14 on Page 13, the expression $\frac{1}{13}(2 - 3\mathrm{i})$ appears as a factor of $a + b\mathrm{i}$.

Example 4.3.12 (*a*)

$$
\begin{aligned}
\text{The inverse of } -2\mathrm{i} &= (0 - 2\mathrm{i})^{-1} && -2\mathrm{i} = 0 - 2\mathrm{i} \\
&= \frac{1}{(0 - 2\mathrm{i}) \cdot (0 + 2\mathrm{i})} \cdot (0 + 2\mathrm{i}) && \overline{0 - 2\mathrm{i}} = 0 + 2\mathrm{i} \\
&= \frac{1}{-4 \cdot \mathrm{i}^2} \cdot 2\mathrm{i} \\
&= \frac{1}{4} \cdot 2\mathrm{i} && \mathrm{i}^2 = -1 \\
&= \frac{1}{2}\mathrm{i}
\end{aligned}
$$

(example cont'd) (*b*)

$$
\begin{aligned}
\text{The inverse of } \tfrac{1}{2}\mathrm{i} &= \left(0+\tfrac{1}{2}\mathrm{i}\right)^{-1} \\
&= \frac{1}{\left(0+\frac{1}{2}\mathrm{i}\right)\cdot\left(0-\frac{1}{2}\mathrm{i}\right)}\cdot\left(0-\tfrac{1}{2}\mathrm{i}\right) && \overline{0+\tfrac{1}{2}\mathrm{i}} = 0-\tfrac{1}{2}\mathrm{i} \\
&= \frac{1}{-\frac{1}{4}\cdot\mathrm{i}^2}\cdot(-\tfrac{1}{2})\mathrm{i} \\
&= \frac{1}{\frac{1}{4}}\cdot(-\tfrac{1}{2})\mathrm{i} && \mathrm{i}^2=-1 \\
&= 4\cdot(-\tfrac{1}{2})\mathrm{i} \\
&= -2\mathrm{i}
\end{aligned}
$$

Remark For every non-zero complex number u, the complex number u^{-1} is also non-zero. Thus we may consider the inverse of u^{-1}. It can be shown that

$$\left(u^{-1}\right)^{-1} = u$$

that is, the inverse of (the inverse of u) is equal to u. We say that u and u^{-1} are inverses of each other.

Example 4.3.13

$$
\begin{aligned}
\text{The inverse of } 5 &= (5+0\mathrm{i})^{-1} \\
&= \frac{1}{(5+0\mathrm{i})\cdot(5-0\mathrm{i})}\cdot(5-0\mathrm{i}) && \overline{5+0\mathrm{i}} = 5-0\mathrm{i} = 5 \\
&= \frac{1}{25}\cdot 5 \\
&= \frac{1}{5}
\end{aligned}
$$

Remark For every non-zero real number a, its inverse is $\frac{1}{a}$ which is the reciprocal of a. Thus there is no ambiguity of the notation a^{-1}; it can mean the inverse of the complex number a or the reciprocal of the real number a.

In the expression $\frac{1}{u\cdot\overline{u}}\cdot\overline{u}$, if we can cancel $\overline{u}$ in the "numerator" and "denominator", then

$$u^{-1} = \frac{1}{u\cdot\overline{u}}\cdot\overline{u} = \frac{1}{u}$$

Indeed, we do have the cancellation principle (see Page 18). However, we need to define division of complex numbers first.

Division in $\mathbb{C}$ Let u and v be complex numbers with $u \neq 0$. We define $v \div u = v \cdot u^{-1}$.

Remark $v \div u$ is also written as $\frac{v}{u}$. Putting $v = 1$, we get $\frac{1}{u} = u^{-1}$.
v and u are called the *numerator* and *denominator* of the expression $\frac{v}{u}$.

Theorem 4.3.5 Let u and v be complex numbers with $u \neq 0$. Then $v \div u$ is the unique solution to the equation $u \cdot x = v$.

Explanation By direct substitution, we see that $v \div u$ is a solution to the equation. However, this method doesn't tell whether there is any other solution.
In the proof below, we multiply both sides of the equality by u^{-1} (which is non-zero), using the Cancellation Principle for Multiplication in $\mathbb{C}$.

Proof

$$\begin{aligned} u \cdot x &= v \\ u^{-1} \cdot (u \cdot x) &= u^{-1} \cdot v && \text{Multiply both sides by } u^{-1} \\ (u^{-1} \cdot u) \cdot x &= v \cdot u^{-1} && \text{Associative and Commutative Properties of } \times \text{ in } \mathbb{C} \\ 1 \cdot x &= v \cdot u^{-1} && \text{By Theorem 4.3.4} \\ x &= v \div u && \text{Definition of } \div \text{ in } \mathbb{C} \end{aligned}$$

□

Corollary 4.3.6 Let u be a complex number with $u \neq 0$. Then u^{-1} is the unique solution to the equation $u \cdot x = 1$.

