Contents

Chapter 3

Quadratic Equations

3.1 Introduction

Problem 3.1.1 The length of a rectangular carpet is 1 m longer than its width. If the length of the diagonal of the carpet is 5 m, find the dimensions of the carpet.

Explanation The question asks for the length and width of the carpet. Although the length and width are unknowns, if we know one of them, then we know the other.

Idea of Solution Suppose the length of the carpet is $x\,m$. Then the width is $(x - 1)\,m$.

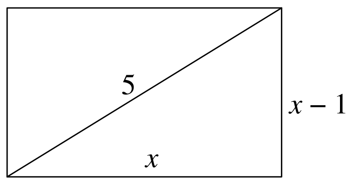


Figure 3.1.1

By the Pythagoras Theorem, we have

$$x^2 + (x - 1)^2 = 5^2 \tag{3.1.1}$$

The question is to find positive real number(s) x satisfying Equation (3.1.1).

Equation (3.1.1) is called a *quadratic equation*. In this chapter, we will discuss several methods for solving quadratic equations; then we will consider some practical problems leading to quadratic equations; and finally we will discuss how the solutions and coefficients of a quadratic equation are related.

Terminology 3.1.1 An expression that can be written in the form ax^2+bx+c, where a, b and c are real numbers with $a \neq 0$, is called a ***quadratic expression** with one variable x*.

Remark For simply, a *quadratic expression with one variable x* will be called a *quadratic expression*.

In the quadratic expression $ax^2 + bx + c$, there are three terms, namely, ax^2, bx and c which are called the *x^2-term*, the *x-term* and the *constant term* respectively; the real numbers a, b and c are called the *coefficients* of the x^2-term, the x-term and the constant term respectively.

Example 3.1.2 The expression $5x^2 + 7x + 8$ is a quadratic expression.

- The x^2-term is $5x^2$; the coefficient of the x^2-term is 5.
- The x-term is $7x$; the coefficient of the term x-term is 7.
- The constant term is 8; the coefficient of the constant term is 8.

Terminology 3.1.2 An equation that can be written in the form

$$ax^2 + bx + c = 0 \tag{3.1.2}$$

where a, b, c are real numbers with $a \neq 0$, is called a ***quadratic equation** with one unknown x*.

Remark For simplicity, a *quadratic equation with one unknown x* will be called a *quadratic equation*. In Chapter **??** Section **??**, we will consider *quadratic equation with complex coefficients* in which the coefficients a, b, c are complex numbers.

When a quadratic equation is written in the form $ax^2 + bx + c = 0$, we say that the quadratic equation is in *standard form*. For example,

(1) $2x^2 + 3x + 4 = 0$

is a quadratic equation in standard form. The following quadratic equations are also considered to be in standard form:

(2) $x^2 - 2x + 5 = 0$

(3) $7x^2 + 11x = 0$

(4) $x^2 - 4 = 0$

This is because the values of a, b and c can be recognized immediately.

The following table gives the values of a, b and c for the quadratic equations given in (1), (2), (3) and (4).

Quadratic Equation	a	b	c
$2x^2 + 3x + 4 = 0$	2	3	4
$x^2 - 2x + 5 = 0$	1	−2	5
$7x^2 + 11x = 0$	7	11	0
$x^2 - 4 = 0$	1	0	−4

The equations given in the next example are also quadratic equations. This is because they can be written in the form $ax^2 + bx + c = 0$.

Example 3.1.3 For each of the following equations, rewrite it in the form $ax^2 + bx + c = 0$ where $a > 0$ and write down the values of a, b and c.

(a) $x^2 + 4x = 5$

(b) $x(x-2) + 3 = 0$

(c) $(3x+1)^2 = 9$

(d) $x^2 + (x-1)^2 = 5^2$

Solution To rewrite the given equations in standard form, we expand, combine and rearrange terms (if necessary).

(a)
$$\begin{aligned} x^2 + 4x &= 5 \\ x^2 + 4x - 5 &= 0 \end{aligned}$$
$a = 1,\ b = 4$ and $c = -5$.

(b)
$$\begin{aligned} x(x-2) + 3 &= 0 \\ x^2 - 2x + 3 &= 0 \end{aligned}$$
$a = 1,\ b = -2$ and $c = 3$.

(c)
$$\begin{aligned} (3x+1)^2 &= 9 \\ 9x^2 + 6x + 1 &= 9 \quad \text{Perfect Square Identity} \\ 9x^2 + 6x - 8 &= 0 \end{aligned}$$
$a = 9,\ b = 6$ and $c = -8$.

(d)
$$\begin{aligned} x^2 + (x-1)^2 &= 5^2 \\ x^2 + x^2 - 2x + 1 &= 25 \quad \text{Perfect Square Identity} \\ 2x^2 - 2x - 24 &= 0 \end{aligned}$$
$a = 2,\ b = -2$ and $c = -24$.

□

Definition 3.1.3 Consider the quadratic equation

$$ax^2 + bx + c = 0 \tag{3.1.3}$$

where a, b and c are real numbers with $a \neq 0$. A ***solution in*** $\mathbb{R}$ to Equation 3.1.3 means a real number, denoted by r, such that if we substitute $x = r$ in (3.1.3), the equality is satisfied, that is, $ar^2 + br + c = 0$.

Remark Since we will not consider complex numbers unless otherwise stated, for convenience, *solutions in* $\mathbb{R}$ will be called *solutions*.

Example 3.1.4 Consider the quadratic equation $x^2 + 3x - 4 = 0$.

- Since $1^2 + 3 \cdot 1 - 4 = 0$, it follows that 1 is a solution to the equation.
- Since $2^2 + 3 \cdot 2 - 4 \neq 0$, it follows that 2 is not a solution to the equation.

Terminology 3.1.4 To *solve* a quadratic equation means to find *all* the solutions (in $\mathbb{R}$) to the equation.

In the next two sections, we will discuss some methods for solving quadratic equations. In Section 3.2, we will consider the *factor method*. In Section 3.3, we will first consider the *complete square method*, and then we will apply the method to derive a formula, called the *quadratic formula*, for solutions to quadratic equations in standard form. In the next chapter, we will discuss another method, called the *graphical method*, for solving quadratic equations.

Exercise 3.1

1. For each of the following quadratic equations, rewrite it in the form $ax^2 + bx + c = 0$ and write down the values of a, b and c.

(a) $x^2 - 3x = 4$	(b) $2x + 1 = 5x^2$
(c) $2x^2 = 5$	(d) $3 - 4x^2 = 8x$
(e) $x(x - 2) = 10 + x$	(f) $(x - 2)(x + 5) = 3x^2$

2. Consider the quadratic equation $x^2 - 3x = 4$.
   (a) Show that 1, 2 and 3 are not solutions to the equation.
   (b) Show that -1 and 4 are solutions to the equation.
   Can you tell whether there is any more solution?