Proof The result can be obtained from Theorem 4.3.5 by putting $v = 1$ and noting that $u^{-1} = 1 \div u$. □

For division, there is also a cancellation principle. To prove the principle, we need a property for inverses of non-zero complex numbers.

Theorem 4.3.7 Let $u, v \in \mathbb{C}$ with $u \neq 0$ and $v \neq 0$. Then $(u \cdot v)^{-1} = u^{-1} \cdot v^{-1}$.

Proof By Corollary 4.3.6, the complex number $(u \cdot v)^{-1}$ is the unique solution to the following equation

$$(u \cdot v) \cdot x = 1 \tag{4.3.2}$$

To prove the equality, it suffices to show that $u^{-1} \cdot v^{-1}$ is a solution to Equation (4.3.2). This can be done by direct substitution.

$$\begin{aligned} \text{L.S.} &= (u \cdot v) \cdot (u^{-1} \cdot v^{-1}) && \text{Substitute } x = u^{-1} \cdot v^{-1} \\ &= (u \cdot u^{-1}) \cdot (v \cdot v^{-1}) && \text{Commutative and Associative Properties of } \times \text{ in } \mathbb{C} \\ &= 1 \cdot 1 && \text{By Theorem 4.3.4} \\ &= \text{R.S.} \end{aligned}$$

□

Cancellation Principle for Division Let $u, v, w \in \mathbb{C}$ with $u \neq 0$ and $v \neq 0$. Then $\frac{w \cdot u}{v \cdot u} = \frac{w}{v}$.

Explanation $\frac{w \cdot u}{v \cdot u}$ means $(w \cdot u) \div (v \cdot u)$.

Proof

$$\begin{aligned}
\frac{w \cdot u}{v \cdot u} &= (w \cdot u) \cdot (v \cdot u)^{-1} && \text{Definition of } \div \text{ in } \mathbb{C} \\
&= (w \cdot u) \cdot (v^{-1} \cdot u^{-1}) && \text{By Theorem 4.3.7} \\
&= (w \cdot v^{-1}) \cdot (u \cdot u^{-1}) && \text{Commutative and Associative Properties of } \times \text{ in } \mathbb{C} \\
&= w \cdot v^{-1} \cdot 1 && \text{By Theorem 4.3.4} \\
&= w \cdot v^{-1} \\
&= \frac{w}{v} && \text{Definition of } \div \text{ in } \mathbb{C}
\end{aligned}$$

To find $v \div u$, we can rewrite it as $\frac{v}{u}$ and then multiply the numerator and denominator by $\overline{u}$.

$$v \div u = \frac{v \cdot \overline{u}}{u \cdot \overline{u}}$$

The equality follows from the Cancellation Principle for Division in $\mathbb{C}$.

Example 4.3.14

$$\begin{aligned}
2 \div (1 + \mathrm{i}) &= \frac{2}{1 + \mathrm{i}} \\
&= \frac{2 \cdot (1 - \mathrm{i})}{(1 + \mathrm{i}) \cdot (1 - \mathrm{i})} && \text{Multiply numerator and denominator by } \overline{1 + \mathrm{i}} = 1 - \mathrm{i} \\
&= \frac{2 \cdot (1 - \mathrm{i})}{1^2 + 1^2} && \text{By Theorem 4.3.3} \\
&= \frac{2 \cdot (1 - \mathrm{i})}{2} \\
&= 1 - \mathrm{i}
\end{aligned}$$

Example 4.3.15

$$\begin{aligned}
\mathrm{i} \div (3 - \mathrm{i}) &= \frac{\mathrm{i}}{3 - \mathrm{i}} \\
&= \frac{\mathrm{i} \cdot (3 + \mathrm{i})}{(3 - \mathrm{i}) \cdot (3 + \mathrm{i})} && \text{Multiply numerator and denominator by } \overline{3 - \mathrm{i}} = 3 + \mathrm{i} \\
&= \frac{3\mathrm{i} + \mathrm{i}^2}{3^2 + 1^2} && \text{By Theorem 4.3.3} \\
&= \frac{3\mathrm{i} - 1}{10} && \mathrm{i}^2 = -1 \\
&= \frac{-1}{10} + \frac{3}{10}\mathrm{i}
\end{aligned}$$

Example 4.3.16

$$\begin{aligned}
\frac{7+8i}{i} &= \frac{(7+8i)\cdot(-i)}{i\cdot(-i)} && \text{Multiply numerator and denominator by } \overline{0+i} = 0-i \\
&= \frac{-7i-8i^2}{-i^2} \\
&= \frac{-7i-8\cdot(-1)}{-(-1)} && i^2=-1 \\
&= \frac{8-7i}{1} \\
&= 8-7i
\end{aligned}$$

Example 4.3.17

$$\begin{aligned}
\frac{5-6i}{3+2i} &= \frac{(5-6i)\cdot(3-2i)}{(3+2i)\cdot(3-2i)} && \overline{3+2i} = 3-2i \\
&= \frac{5\cdot 3+5\cdot(-2i)-6i\cdot 3-6i\cdot(-2i)}{3^2+2^2} && \text{By Theorem 4.3.3} \\
&= \frac{15-10i-18i+12i^2}{13} \\
&= \frac{15-28i+12\cdot(-1)}{13} && i^2=-1 \\
&= \frac{3-28i}{13} \\
&= \frac{3}{13}-\frac{28}{13}i
\end{aligned}$$