3. (a) Consider the quadratic equation $x^2 + bx + c = 0$, where b and c are integers. Suppose that k is an integer, $k \neq 0$ and k is a solution to the equation. Show that k divides c (that is, $\frac{c}{k}$ is an integer).
   (b) Use the result in (a) to find all the integer solutions to the quadratic equation $x(x-2) = 10+x$.
   Can you tell whether there is any more solution, besides the integers you found?

3.2 Factor Method

Before discussing the *factor method*, we consider an example where the quadratic equation is in a special form.

Example 3.2.1 Solve the following quadratic equation:

$$(x-2)(x-5)=0 \tag{3.2.1}$$

Explanation By direct substitution, we see that 2 and 5 are solutions to (3.2.1). Moreover, they are the only solutions. This is because of the Product Zero Principle (see Page **??** in Chapter **??**)

Solution to Example 3.2.1

$$\begin{aligned} (x-2)(x-5) &= 0 \\ x-2=0 \quad \text{or} \quad x-5=0 & \quad \text{Product Zero Principle} \\ x=2 \quad \text{or} \quad x=5 & \quad \text{Solve linear equations separately} \end{aligned}$$

The solutions to Equation (3.2.1) are 2 and 5. □

Remark Sometimes, we omit writing "*The solutions to (3.2.1) are* 2 *and* 5.". This is because once we get "$x = 2$ *or* $x = 5$", we can write down the solutions immediately.

The word *solution* in "*Solution to Example 3.2.1*" means the procedure to get the answer for the example, whereas the word *solutions* in "*The solutions to Equation* (3.2.1)" means the real numbers satisfying the equation.

In the third step of the Solution to Example 3.2.1, we write

(1) $x = 2$ or $x = 5$;

but in the answer, we write

(2) the solutions are 2 and 5.

(1) is to describe the solutions and (2) is to list the solutions. Given an equation (with one unknown), the *solution set* of the equation is the set of all real numbers that are solutions to the equation. The solution set of Equation (3.2.1) can be written in the following two forms:

(1) $\{x \in \mathbb{R} : x = 2 \text{ or } x = 5\}$ using description;

(2) $\{2, 5\}$ using listing.

Example 3.2.2 Solve the quadratic equation $(2x+3)(x-7)=0$.

Explanation Apply the Product Zero Principle as in Example 3.2.1.

Solution

$$\begin{aligned} (2x+3)(x-7) &= 0 \\ 2x+3=0 \quad \text{or} \quad x-7=0 & \quad \text{Product Zero Principle} \\ x=-\frac{3}{2} \quad \text{or} \quad x=7 & \end{aligned}$$

The solutions are $-\frac{3}{2}$ and 7. □

Example 3.2.3 Solve the equation $x(x+3)=0$.

Explanation The equation is in the form $a \cdot b = 0$ where $a = x$ and $b = x+3$.

Solution

$$\begin{aligned} x(x+3) &= 0 \\ x=0 \quad \text{or} \quad x+3=0 & \quad \text{Product Zero Principle} \\ x=0 \quad \text{or} \quad x=-3 & \end{aligned}$$

The solutions are 0 and −3. □

Given a quadratic equation *in standard form*

$$ax^2+bx+c=0,$$

if we can factorize the left-side:

$$ax^2+bx+c=(px+q)(rx+s)$$

then the equation can be solved using the Product Zero Principle as in the above examples. This method is called the *factor method*.

3.2.1 Expansion and Factorization

In this section, we give a review on factorization of ax^2+bx+c. Since factorization is the reverse process of expansion, students who are familiar with expanding $(px+q)(rx+s)$ will be able to perform factorization in the head.

Terminology 3.2.1 To *expand* $(px+q)(rx+s)$, where p, q, r, s are real numbers, means to obatin an expression in the form ax^2+bx+c such that $(px+q)(rx+s)=ax^2+bx+c$.

Explanation $(px+q)(rx+s)=ax^2+bx+c$ means that the expression on the left-side and the right-side of the equality sign are equal for every real number x. Using the concept of functions (see Chapter **??**), the equality means that the functions represented by $(px+q)(rx+s)$ and ax^2+bx+c are equal.

Example 3.2.4 Expand $(2x+3)(5x+7)$.

Explanation The following shows a long way to expand the given expression using the Distributive Property of Multiplication over Addition:

$$\begin{aligned} (2x+3)(5x+7) &= (2x+3)\cdot 5x + (2x+3)\cdot 7 && \text{Distributive Property} \\ &= 2x\cdot 5x + 3\cdot 5x + 2x\cdot 7 + 3\cdot 7 && \text{Distributive Property} \\ &= 10x^2 + 15x + 14x + 21 \\ &= 10x^2 + 29x + 21 \end{aligned}$$

Note that

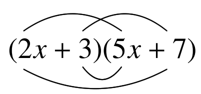
- the term $10x^2$ is obtained by multiplying $2x$ and $5x$
- the term $29x$ is obtained by adding $2x\cdot 7$ and $3\cdot 5x$
- the constant term 21 is obtained by multiplying 3 and 7.

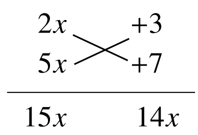
Solution

$$\begin{aligned} (2x+3)(5x+7) &= 10x^2 + 15x + 14x + 21 \\ &= 10x^2 + 29x + 21 \end{aligned}$$

□

Remark With enough practice, students should be able to write down the answer without the intermediate step. The following diagrams are useful in this regard.

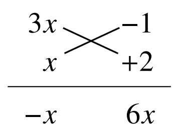




Example 3.2.5 Expand $(3x-1)(x+2)$.

Solution

$$\begin{aligned} (3x-1)(x+2) &= 3x^2 - x + 6x - 2 \\ &= 3x^2 + 5x - 2 \end{aligned}$$

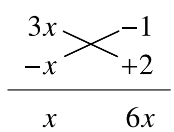


□

Example 3.2.6 Expand $(3x-1)(2-x)$.

Solution

$$\begin{aligned} (3x-1)(2-x) &= (3x-1)(-x+2) \\ &= -3x^2 + x + 6x - 2 \\ &= -3x^2 + 7x - 2 \end{aligned}$$



□

Example 3.2.7 Expand $(3x+1)^2$.

Solution

$$\begin{aligned}(3x+1)^2 &= (3x+1)(3x+1)\\ &= 9x^2+3x+3x+1\\ &= 9x^2+6x+1\end{aligned}$$

$$\begin{array}{cc} 3x & +1 \\ 3x & +1 \\ \hline 3x & 3x \end{array}$$

□

Remark Alternatively, we can use the Perfect Square Identity to get

$$\begin{aligned}(3x+1)^2 &= (3x)^2+2(3x)(1)+1^2\\ &= 9x^2+6x+1\end{aligned}$$

Example 3.2.8 Expand $(2x-5)^2$.