Exercise 4.3

1. For each of the following, find the negative of the given complex number.

   (*a*) $3+7i$ (*b*) $4-\sqrt{3}i$ (*c*) $-6i$ (*d*) 0
   (*e*) -12.34 (*f*) $2i-3$ (*g*) $123+\pi i$ (*h*) $-123-\pi i$

2. Perform the following subtraction of complex numbers.

   (*a*) $4i-3i$ (*b*) $5i-13i$
   (*c*) $2i-(3-5i)$ (*d*) $(4+i)-8i$
   (*e*) $2-(4+7i)$ (*f*) $(15-16i)-17$
   (*g*) $(3+12i)-(7+5i)$ (*h*) $(4+5i)-(11+7i)$
   (*i*) $(12-3i)-(4+5i)$ (*j*) $(4+5i)-(12-3i)$

3. Simplify the following

   (*a*) $(3-4i)+(4+i)$ (*b*) $(5+6i)+(i-5)$
   (*c*) $(1-2i)-3(4-i)$ (*d*) $2(3i-4)+5(1+2i)$
   (*e*) $2i(3-i)$ (*f*) $-3i(4i-5)$
   (*g*) $(1+i)(4-i)$ (*h*) $(-2+5i)(3i-7)$

4. Suppose x and y are real numbers satisfying $(a + b\mathrm{i}) - (12 + \mathrm{i}) = 2b - 3a\mathrm{i}$. Find the values of a and b.
5. Suppose u is a number satisfying $u + 12\mathrm{i} = 3 + 4\mathrm{i}$. Find the value of u.
6. Simplify the following

   (a) $(12 + 5\mathrm{i})(12 - 5\mathrm{i})$ (b) $(4 - \sqrt{5}\mathrm{i})(4 + \sqrt{5}\mathrm{i})$

   (c) $(2 - 3\mathrm{i})^2$ (d) $(4 + \mathrm{i})^2$
7. For each of the following, find the conjugate of the given complex number.

   (a) $2 + 5\mathrm{i}$ (b) $4 - \mathrm{i}$ (c) $-3\mathrm{i}$ (d) 0

   (e) -123 (f) i (g) $-12 - \pi\mathrm{i}$ (h) $-12 + \pi\mathrm{i}$
8. Perform the following division of complex numbers.

   (a) $(4 + 6\mathrm{i}) \div 2\mathrm{i}$ (b) $(2 - \mathrm{i}) \div (-\mathrm{i})$ (c) $(-6 + 9\mathrm{i}) \div 3$

   (d) $\dfrac{1}{2 + 3\mathrm{i}}$ (e) $\dfrac{2\mathrm{i}}{1 + \mathrm{i}}$ (f) $\dfrac{10 - 5\mathrm{i}}{1 + 2\mathrm{i}}$

   (g) $\dfrac{-3 + 2\mathrm{i}}{6 - 4\mathrm{i}}$ (h) $\dfrac{6 - 4\mathrm{i}}{-3 + 2\mathrm{i}}$ (i) $\dfrac{5 + 7\mathrm{i}}{3 + \mathrm{i}}$
9. Suppose a is a real number satisfying $\dfrac{5 + 5\mathrm{i}}{1 + a\mathrm{i}} = 2\mathrm{i} - 1$. Find the value of a.
10. Simplify the following

    (a) $\dfrac{18\mathrm{i}^5}{12\mathrm{i}^3}$ (b) $2\mathrm{i} - 3 + \dfrac{1}{2\mathrm{i} - 3}$ (c) $\dfrac{5 - 7\mathrm{i}}{-\mathrm{i}^3} + \mathrm{i}^2 \cdot (2 - 3\mathrm{i}^5)$

    (d) $\dfrac{1}{(2 + 3\mathrm{i})^2}$ (e) $\left(\dfrac{2 + \mathrm{i}}{3 - \mathrm{i}}\right)^2$ (f) $(12 + 5\mathrm{i})^2 - (12 - 5\mathrm{i})^2$

4.4 Quadratic Equations with Complex Coefficients

Theorem 4.2.1 tells that the complex number i is a solution to the equation $z^2 = -1$ (where the unknown complex number is denoted by z). For every complex number η, solutions to the equation $z^2 = \eta$ are called *square roots* of η. In this section, we will show that every complex number has square root(s), and more generally, we will show that every quadratic equation with complex coefficients has solution(s) in $\mathbb{C}$.

Terminology 4.4.1 An equation that can be written in the form

$$az^2 + bz + c = 0$$

where $a, b, c \in \mathbb{C}$ with $a \neq 0$, is called a ***quadratic equation with complex coefficients*** (*with one unknown z*).

Remark Because $\mathbb{R} \subseteq \mathbb{C}$, quadratic equations given in Chapter **??** Terminology **??** are also quadratic equation with complex coefficients.

Definition 4.4.2 Consider the following quadratic equation with complex coefficients

$$az^2 + bz + c = 0 \tag{4.4.1}$$

where $a, b, c \in \mathbb{C}$ with $a \neq 0$. By a ***solution in*** $\mathbb{C}$ to Equation (4.4.1), we mean a complex number, denoted by η, such that if we substitute $z = \eta$ in (4.4.1), the equality is satisfied, that is, $a\eta^2 + b\eta + c = 0$.

To find solutions in $\mathbb{C}$ to equations in the form $z^2 - \eta^2 = 0$, where η is a complex number, we can use the factor method. This is because the Product Zero Principle is also valid in $\mathbb{C}$.

Example 4.4.1 Find the solutions in $\mathbb{C}$ to the equation $z^2 - 4 = 0$.

Explanation The equation has two solutions in $\mathbb{R}$, namely, 2 and -2. In fact, they are the only solutions in $\mathbb{C}$.

Solution

$$\begin{aligned} z^2 - 4 &= 0 \\ (z+2)(z-2) &= 0 && \text{Factorization} \\ z+2=0 \quad \text{or} \quad & z-2=0 && \text{Product Zero Principle} \\ z=-2 \quad \text{or} \quad & z=2 \end{aligned}$$

The solutions in $\mathbb{C}$ are -2 and 2. □

Remark In general, if p is a positive real number, then $z^2 - p = (z+\sqrt{p})(z-\sqrt{p})$, hence the solutions in $\mathbb{C}$ to the equation $z^2 - p = 0$ are the solutions in $\mathbb{R}$ to the equation.

Example 4.4.2 Find the solutions in $\mathbb{C}$ to the equation $z^2 + 4 = 0$.

Explanation Note that $(z+2\mathrm{i})(z-2\mathrm{i}) = z^2 - (2\mathrm{i})^2 = z^2 - 4\mathrm{i}^2 = z^2 - 4 \cdot (-1) = z^2 + 4$. The equation can be solved by the factor method.

Solution

$$\begin{aligned} z^2 + 4 &= 0 \\ (z+2\mathrm{i})(z-2\mathrm{i}) &= 0 && \text{Factorization} \\ z+2\mathrm{i}=0 \quad \text{or} \quad & z-2\mathrm{i}=0 && \text{Product Zero Principle} \\ z=-2\mathrm{i} \quad \text{or} \quad & z=2\mathrm{i} \end{aligned}$$

The solutions in $\mathbb{C}$ are $2\mathrm{i}$ and $-2\mathrm{i}$. □

Remark In general, if p is a positive real number, then $z^2 + p = (z+\sqrt{p}\mathrm{i})(z-\sqrt{p}\mathrm{i})$, hence solutions in $\mathbb{C}$ to the equation $z^2 + p = 0$ can be found by the factor method.

The next example illustrates how to find solutions to equations in the form $z^2 = \eta$, where η is a complex number.

Example 4.4.3 Find the solutions in $\mathbb{C}$ to the equation $z^2 = -3 + 4\mathrm{i}$.

Solution Write $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$. To solve the equation means to find $x, y \in \mathbb{R}$ such that

$$(x + y\mathrm{i})^2 = -3 + 4\mathrm{i}$$

that is,

$$\begin{aligned} x^2 + 2 \cdot x \cdot y\mathrm{i} + (y\mathrm{i})^2 &= -3 + 4\mathrm{i} \\ x^2 + 2xy\mathrm{i} - y^2 &= -3 + 4\mathrm{i} \\ (x^2 - y^2) + 2xy\mathrm{i} &= -3 + 4\mathrm{i} \end{aligned}$$

By Equality of Complex Numbers,

$$x^2 - y^2 = -3 \tag{4.4.2}$$

$$2xy = 4 \tag{4.4.3}$$

Note that by (4.4.3), $x \neq 0$ and

$$y = \frac{2}{x} \tag{4.4.4}$$

Substituting (4.4.4) into (4.4.2) gives

$$\begin{aligned} x^2 - \left(\frac{2}{x}\right)^2 &= -3 \\ x^2 - \frac{4}{x^2} &= -3 \\ x^4 - 4 &= -3x^2 && \text{Multiply both sides by } x^2 \\ x^4 + 3x^2 - 4 &= 0 \\ (x^2)^2 + 3x^2 - 4 &= 0 && \text{Treat } x^2 \text{ as an unknown} \\ (x^2 - 1)(x^2 + 4) &= 0 && \text{Factorization} \\ x^2 - 1 = 0 \quad &\text{or} \quad x^2 + 4 = 0 && \text{Product Zero Principle} \\ x^2 = 1 \quad &\text{or} \quad x^2 = -4 \ \text{(rejected since } x^2 \text{ can't be negative)} \end{aligned}$$

Therefore $x = 1$ or $x = -1$.

- Putting $x = 1$ into (4.4.4), we get $y = 2$, that is, $z = 1 + 2\mathrm{i}$.
- Putting $x = -1$ into (4.4.4), we get $y = -2$, that is, $z = -1 - 2\mathrm{i}$.