Solution

$$\begin{aligned}(2x-5)^2 &= (2x-5)(2x-5)\\ &= 4x^2-10x-10x+25\\ &= 4x^2-20x+25\end{aligned}$$

$$\begin{array}{cc} 2x & -5 \\ 2x & -5 \\ \hline -10x & -10x \end{array}$$

□

Remark Alternatively, we can use the Perfect Square Identity to get

$$\begin{aligned}(2x-5)^2 &= (2x)^2-2\cdot 2x\cdot 5+5^2\\ &= 4x^2-20x+25\end{aligned}$$

Example 3.2.9 Expand $(3x-2)(3x+2)$.

Solution

$$\begin{aligned}(3x-2)(3x+2) &= 9x^2-6x+6x-4\\ &= 9x^2-4\end{aligned}$$

$$\begin{array}{cc} 3x & -2 \\ 3x & +2 \\ \hline -6x & 6x \end{array}$$

□

Remark Alternatively, we can use the Identity for Difference of Squares to get

$$\begin{aligned}(3x-2)(3x+2) &= (3x)^2-2^2\\ &= 9x^2-4\end{aligned}$$

Example 3.2.10 Expand $7x(6-5x)$.

Explanation Apply the Distributive Property of Multiplication over Addition directly.

Solution $7x(6-5x) = 42x-35x^2$ □

Remark We may write $7x$ as $7x+0$ and apply the method as in the previous few examples. However, the procedure is much longer.

$$\begin{aligned}7x(6-5x) &= (7x+0)(-5x+6)\\ &= -35x^2+0x+42x+0\\ &= -35x^2+42x\end{aligned}$$

$$\begin{array}{cc} 7x & +0 \\ -5x & +6 \\ \hline 0x & 42x \end{array}$$

Terminology 3.2.2 To *factorize* a quadratic expression ax^2+bx+c, where a, b, c are real numbers with $a \neq 0$, means to obtain an expression in the form $(px+q)(rx+s)$, where p, q, r, s are real numbers, such that $ax^2+bx+c = (px+q)(rx+s)$.

Remark For a factor $px+q$,

- if $p = 1$, we can write $(1x+q)$ as $(x+q)$;
- if $q = 0$, we can write $(px+0)$ as px.

Below, we consider some examples on factorization of quadratic expressions. The first example illustrates how to factorize ax^2+bx.

Example 3.2.11 Factorize $2x^2 - 6x$.

Explanation Note that $2x$ is a factor of both terms. We can extract the common factor $2x$.

Solution $2x^2 - 6x \;=\; 2x\,(x-3)$ □

The next two examples illustrate how to factorize $ax^2 + c$ where a and c have opposite signs (that is, one is positive and the other negative). The method is to apply the Identity for Difference of Squares.

Example 3.2.12 Factorize $4x^2 - 25$.

Solution
$$\begin{aligned} 4x^2 - 25 &= (2x)^2 - 5^2 \\ &= (2x+5)(2x-5) \end{aligned}$$
□

Example 3.2.13 Factorize $16 - 9x^2$.

Solution
$$\begin{aligned} 16 - 9x^2 &= 4^2 - (3x)^2 \\ &= (4+3x)(4-3x) \end{aligned}$$
□

Before discussing factorization of quadratic expressions in general, we consider a result, called the *Compare Coefficient Principle for Quadratic Expressions* or simply *Compare Coefficient Principle*, which states that if two quadratic expressions are equal, then the corresponding coefficients are equal.

Compare Coefficient Principle Suppose that $a_1x^2 + b_1x + c = a_2x^2 + b_2x + c_2$, where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers with $a_1 \neq 0$ and $a_2 \neq 0$. Then $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$.

Explanation The equality $a_1x^2 + b_1x + c = a_2x^2 + b_2x + c_2$ means that the equality holds for every real number x.

Putting $x = 0$, we get $c_1 = c_2$.
Putting $x = 1$ and $x = 2$ respectively, we get two equalities from which we get $a_1 = a_2$ and $b_1 = b_2$.

Remark When we apply the Compare Coefficient Principle, sometimes we say "*compare the x-term, we get* $b_1 = b_2$" etc.

Next we consider factorization of $ax^2 + bx + c$ where $a = 1$ and b, c are integers such that

$$x^2 + bx + c = (x + p)(x + q)$$

where p and q are integers. Expanding the right-side, we get

$$x^2 + bx + c = x^2 + (p + q)x + pq.$$

Comparing the x-term and the constant term, we get

$$p + q = b \qquad \text{and} \qquad pq = c.$$

Since p and q are integers, it follows that there are only finitely many pairs (p, q) satisfying the second condition $pq = c$; we can check which one satisfies the first condition by direct substitution.

Example 3.2.14 Factorize $x^2 + 7x + 10$.

Explanation We want to find integers p and q such that

$$p + q = 7 \qquad \text{and} \qquad pq = 10.$$

Note that both p and q are positive. We may assume that $p \leq q$ (this is because if $p > q$, we can interchange the role of p and q). From the condition $pq = 10$, we get $(p, q) = (1, 10)$ or $(2, 5)$.

- If $(p, q) = (1, 10)$, then $p + q = 11 \neq 7$.
- If $(p, q) = (2, 5)$, then $p + q = 7$.

Solution $x^2 + 7x + 10 = (x + 2)(x + 5)$

x $+2$
x $+5$
$2x$ $5x$

□

Remark $(-x - 2)(-x - 5)$ is also a possible answer.

Example 3.2.15 Factorize $x^2 - 7x + 12$.

Explanation We want to find integers p and q such that

$$p + q = -7 \qquad \text{and} \qquad pq = 12.$$

Solution $x^2 - 7x + 12 = (x - 3)(x - 4)$

$$\begin{array}{cc} x & -3 \\ x & -4 \\ \hline -3x & -4x \end{array}$$

□

Example 3.2.16 Factorize $x^2 + 4x - 21$.

Explanation We want to find integers p and q such that

$$p + q = 4 \qquad \text{and} \qquad pq = -21.$$

Solution $x^2 + 4x - 21 = (x + 7)(x - 3)$

$$\begin{array}{cc} x & +7 \\ x & -3 \\ \hline 7x & -3x \end{array}$$

□

Example 3.2.17 Factorize $x^2 - 2x - 15$.

Explanation We want to find integers p and q such that

$$p + q = -2 \qquad \text{and} \qquad pq = -15.$$

Solution $x^2 - 2x - 15 = (x + 3)(x - 5)$

$$\begin{array}{cc} x & +3 \\ x & -5 \\ \hline 3x & -5x \end{array}$$

□

Example 3.2.18 Factorize $x^2 + 6x + 9$.

Explanation We may use the method as in the preceding examples. Alternatively, we can apply the Perfect Square Identity.

Solution $x^2 + 6x + 9 = (x + 3)^2$ □

Example 3.2.19 Factorize $x^2 - 8x + 16$.

Explanation Apply the Perfect Square Identity.

Solution $x^2 - 8x + 16 = (x - 4)^2$ □

Next we consider factorization of $ax^2 + bx + c$, where a, b, c are integers with $a \neq 0$ and $a \neq 1$, such that

$$ax^2 + bx + c = (px + q)(rx + s)$$

with p, q, r, s being integers. Expanding the right-side, we get

$$ax^2 + bx + c = prx^2 + (ps + qr)x + qs.$$

Comparing the coefficients of the x^2-term, x-term and constant term, we get

$$pr = a, \qquad ps + qr = b \qquad \text{and} \qquad qs = c.$$

Example 3.2.20 Factorize $2x^2 + 7x + 3$.

Explanation We want to find integers p, q, r, s such that

$$pr = 2, \qquad ps + qr = 7 \qquad \text{and} \qquad qs = 3.$$

We may assume that p and r are positive (this is because if both of them are negative, we may multiply the factor $(px + q)$ by -1 and multiply the factor $(rx + s)$ by -1, see the Remark of the solution to Example 3.2.14). We may also assume that $p \leq r$. From the condition $pr = 2$, we get $p = 1$ and $r = 2$. Note that both q and s must be positive. From the condition $qs = 3$, we get $(q, s) = (1, 3)$ or $(3, 1)$

- If $(q, s) = (1, 3)$, then $ps + qr = 1 \cdot 3 + 1 \cdot 2 = 5 \neq 7$.
- If $(q, s) = (3, 1)$, then $ps + qr = 1 \cdot 1 + 3 \cdot 2 = 7$.

Solution $2x^2 + 7x + 3 \;=\; (x + 3)(2x + 1)$

$$\begin{array}{cc} x & +3 \\ 2x & +1 \\ \hline 6x & x \end{array}$$

□

To factorize $ax^2 + bx + c$, if the coefficients of the x^2-term and the constant term have many integer factors, it may be time consuming to find p, q, r and s by try and error. Below we use an example to describe an alternative method for factorization. The quadratic expression is basically the same as the last one except that $7x$ is written as $x + 6x$.

Example 3.2.21 Factorize $2x^2 + x + 6x + 3$ by grouping and extracting common factor.

Solution

$$\begin{aligned} 2x^2 + x + 6x + 3 &= (2x^2 + x) + (6x + 3) && \text{Grouping} \\ &= x(2x + 1) + 3(2x + 1) && \text{Extract common factor} \\ &= (2x + 1)(x + 3) && \text{Extract common factor} \end{aligned}$$

□

Example 3.2.21 shows that to factorize the expression $2x^2 + 7x + 3$, we can write $7x$ as $x + 6x$. If we write $7x$ as $2x + 5x$, the method of grouping and extracting common factor doesn't work:

$$\begin{aligned} 2x^2 + 7x + 3 &= (2x^2 + 2x) + (5x + 3) \\ &= 2x(x + 1) + (5x + 3) \\ &\quad \text{can't further extract common factor} \end{aligned}$$

The reason why writing $7x = 1x + 6x$ works is due to the fact that $1 \cdot 6 = 2 \cdot 3$ (product of the coefficient of the x^2-term and constant term). In general, if we can find m and n such that

$$m + n = b \qquad \text{and} \qquad m \cdot n = a \cdot c,$$

then we can use the method in Example 3.2.21 to factorize $ax^2 + bx + c$.

Example 3.2.22 Factorize $3x^2 + 17x + 10$.

Solution We want to find integers m and n such that

$$m + n = 17 \qquad \text{and} \qquad mn = 3 \cdot 10 = 30$$

We can take $m = 2$ and $n = 15$. Writing $17x = 2x + 15x$, we get

$$\begin{aligned} 3x^2 + 17x + 10 &= 3x^2 + 2x + 15x + 10 \\ &= (3x^2 + 2x) + (15x + 10) \\ &= x(3x + 2) + 5(3x + 2) \\ &= (3x + 2)(x + 5) \end{aligned}$$

$$\begin{array}{cc} 3x & +2 \\ x & +5 \\ \hline 2x & 15x \end{array}$$

□

Example 3.2.23 Factorize $5x^2 + 7x - 6$.

Solution We want to find integers m and n such that

$$m + n = 7 \qquad \text{and} \qquad mn = 5 \cdot -6 = -30$$

We can take $m = 10$ and $n = -3$. Writing $7x = 10x - 3x$, we get

$$\begin{aligned} 5x^2 + 7x - 6 &= 5x^2 + 10x - 3x - 6 \\ &= (5x^2 + 10x) - (3x + 6) \qquad \text{Note the sign of the last term} \\ &= 5x(x + 2) - 3(x + 2) \\ &= (x + 2)(5x - 3) \end{aligned}$$

$$\begin{array}{cc} x & +2 \\ 5x & -3 \\ \hline 10x & -3x \end{array}$$

□

Remark If we write $7x = -3x + 10x$, then we get

$$\begin{aligned} 5x^2 + 7x - 6 &= (5x^2 - 3x) + (10x - 6) \\ &= x(5x - 3) + 2(5x - 3) \\ &= (5x - 3)(x + 2) \end{aligned}$$

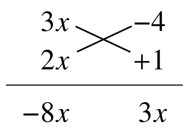
Example 3.2.24 Factorize $6x^2 - 5x - 4$.

Solution We want to find integers m and n such that

$$m + n = -5 \qquad \text{and} \qquad mn = 6 \cdot -4 = -24$$

We can take $m = -8$ and $n = 3$. Writing $-5x = -8x + 3x$, we get

$$\begin{aligned} 6x^2 - 5x - 4 &= (6x^2 - 8x) + (3x - 4) \\ &= 2x(3x - 4) + 1(3x - 4) \\ &= (3x - 4)(2x + 1) \end{aligned}$$



□

Example 3.2.25 Factorize $4x^2 + 20x + 25$.

Explanation We may apply the method as in the preceding examples. Alternatively, we can apply the Perfect Square Identity.

Solution

$$\begin{aligned} 4x^2 + 20x + 25 &= (2x)^2 + 2 \cdot (2x) \cdot 5 + 5^2 \\ &= (2x + 5)^2 \end{aligned}$$

□

Example 3.2.26 Factorize $9x^2 - 12x + 4$.

Explanation Apply the Perfect Square Identity.

Solution

$$\begin{aligned} 9x^2 - 12x + 4 &= (3x)^2 - 2 \cdot (3x) \cdot 2 + 2^2 \\ &= (3x + 2)^2 \end{aligned}$$

□

The quadratic expressions given in Example 3.2.11 through Example 3.2.26 are chosen such that they can be factorized as $(px+q)(rx+s)$ where p, q, r, s are integers. There are quadratic expressions that cannot be factorized in this way, for example, $x^2 + 7x + 5$: we can't find integers p, q, r, s such that $x^2 + 7x + 5 = (px + q)(rx + s)$.

In general, if a, b, c are integers with $a \neq 0$ and $b^2 - 4ac \neq k^2$ for all integers k, then we can't factorize the quadratic expression $ax^2 + bx + c$ using integer coefficients. Readers will understand why after learning the *quadratic formula* in Section 3.3.