The solutions in $\mathbb{C}$ to the equation are $1 + 2\mathrm{i}$ and $-1 - 2\mathrm{i}$. □

4.4.1 Square Roots of Complex Numbers

The following two results together mean that every complex number has square root(s). The idea for the proof of Theorem 4.4.2 can be found in Example 4.4.1, Example 4.4.2 and Example 4.4.3.

Theorem 4.4.1 The equation $z^2 = 0$ has exactly one solution in $\mathbb{C}$, namely, 0.

Proof The result follows from the Product Zero Principle in $\mathbb{C}$. □

Theorem 4.4.2 Let η be a complex number with $\eta \neq 0$. Then the equation

$$z^2 = \eta \tag{4.4.5}$$

has exactly two solutions in $\mathbb{C}$; moreover, the two solutions are the negatives of each other.

Proof Write $\eta = \alpha + \beta\mathrm{i}$, where $\alpha, \beta \in \mathbb{R}$ with α, β not both 0. To solve the equation $z^2 = \alpha + \beta\mathrm{i}$, we consider three cases: (*Case 1*) $\beta = 0$ and $\alpha > 0$

(*Case 2*) $\beta = 0$ and $\alpha < 0$

(*Case 3*) $\beta \neq 0$

(*Case 1*) In this case, the equation can be solved as follows:

$$\begin{aligned}
z^2 &= \alpha \\
z^2 - \alpha &= 0 \\
z^2 - (\sqrt{\alpha})^2 &= 0 \\
(z + \sqrt{\alpha})(z - \sqrt{\alpha}) &= 0 \\
z + \sqrt{\alpha} = 0 \quad &\text{or} \quad z - \sqrt{\alpha} = 0 \qquad \text{Product Zero Principle} \\
z = -\sqrt{\alpha} \quad &\text{or} \quad z = \sqrt{\alpha}
\end{aligned}$$

(*Case 2*) In this case, α can be written as $-p$ where p is a positive real number. The equation can be solved as follows:

$$\begin{aligned}
z^2 &= -p \\
z^2 + p &= 0 \\
(z + \sqrt{p}\mathrm{i})(z - \sqrt{p}\mathrm{i}) &= 0 \qquad \text{Factorization} \\
z + \sqrt{p}\mathrm{i} = 0 \quad &\text{or} \quad z - \sqrt{p}\mathrm{i} = 0 \qquad \text{Product Zero Principle} \\
z = -\sqrt{p}\mathrm{i} \quad &\text{or} \quad z = \sqrt{p}\mathrm{i}
\end{aligned}$$

(pf cont'd) (*Case 3*) In this case, we write $z = x + y\mathrm{i}$, where $x, y \in \mathbb{R}$. To solve the equation means to find $x, y \in \mathbb{R}$ such that

$$(x + y\mathrm{i})^2 = \alpha + \beta\mathrm{i}$$

that is,

$$\begin{aligned} x^2 + 2 \cdot x \cdot y\mathrm{i} + (y\mathrm{i})^2 &= \alpha + \beta\mathrm{i} \\ x^2 + 2xy\mathrm{i} - y^2 &= \alpha + \beta\mathrm{i} \\ (x^2 - y^2) + 2xy\mathrm{i} &= \alpha + \beta\mathrm{i} \end{aligned}$$

By Equality of Complex Numbers,

$$x^2 - y^2 = \alpha \tag{4.4.6}$$

$$2xy = \beta \tag{4.4.7}$$

Since $\beta \neq 0$, it follows from (4.4.7) that $x \neq 0$ and

$$y = \frac{\beta}{2x} \tag{4.4.8}$$

Substituting (4.4.8) into (4.4.6) gives

$$\begin{aligned} x^2 - \left(\frac{\beta}{2x}\right)^2 &= \alpha \\ x^2 - \frac{\beta^2}{4x^2} &= \alpha \\ 4x^4 - \beta^2 &= 4\alpha x^2 && \text{Multiply both sides by } 4x^2 \\ 4x^4 - 4\alpha x^2 - \beta^2 &= 0 \end{aligned}$$

By treating x^2 as an unknown, the above equation is a quadratic equation with real coefficients:

$$4(x^2)^2 - 4\alpha(x^2) - \beta^2 = 0$$

By the quadratic formula,

$$\begin{aligned} x^2 &= \frac{-(-4\alpha) \pm \sqrt{(-4\alpha)^2 - 4 \cdot 4 \cdot (-\beta^2)}}{2 \cdot 1} \\ &= \frac{4\alpha \pm \sqrt{16\alpha^2 + 16\beta^2}}{2} \\ &= \frac{4\alpha \pm 4\sqrt{\alpha^2 + \beta^2}}{2} \\ &= 2\left(\alpha \pm \sqrt{\alpha^2 + \beta^2}\right) \end{aligned}$$