Although the quadratic expression $x^2 + 7x + 5$ can't be factorized using integer coefficients, it can be factorized using real coefficients:

$$x^2 + 7x + 5 = \left(x + \frac{7+\sqrt{29}}{2}\right)\left(x + \frac{7-\sqrt{29}}{2}\right)$$

Check

$$\begin{aligned}\left(x + \frac{7+\sqrt{29}}{2}\right)\left(x + \frac{7-\sqrt{29}}{2}\right) &= \left(x + \frac{7}{2} + \frac{\sqrt{29}}{2}\right)\left(x + \frac{7}{2} - \frac{\sqrt{29}}{2}\right) \\ &= \left(x + \frac{7}{2}\right)^2 - \left(\frac{\sqrt{29}}{2}\right)^2 \\ &= x^2 + 2 \cdot x \cdot \frac{7}{2} + \left(\frac{7}{2}\right)^2 - \left(\frac{\sqrt{29}}{2}\right)^2 \\ &= x^2 + 7x + \frac{49}{4} - \frac{29}{4} \\ &= x^2 + 7x + 5\end{aligned}$$

Exercise 3.2.1

1. Expand the following expressions.

   (*a*) $x(x+2)$ (*b*) $3x(x-4)$ (*c*) $x(6-x)$

   (*d*) $2x(3x+4)$ (*e*) $5x(8-3x)$

2. Expand the following expressions.

   (*a*) $(x+2)^2$ (*b*) $(x-5)^2$ (*c*) $(2x+3)^2$ (*d*) $(3x-7)^2$

3. Expand the following expressions.

   (*a*) $(x+7)(x-7)$ (*b*) $(2x+5)(2x-5)$

   (*c*) $(3x-4)(3x+4)$ (*d*) $(1-5x)(1+5x)$

4. Expand the following expressions.

   (*a*) $(x+5)(x+2)$ (*b*) $(x+5)(x-7)$ (*c*) $(x-3)(4-x)$

   (*d*) $(x-2)(x+6)$ (*e*) $(2x+3)(x+5)$ (*f*) $(8x-1)(x-6)$

   (*g*) $(4x+3)(6-5x)$

5. Factorize the following expressions.

   (*a*) $x^2 + 4x$ (*b*) $5x^2 - 7x$ (*c*) $6x - 9x^2$ (*d*) $x^2 - 36$

   (*e*) $25 - x^2$ (*f*) $3x^2 - 48$ (*g*) $63 - 7x^2$ (*h*) $4x^2 - 25$

   (*i*) $18x^2 - 32$ (*j*) $48 - 75x^2$

6. Factorize the following expressions.

(a)	$x^2 - 11x + 30$	(b)	$x^2 + 9x + 18$	(c)	$x^2 - 3x - 28$
(d)	$x^2 + 7x - 18$	(e)	$x^2 + 10x + 24$	(f)	$x^2 + 2x + 24$
(g)	$2x^2 + 5x + 3$	(h)	$3x^2 - 5x + 2$	(i)	$2x^2 - x - 10$
(j)	$6x^2 - x - 1$	(k)	$6x^2 - 11x + 3$	(l)	$10x^2 + 9x + 2$
(m)	$2x^2 + 10x + 12$	(n)	$9x^2 - 3x - 12$	(o)	$20x^2 + 2x - 4$

7. Factorize the following expressions.

(a)	$x^2 - 6x + 9$	(b)	$x^2 + 10x + 25$	(c)	$4x^2 + 4x + 1$
(d)	$9x^2 + 12x + 4$	(e)	$3x^2 + 12x + 12$	(f)	$5x^2 + 30x + 45$
(g)	$18x^2 - 60x + 50$				

3.2.2 Solving Quadratic Equations by Factorization

Below we apply the factor method to solve some quadratic equations. In Example 3.2.27 through Example 3.2.34, the given quadratic equations are in standard form.

Example 3.2.27 Use the factor method to solve the equation $2x^2 - 6x = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.11.

Solution

$$\begin{aligned} 2x^2 - 6x &= 0 \\ 2x(x-3) &= 0 && \text{Factorization} \\ 2x = 0 \quad \text{or} &\quad x - 3 = 0 && \text{Product Zero Principle} \\ x = 0 \quad \text{or} &\quad x = 3 \end{aligned}$$

The solutions are 0 and 3. □

Example 3.2.28 Use the factor method to solve the equation $4x^2 - 25 = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.12.

Solution

$$\begin{aligned} 4x^2 - 25 &= 0 \\ (2x+5)(2x-5) &= 0 && \text{Factorization} \\ 2x + 5 = 0 \quad \text{or} &\quad 2x - 5 = 0 && \text{Product Zero Principle} \\ x = -\frac{5}{2} \quad \text{or} &\quad x = \frac{5}{2} \end{aligned}$$

The solutions are $-\frac{5}{2}$ and $\frac{5}{2}$. □

Example 3.2.29 Use the factor method to solve the equation $x^2 + 7x + 10 = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.14.

Solution

$$\begin{aligned} x^2 + 7x + 10 &= 0 \\ (x+2)(x+5) &= 0 && \text{Factorization} \\ x+2=0 \quad \text{or} \quad x+5 &= 0 && \text{Product Zero Principle} \\ x=-2 \quad \text{or} \quad x &= -5 \end{aligned}$$

The solutions are -2 and -5. □

Example 3.2.30 Use the factor method to solve the equation $x^2 - 2x - 15 = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.17.

Solution

$$\begin{aligned} x^2 - 2x - 15 &= 0 \\ (x+3)(x-5) &= 0 && \text{Factorization} \\ x+3=0 \quad \text{or} \quad x-5 &= 0 && \text{Product Zero Principle} \\ x=-3 \quad \text{or} \quad x &= 5 \end{aligned}$$

The solutions are -3 and 5. □

Example 3.2.31 Use the factor method to solve the equation $x^2 - 8x + 16 = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.19.

Solution

$$\begin{aligned} x^2 - 8x + 16 &= 0 \\ (x-4)^2 &= 0 && \text{Factorization} \\ x-4 &= 0 && \text{Square-root of 0} \\ x &= 4 \end{aligned}$$

The solution is 4. □

Example 3.2.32 Use the factor method to solve the equation $2x^2 + 7x + 3 = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.20.

Solution

$$\begin{aligned} 2x^2 + 7x + 3 &= 0 \\ (x+3)(2x+1) &= 0 && \text{Factorization} \\ x+3=0 \quad \text{or} \quad 2x+1 &= 0 && \text{Product Zero Principle} \\ x=-3 \quad \text{or} \quad x &= -\frac{1}{2} \end{aligned}$$

The solutions are -3 and $-\frac{1}{2}$. □

Example 3.2.33 Use the factor method to solve the equation $6x^2 - 5x - 4 = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.24.

Solution

$$\begin{aligned} 6x^2 - 5x - 4 &= 0 \\ (3x-4)(2x+1) &= 0 && \text{Factorization} \\ 3x - 4 = 0 \quad &\text{or} \quad 2x + 1 = 0 && \text{Product Zero Principle} \\ x = \frac{4}{3} \quad &\text{or} \quad x = -\frac{1}{2} \end{aligned}$$

The solutions are $\frac{4}{3}$ and $-\frac{1}{2}$. □

Example 3.2.34 Use the factor method to solve the equation $4x^2 + 20x + 25 = 0$.