Since $x^2 \geq 0$ and $\alpha - \sqrt{\alpha^2 + \beta^2} < 0$ (because $\beta \neq 0$), it follows that

$$x^2 \neq 2\left(\alpha - \sqrt{\alpha^2 + \beta^2}\right)$$

and so $x^2 = 2\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)$ which gives

$$x = \pm\sqrt{2\left(\alpha + \sqrt{\alpha^2 + \beta^2}\right)}$$

(pf cont'd) Putting the values of x into (4.4.8),

- when $x = \sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)}$, we get $y = \dfrac{\beta}{2\sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)}}$ and

  so $z = \sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)} + \dfrac{\beta}{2\sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)}}\mathrm{i}$,

- when $x = -\sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)}$, we get $y = \dfrac{-\beta}{2\sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)}}$ and

  so $z = -\sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)} - \dfrac{\beta}{2\sqrt{2\left(\alpha + \sqrt{\alpha^2+\beta^2}\right)}}\mathrm{i}$.

In all the three cases, Equation (4.4.5) has exactly two solutions in $\mathbb{C}$. Moreover, the two solutions are the negatives of each other. □

Definition 4.4.3 Let η be a complex number. We call ***a square root of*** η to mean a complex number whose square is equal to η.

Explanation A square root of η means a solution to the equation $z^2 = \eta$.

Remark By Theorem 4.4.2, every non-zero complex number has exactly two square roots which are the negatives of each other.

Example 4.4.4 The square roots of 9 are 3 and −3.

Check

$$\begin{aligned} 3^2 &= 9 \\ (-3)^2 &= 9 \end{aligned}$$

Example 4.4.5 The square roots of −1 are i and −i.

Check

$$\begin{aligned} \mathrm{i}^2 &= -1 \\ (-\mathrm{i})^2 &= (-1\cdot\mathrm{i})\cdot(-1\cdot\mathrm{i}) \\ &= (-1)^2\cdot\mathrm{i}^2 = -1 \end{aligned}$$

Example 4.4.6 The square roots of −25 are 5i and −5i.

Proof

$$\begin{aligned} (5\mathrm{i})^2 &= 5\mathrm{i}\cdot 5\mathrm{i} = 5^2\cdot\mathrm{i}^2 \\ &= 25\cdot(-1) = -25 \end{aligned}$$

Hence 5i is a square root of 25i.

By Theorem 4.4.2, the other square root of −25 is −5i. □

Example 4.4.7 The square roots of $(24 - 10\mathrm{i})$ are $(5 - \mathrm{i})$ and $(-5 + \mathrm{i})$.

Proof
$$\begin{aligned}(5-\mathrm{i})^2 &= 5^2 - 2\cdot 5\cdot \mathrm{i} + \mathrm{i}^2\\ &= 25 - 10\mathrm{i} + (-1)\\ &= 24 - 10\mathrm{i}\end{aligned}$$

Hence $(5 - \mathrm{i})$ is a square root of $(24 - 10\mathrm{i})$.

By Theorem 4.4.2, the other square root of $(24 - 10\mathrm{i})$ is $-(5 - \mathrm{i}) = -5 + \mathrm{i}$.

□

Remark Given a complex number η, to find the square roots of η, we can change the equation $z^2 = \eta$ to two equations with two real unknowns (see Example 4.4.3 or the proof of Theorem 4.4.2). Alternatively, we can use polar representation for complex numbers *(will not be discussed in this course)*.

For a non-negative real number, we have defined its principle square root to be its square root that is non-negative. We want to extend the definition of principle square roots to complex numbers. Note that for every non-zero complex number, its square roots are the negatives of each other. Thus there are two possibilities for the imaginary parts of its square roots:

- The imaginary part of both square roots are zero (that is, both square roots are real numbers).
- The imaginary part of both square roots are non-zero (hence one of the square root has positive imaginary part and the other has negative imaginary part).

Definition 4.4.4 Let $\eta \in \mathbb{C}$. We call the ***principle square root of*** η, denoted by $\sqrt{\eta}$, to mean the square root of η that satisfies either one of the following conditions:

(1) Imaginary part of $\sqrt{\eta} > 0$

(2) Imaginary part of $\sqrt{\eta} = 0$ and Real part of $\sqrt{\eta} \geq 0$

Example 4.4.8 The principle square root of $(24-10\mathrm{i})$ is $(-5+\mathrm{i})$, that is, $\sqrt{24 - 10\mathrm{i}} = -5+\mathrm{i}$.

Check By Example 4.4.7, $(-5 + \mathrm{i})$ is a square root of $(24 - 10\mathrm{i})$.

The imaginary part of $(-5 + \mathrm{i})$ is equal to 1 which is positive.

Example 4.4.9 The principle square root of -25 is $5\mathrm{i}$, that is, $\sqrt{-25} = 5\mathrm{i}$.

Check By Example 4.4.6, $5\mathrm{i}$ is a square root of -25.

The imaginary part of $5\mathrm{i}$ is equal to 5 which is positive.

Example 4.4.10 The principle square root of -1 is i, that is, $\sqrt{-1} = \mathrm{i}$.

Check By Example 4.4.5, i is a square root of -1.

The imaginary part of i is equal to 1 which is positive.

Example 4.4.11 The principle square root of 9 is 3, that is, $\sqrt{9} = 3$.

Check By Example 4.4.4, 3 is a square root of 9.

The imaginary part of 3 is equal to 0 and the real part of 3 is equal to 3 which is non-negative.

Remark For every non-negative real number a, the principle square root of a as a complex number is the same as the principle square root of a as a real number.

Theorem 4.4.3 Let p be a positive real number. Then the principle square root of $-p$ is equal to $\sqrt{p}\,\mathrm{i}$, that is, $\sqrt{-p} = \sqrt{p}\,\mathrm{i}$.

Proof Note that
$$\begin{aligned}\left(\sqrt{p}\,\mathrm{i}\right)^2 &= \left(\sqrt{p}\right)^2 \cdot \mathrm{i}^2 \\ &= p \cdot (-1) \\ &= -p\end{aligned}$$

Thus $\sqrt{p}\,\mathrm{i}$ is a square root of $-p$.

Moreover, the imaginary part of $\sqrt{p}\,\mathrm{i}$ is equal to $\sqrt{p}$ which is positive.

Hence $\sqrt{-p} = \sqrt{p}\,\mathrm{i}$. □

Example 4.4.12
$$\sqrt{-5} = \sqrt{5}\,\mathrm{i}$$
$$\sqrt{-4} = \sqrt{4}\,\mathrm{i} = 2\mathrm{i}$$
$$\sqrt{-8} = \sqrt{8}\,\mathrm{i} = 2\sqrt{2}\,\mathrm{i}$$

Caution For real numbers a and b, we always have $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$. However, for complex numbers u and v, we may have $\sqrt{u \cdot v} \neq \sqrt{u} \cdot \sqrt{v}$.

Example 4.4.13 Denote $u = -1$ and denote $v = -4$. Then

$$\begin{aligned}\sqrt{u \cdot v} &= \sqrt{(-1)\cdot(-4)} \\ &= \sqrt{4} \\ &= 2\end{aligned} \qquad \text{and} \qquad \begin{aligned}\sqrt{u} \cdot \sqrt{v} &= \sqrt{-1} \cdot \sqrt{-4} \\ &= \mathrm{i} \cdot 2\mathrm{i} \\ &= 2 \cdot (-1) \\ &= -2\end{aligned}$$

Thus, $\sqrt{u \cdot v} \neq \sqrt{u} \cdot \sqrt{v}$.

Remark In this case, we have $\sqrt{u \cdot v} = -1 \cdot \sqrt{u} \cdot \sqrt{v}$.

Theorem 4.4.4 Let $u, v \in \mathbb{C}$. Then $\sqrt{u \cdot v} = \sqrt{u} \cdot \sqrt{v}$ or $\sqrt{u \cdot v} = -\sqrt{u} \cdot \sqrt{v}$.

Proof For the case where $u = 0$ or $v = 0$, the result is obvious.

For the case where $u \neq 0$ and $v \neq 0$, it is straightforward to check that both $\sqrt{u}\sqrt{v}$ and $-\sqrt{u}\sqrt{v}$ are the square roots of uv, hence one of them is equal to the principle square root of uv. □

Exercise 4.4.1

1. For each of the following, rewrite it in the form a or $a\mathrm{i}$ where a is a real number.

   (a) $\sqrt{-16}$ (b) $\sqrt{-27}$ (c) $\sqrt{-3} \cdot \sqrt{-12}$

   (d) $\sqrt{\mathrm{i}^6}$ (e) $\sqrt{\left(-\frac{3}{2}\right)^2}$ (f) $\sqrt{(\pi\mathrm{i})^2}$

   (g) $\sqrt{\mathrm{i}^3 \cdot \mathrm{i}^7}$ (h) $\sqrt{-3\mathrm{i}^2}$ (i) $\sqrt{-25} - \sqrt{-9}$

   (j) $\sqrt{-25} \cdot \sqrt{-9}$ (k) $\sqrt{(-25) \cdot (-9)}$ (l) $\dfrac{\sqrt{-25}}{\sqrt{-9}}$

2. It is known that $(3 - \mathrm{i})^2 = 8 - 6\mathrm{i}$. For each of the following complex numbers, express its principle square root in the form $a + b\mathrm{i}$ where a and b are real numbers.

   (a) $8 - 6\mathrm{i}$ (b) $32 - 24\mathrm{i}$ (c) $4 - 3\mathrm{i}$ (d) $-8 + 6\mathrm{i}$

3. It is known that $\sqrt{\mathrm{i}} = \dfrac{1 + \mathrm{i}}{\sqrt{2}}$. Express each of the following in the form $a + b\mathrm{i}$ where a and b are real numbers.