Explanation Factorization of the left-side of the equation is done in Example 3.2.25.

Solution

$$\begin{aligned} 4x^2 + 20x + 25 &= 0 \\ (2x+5)^2 &= 0 && \text{Factorization} \\ 2x + 5 &= 0 && \text{Square-root of 0} \\ x &= -\frac{5}{2} \end{aligned}$$

The solution is $-\frac{5}{2}$. □

Below, we consider quadratic equations not in standard form. To solve such equations, we shouldn't use factorization in the first step. For the quadratic equation in Example 3.2.35, factorizing $x^2 + 3x$ doesn't help:

$$\begin{aligned} x^2 + 3x &= 4 \\ x(x+3) &= 4 \end{aligned}$$

There is no Product 4 *Principle*. Instead, the first step is to rewrite the equation in standard form; then we can use the factor method as in the preceding examples.

Example 3.2.35 Solve the equation $x^2 + 3x = 4$.

Solution

$$\begin{aligned} x^2 + 3x &= 4 \\ x^2 + 3x - 4 &= 0 && \text{Rewrite in standard form} \\ (x+4)(x-1) &= 0 && \text{Factorization} \\ x + 4 = 0 \quad &\text{or} \quad x - 1 = 0 && \text{Product Zero Principle} \\ x = -4 \quad &\text{or} \quad x = 1 \end{aligned}$$

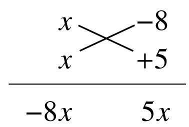
x	$+4$
x	-1
$4x$	$-x$

The solutions are -4 and 1. □

Example 3.2.36 Solve the equation $x(x-3) = 40$.

Solution

$$\begin{aligned} x(x-3) &= 40 \\ x^2 - 3x &= 40 \\ x^2 - 3x - 40 &= 0 && \text{Rewrite in standard form} \\ (x-8)(x+5) &= 0 && \text{Factorization} \\ x - 8 = 0 \quad &\text{or} \quad x + 5 = 0 && \text{Product Zero Principle} \\ x = 8 \quad &\text{or} \quad x = -5 \end{aligned}$$

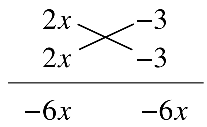


The solutions are 8 and −5. □

Example 3.2.37 Solve the equation $4x^2 + 9 = 12x$.

Solution

$$\begin{aligned} 4x^2 + 9 &= 12x \\ 4x^2 - 12x + 9 &= 0 && \text{Rewrite in standard form} \\ (2x-3)^2 &= 0 && \text{Factorization} \\ 2x - 3 &= 0 && \text{Square-root of 0} \\ x &= \frac{3}{2} \end{aligned}$$

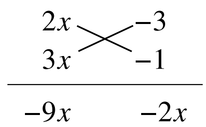


The solution is $\frac{3}{2}$. □

Example 3.2.38 Solve the equation $6x^2 = 11x - 3$.

Solution

$$\begin{aligned} 6x^2 &= 11x - 3 \\ 6x^2 - 11x + 3 &= 0 && \text{Rewrite in standard form} \\ 6x^2 - 2x - 9x + 3 &= 0 && (-2) + (-9) = -11 \text{ and } (-2)\cdot(-9) = 6\cdot 3 \\ 2x(3x-1) - 3(3x-1) &= 0 \\ (2x-3)(3x-1) &= 0 \\ 2x - 3 = 0 \quad &\text{or} \quad 3x - 1 = 0 && \text{Product Zero Principle} \\ x = \frac{3}{2} \quad &\text{or} \quad x = \frac{1}{3} \end{aligned}$$

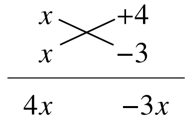


The solutions are $\frac{3}{2}$ and $\frac{1}{3}$. □

Example 3.2.39 Solve the equation $3x^2 - 10 = 26 - 3x$.

Solution

$$\begin{aligned} 3x^2 - 10 &= 26 - 3x && \\ 3x^2 + 3x - 36 &= 0 && \text{Rewrite in standard form} \\ 3(x^2 + x - 12) &= 0 && \text{Extract common factor} \\ x^2 + x - 12 &= 0 && \text{Multiply both sides by } \tfrac{1}{3} \\ (x+4)(x-3) &= 0 && \\ x + 4 = 0 \quad \text{or} &\quad x - 3 = 0 && \text{Product Zero Principle} \\ x = -4 \quad \text{or} &\quad x = 3 && \end{aligned}$$



The solutions are −4 and 3. □

Quadratic equations in this section are chosen such that when they are written in standard form $ax^2 + bx + c = 0$, the quadratic expression $ax^2 + bx + c$ can be factorized using integer coefficients. For the case where the quadratic expression can't be factorized using integer coefficients, it is still possible that the quadratic equation $ax^2 + bx + c = 0$ has solutions. For example, the solutions to the quadratic equation $x^2 - 8x - 5 = 0$ are $4 + \sqrt{21}$ and $4 - \sqrt{21}$. For details, see Example 3.3.5 in Section 3.3.

Exercise 3.2.2

1. Use the Product Zero Theorem to solve the following quadratic equations.

(*a*)	$(x-7)(x-3) = 0$	(*b*)	$(x+4)(2x-5) = 0$
(*c*)	$x(x-6) = 0$	(*d*)	$2x(3x+4) = 0$
(*e*)	$(x+3)^2 = 0$	(*f*)	$4(6-5x)^2 = 0$

2. Use the factor method to solve the following quadratic equations.

(*a*)	$11x - x^2 = 0$	(*b*)	$5x^2 + 15x = 0$
(*c*)	$x^2 - 36 = 0$	(*d*)	$4x^2 - 9 = 0$
(*e*)	$98 - 2x^2 = 0$	(*f*)	$16 - 8x + x^2 = 0$
(*g*)	$x^2 + 10x + 25 = 0$	(*h*)	$x^2 - 10x + 21 = 0$
(*i*)	$x^2 - x - 12 = 0$	(*j*)	$x^2 + 3x - 18 = 0$
(*k*)	$x^2 + 17x + 60 = 0$	(*l*)	$2x^2 + x - 3 = 0$
(*m*)	$4x^2 - x - 5 = 0$	(*n*)	$x^2 + 6x + 9 = 0$
(*o*)	$16x^2 - 40x + 25 = 0$	(*p*)	$x^2 = 6x - 8$
(*q*)	$2x^2 + 5x = 3$	(*r*)	$2x^2 = 6x + 8$
(*s*)	$4x + 4x^2 = 1$	(*t*)	$3(x^2 + 1) = x + 5$
(*u*)	$(x+2)(x-3) = 6$	(*v*)	$(2x-1)(x+2) = 25$

3.3 Complete Square Method and Quadratic Formula

To solve quadratic equations in the form

$$x^2 + bx = d$$

we can add a suitable number s to both sides of the equation to get

$$x^2 + bx + s = d + s$$

such that the left-side is a perfect square, that is,

$$x^2 + bx + s = (x + t)^2$$

for some number t. Such a method is called the *complete square method*.

Using the complete square method, we can derive a formula, called the *quadratic formula* (see Theorem 3.3.1), which can be used to obtain solution(s), if there is any, for quadratic equations written in standard form.

3.3.1 Complete Square Method

Before discussing the *complete square method*, first we consider a simple example.

Example 3.3.1 Solve the quadratic equation $x^2 = 4$.

Solution It is clear 2 and -2 are solutions to the equations. Moreover, if $x \neq 2$ and $x \neq -2$, then $x^2 \neq 4$. Hence, the solutions are 2 and -2. □

Remark Alternatively, we may write down the solutions immediately by applying Lemma **??** with $a = 2$ or the Square-root Theorem in Chapter **??**.

To solve quadratic equations like that in Example 3.3.1, we can use the *Square-root Theorem* (see Page **??** in Chapter **??**).

Example 3.3.2 Solve the equation $(x - 1)^2 = 25$.

Explanation By treating $x - 1$ as an unknown, we can apply the Square-root Theorem.

Solution

$$\begin{aligned} (x-1)^2 &= 25 \\ x - 1 = 5 \quad \text{or} \quad x - 1 &= -5 \qquad \text{By Square-root Theorem} \\ x = 6 \quad \text{or} \quad x &= -4 \end{aligned}$$

The solutions are 6 and -4. □

Example 3.3.3 Solve the equation $x^2 + 6x + 9 = 49$.

Explanation We can solve the equation by rewriting it in a form similar to that in Example 3.3.2

Solution
$$\begin{aligned} x^2 + 6x + 9 &= 49 \\ (x+3)^2 &= 49 && \text{By Perfect Square Identity} \\ x + 3 = 7 \quad \text{or} \quad & x + 3 = -7 && \text{By Square-root Theorem} \\ x = 4 \quad \text{or} \quad & x = -10 \end{aligned}$$

The solutions are 4 and −10. □

In Example 3.3.3, the left-side of the quadratic equation can be written as $(x + k)^2$ using the Perfect Square Identity. To solve a quadratic equation in the form $x^2 + bx = d$, we can add a suitable number s to both sides such that $x^2 + bx + s$ is a perfect square. This method is called the *complete square method*. The following example illustrates the procedures.

Example 3.3.4 Use the complete square method to solve the equation $x^2 + 10x = 11$.

Explanation By adding 25 to $x^2 + 10x$, we get $x^2 + 10x + 25 = (x + 5)^2$, a perfect square. In order to keep the equality, we have to add 25 to both sides.

Solution
$$\begin{aligned} x^2 + 10x &= 11 \\ x^2 + 10x + 25 &= 11 + 25 && \text{Add 25 to both sides} \\ (x+5)^2 &= 36 && \text{By Perfect Square Identity} \\ x + 5 = 6 \quad \text{or} \quad & x + 5 = -6 && \text{By Square-root Theorem} \\ x = 1 \quad \text{or} \quad & x = -11 \end{aligned}$$

The solutions are 1 and −11. □

To solve quadratic equations in the form
$$x^2 + bx = d$$
using the complete square method, the main step is to add a suitable number s to both sides of the equation such that $x^2 + bx + s$ is a perfect square, that is,
$$x^2 + bx + s = (x + t)^2 \tag{3.3.1}$$
for some number t. To find such a number s, we expand the right-side of (3.3.1) to get
$$x^2 + bx + s = x^2 + 2tx + t^2$$
and then comparing the x-term and the constant term to get

$$b = 2t \qquad \text{and} \qquad s = t^2.$$

Hence $t = \frac{b}{2}$ and so $s = \left(\frac{b}{2}\right)^2$.

Example 3.3.5 Use the complete square method to solve the equation $x^2 - 8x = 5$.

Explanation The number to be added to both sides is $\left(\frac{-8}{2}\right)^2 = 16$.

Solution

$$\begin{aligned} x^2 - 8x &= 5 \\ x^2 - 8x + 16 &= 5 + 16 && \text{Add 16 to both sides} \\ (x-4)^2 &= 21 && \text{By Perfect Square Identity} \\ x - 4 = \sqrt{21} \quad &\text{or} \quad x - 4 = -\sqrt{21} && \text{By Square-root Theorem} \\ x = \sqrt{21} + 4 \quad &\text{or} \quad x = -\sqrt{21} + 4 \end{aligned}$$

The solutions are $\sqrt{21} + 4$ and $-\sqrt{21} + 4$. □

To solve quadratic equations in the form $x^2 + bx + c = 0$, we can first rewrite the equation as

$$x^2 + bx = -c$$

and then add $\left(\frac{b}{2}\right)^2$ to both sides of the equation.

Example 3.3.6 Use the complete square method to solve the equation $x^2 + 16x + 55 = 0$.

Solution

$$\begin{aligned} x^2 + 16x + 55 &= 0 \\ x^2 + 16x &= -55 \\ x^2 + 16x + 64 &= -55 + 64 && \text{To get a perfect square} \\ (x+8)^2 &= 9 \\ x + 8 = 3 \quad &\text{or} \quad x + 8 = -3 \\ x = -5 \quad &\text{or} \quad x = -11 \end{aligned}$$

The solutions are -5 and -11. □

Example 3.3.7 Use the complete square method to solve the equation $x^2 - 6x + 11 = 0$.

Solution

$$\begin{aligned} x^2 - 6x + 11 &= 0 \\ x^2 - 6x &= -11 \\ x^2 - 6x + 9 &= -11 + 9 && \text{To get a perfect square} \\ (x-3)^2 &= -2 \end{aligned}$$

Since $-2 < 0$, it follows from the Square-root Theorem that the equation has no solution. □

To solve quadratic equations in standard form $ax^2+bx+c=0$ using the complete square method, first we multiply both sides of the equation by $\frac{1}{a}$ so that the coefficient of the x^2-term is 1; then we apply the procedures as in the previous examples.

Example 3.3.8 Solve the equation $2x^2+3x-5=0$.