   (a) $\sqrt{2\mathrm{i}}$ (b) $\sqrt{-2\mathrm{i}}$ (c) $\sqrt{3\mathrm{i}^5}$ (d) $\sqrt{4\mathrm{i}^{11}}$

4.4.2 Quadratic Formula

The complete square method discussed in Chapter **??** Section **??** works also for quadratic equations with complex coefficients. In Theorem **??**, we have to consider three cases: $\Delta > 0$, $\Delta = 0$ and $\Delta < 0$. This is because every positive real number has exactly two square roots, the real number 0 has exactly one square root and every negative real number has no square roots. For quadratic equations with complex coefficients, in view of Theorem 4.4.1 and Theorem 4.4.2, we only need to consider two cases: $\Delta \neq 0$ and $\Delta = 0$.

Theorem 4.4.5 Consider the following quadratic equation with with complex coefficients

$$az^2 + bz + c = 0 \tag{4.4.9}$$

where $a, b, c \in \mathbb{C}$ with $a \neq 0$. The solutions in $\mathbb{C}$ to Equation (4.4.9) are given as follows:

(1) If $b^2 - 4ac \neq 0$, then Equation (4.4.9) has exactly two solutions, namely

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(2) If $b^2 - 4ac = 0$, then Equation (4.4.9) has exactly one solution, namely $\dfrac{-b}{2a}$.

Remark Similar to that for quadratic equations with real coefficients, we also write Δ to denote the complex number $b^2 - 4ac$.

Example 4.4.14 Find the solutions in $\mathbb{C}$ to the equation $2z^2 + 3z + 4 = 0$.

Solution Note that $\Delta = 3^2 - 4 \cdot 2 \cdot 4 = -23$.

By Theorem 4.4.5, $z = \dfrac{-3 \pm \sqrt{-23}}{2 \cdot 2} = \dfrac{-3 \pm \sqrt{23}\,\mathrm{i}}{4}$.

The solutions in $\mathbb{C}$ are $\dfrac{-3}{4} + \dfrac{\sqrt{23}}{4}\mathrm{i}$ and $\dfrac{-3}{4} - \dfrac{\sqrt{23}}{4}\mathrm{i}$. □

Example 4.4.15 Find the solutions in $\mathbb{C}$ to the equation $z^2 + \sqrt{3}\,\mathrm{i}z - \mathrm{i} = 0$.

Solution Note that $\Delta = \left(\sqrt{3}\,\mathrm{i}\right)^2 - 4 \cdot 1 \cdot (-\mathrm{i}) = -3 + 4\mathrm{i}$.

By Example 4.4.3, the square roots of $(-3 + 4\mathrm{i})$ are $(1 + 2\mathrm{i})$ and $(-1 - 2\mathrm{i})$.

Hence by Definition 4.4.4, $\sqrt{-3 + 4\mathrm{i}} = 1 + 2\mathrm{i}$.

By Theorem 4.4.5, $z = \dfrac{-\sqrt{3}\,\mathrm{i} \pm \sqrt{-3 + 4\mathrm{i}}}{2} = \dfrac{-\sqrt{3}\,\mathrm{i} \pm (1 + 2\mathrm{i})}{2}$,

that is,

$$z = \frac{-\sqrt{3}\,\mathrm{i} + (1 + 2\mathrm{i})}{2} = \frac{1 + (2 - \sqrt{3})\mathrm{i}}{2} \quad \text{or} \quad z = \frac{-\sqrt{3}\,\mathrm{i} - (1 + 2\mathrm{i})}{2} = \frac{-1 - (2 + \sqrt{3})\mathrm{i}}{2}$$

The solutions in $\mathbb{C}$ are $\dfrac{1}{2} + \dfrac{2 - \sqrt{3}}{2}\mathrm{i}$ and $-\dfrac{1}{2} - \dfrac{2 + \sqrt{3}}{2}\mathrm{i}$. □

Remark To apply the quadratic formula, we have to know the principle square root of Δ. If Δ is a real number, it is easy to find $\sqrt{\Delta}$. If Δ is not a real number, to find $\sqrt{\Delta}$, we can use the method in Example 4.4.3.

Exercise 4.4.2

1. For each of the following complex numbers, find its square roots and hence write down its principle square root.

   (*a*) 36 (*b*) -49 (*c*) -2

   (*d*) $3-4\mathrm{i}$ (*e*) $-8+6\mathrm{i}$ (*f*) $45-28\mathrm{i}$

   (*g*) $15-8\mathrm{i}$

2. For each of the following quadratic equations with complex coefficients, find its solution in $\mathbb{C}$.

   (*a*) $z^2-2z+3=0$ (*b*) $2z^2+3z+4=0$

   (*c*) $\mathrm{i}z^2-3z+4\mathrm{i}=0$ (*d*) $3z^2-2\mathrm{i}z+1=0$

3. For each of the following quadratic equations with complex coefficients, find its solution in $\mathbb{C}$.

   (*a*) $z^2-\sqrt{3}z+\mathrm{i}=0$ (*b*) $(1+2\mathrm{i})z^2+(2+3\mathrm{i})z+1+3\mathrm{i}=0$

   (*c*) $(3-4\mathrm{i})z^2+7z-1+\mathrm{i}=0$ (*d*) $-\mathrm{i}z^2+8z+(6+24\mathrm{i})=0$

   Hint: Results in Question 1 may be useful.