Solution

$$\begin{aligned}
2x^2+3x-5 &= 0 \\
x^2+\frac{3}{2}x-\frac{5}{2} &= 0 \qquad \text{Multiply both sides by } \frac{1}{2} \\
x^2+\frac{3}{2}x &= \frac{5}{2} \\
x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2 &= \frac{5}{2}+\left(\frac{3}{4}\right)^2 \\
\left(x+\frac{3}{4}\right)^2 &= \frac{49}{16}
\end{aligned}$$

$$x+\frac{3}{4}=\frac{7}{4} \quad \text{or} \quad x+\frac{3}{4}=-\frac{7}{4}$$

$$x=1 \quad \text{or} \quad x=-\frac{5}{2}$$

The solutions are 1 and $-\frac{5}{2}$. □

Exercise 3.3.1

1. Use the Square-root Theorem to solve the following quadratic equations.

   (*a*) $(x+3)^2=4$ (*b*) $(2x-5)^2=9$

   (*c*) $(3x-7)^2=0$ (*d*) $(5-x)^2=-3$

2. Use the complete square method to solve the following quadratic equations.

   (*a*) $x^2-2x=3$ (*b*) $x^2+6x=16$

   (*c*) $x^2-4x+3=0$ (*d*) $x^2+10x-24=0$

   (*e*) $x^2+8x+20=0$ (*f*) $2x^2-3x-5=0$

3.3.2 Quadratic Formula

To obtain a formula for the solutions, if any, to a quadratic equation in standard form

$$ax^2+bx+c=0$$

where a, b, c are real numbers and $a \neq 0$, we can apply the method in Example 3.3.8. Using the complete square method, we get an equation in the following form.

$$(x + \text{something})^2 = \text{a real number}$$

Then we can apply the Square-root Theorem. For this, we have to consider the three cases where "a real number" is positive, 0 or negative. Below are the details.

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
x^2 + \frac{b}{a}x + \frac{c}{a} &= 0 && \text{Multiply both sides by } \frac{1}{a} \\
x^2 + \frac{b}{a}x &= \frac{-c}{a} && \text{Add } -\frac{c}{a} \text{ to both sides} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= \frac{-c}{a} + \left(\frac{b}{2a}\right)^2 && \text{Add } \left(\frac{b}{2a}\right)^2 \text{ to both sides} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{-c}{a} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{-c \cdot 4a}{4a^2} + \frac{b^2}{4a^2} \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} && (\dagger)
\end{aligned}$$

(1) If $b^2 - 4ac > 0$, then $\frac{b^2 - 4ac}{4a^2}$ is a positive real number and it can be written as $\left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2$. Hence by Lemma **??** in Chapter **??**,

$$x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a}$$

that is,

$$x = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{or} \qquad x = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

Therefore, the solutions are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(2) If $b^2 - 4ac = 0$, then Equation (†) reduces to $\left(x + \frac{b}{2a}\right)^2 = 0$. Hence by the Square-root Theorem,

$$x + \frac{b}{2a} = 0$$

Therefore, the solution is $-\frac{b}{2a}$.

(3) If $b^2 - 4ac < 0$, then $\frac{b^2 - 4ac}{4a^2}$ is a negative real number. Hence by the Square-root Theorem, Equation (†) has no solution. Therefore, the original quadratic equation has no solution.

The above results are summarized in the following

Theorem 3.3.1 Consider the quadratic equation

$$ax^2 + bx + c = 0 \tag{3.3.2}$$

where a, b and c are real numbers with $a \neq 0$. The solutions (in $\mathbb{R}$) to Equation (3.3.2) are given as follows:

(1) If $b^2 - 4ac > 0$, then (3.3.2) has exactly two solutions , namely

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

(2) If $b^2 - 4ac = 0$, then (3.3.2) has exactly one solution, namely $\frac{-b}{2a}$.

(3) If $b^2 - 4ac < 0$, then (3.3.2) has no (real) solution.

Notation 3.3.1 We write $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to denote that

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Terminology 3.3.2 The expression $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ is called the ***quadratic formula***.

Remark For the case where $b^2 - 4ac = 0$, if we apply the quadratic formula, we get

$$x = \frac{-b \pm \sqrt{0}}{2a} = \frac{-b \pm 0}{2a}.$$

From this, we see that the solution to (3.3.2) is $\frac{-b}{2a}$.

For the case where $b^2 - 4ac < 0$, if we apply the quadratic formula, we get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{4ac - b^2}\,\mathrm{i}}{2a}$$

From this, we see that (3.3.2) has two complex solutions that are not real numbers and so the equation has no real solutions.

Example 3.3.9 Use the quadratic formula to solve the equation $x^2 - 2x - 15 = 0$.

Solution Note that $(-2)^2 - 4 \cdot 1 \cdot (-15) = 4 + 60 = 64 > 0$.

By the quadratic formula, $x = \frac{-(-2) \pm \sqrt{64}}{2 \cdot 1} = \frac{2 \pm 8}{2}$.

That is, $x = \frac{2 + 8}{2} = 5$ or $x = \frac{2 - 8}{2} = -3$.

The solutions are 5 and -3. □

Example 3.3.10 Use the quadratic formula to solve the equation $2x^2 + 7x + 3 = 0$.

Solution Note that $7^2 - 4 \cdot 2 \cdot 3 = 49 - 24 = 25 > 0$.

By the quadratic formula, $x = \dfrac{-7 \pm \sqrt{25}}{2 \cdot 2} = \dfrac{-7 \pm 5}{4}$.

That is, $x = \dfrac{-7+5}{4} = -\dfrac{1}{2}$ or $x = \dfrac{-7-5}{4} = -3$.

The solutions are $-\dfrac{1}{2}$ and -3. □

For quadratic equations not in standard form, to apply the quadratic formula, we rewrite the equations in standard form.

Example 3.3.11 Use the quadratic formula to solve the equation $x^2 + 5 = 7x$.

Solution

$$\begin{aligned} x^2 + 5 &= 7x \\ x^2 - 7x + 5 &= 0 \qquad \text{Rewrite in standard form} \end{aligned}$$

Note that $(-7)^2 - 4 \cdot 1 \cdot 5 = 49 - 20 = 29 > 0$.

By the quadratic formula, $x = \dfrac{-(-7) \pm \sqrt{29}}{2 \cdot 1} = \dfrac{7 \pm \sqrt{29}}{2}$.

The solutions are $\dfrac{7+\sqrt{29}}{2}$ and $\dfrac{7-\sqrt{29}}{2}$. □

Example 3.3.12 Use the quadratic formula to solve the equation $4x^2 + 9 = 12x$.

Solution

$$\begin{aligned} 4x^2 + 9 &= 12x \\ 4x^2 - 12x + 9 &= 0 \qquad \text{Rewrite in standard form} \end{aligned}$$

Note that $(-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$.

By the quadratic formula, $x = \dfrac{-(-12) \pm \sqrt{0}}{2 \cdot 4}$.

That is, $x = \dfrac{12}{8} = \dfrac{3}{2}$.

The solution is $\dfrac{3}{2}$. □

Example 3.3.13 Solve the equation $x(6-x) = 11$.

Solution
$$\begin{aligned} x(6-x) &= 11 \\ 6x - x^2 &= 11 \\ 0 &= x^2 - 6x + 11 \qquad \text{Rewrite in standard form} \end{aligned}$$

Note that $(-6)^2 - 4 \cdot 1 \cdot 11 = 36 - 44 = -8 < 0$.
It follows from the quadratic formula (or Theorem 3.3.1) that the equation has no (real) solutions. □

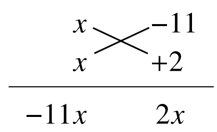
Although all quadratic equations can be solved using the quadratic formula, we may choose a faster method according to different conditions. In general, we may solve a quadratic equation by

- the factor method if the quadratic expression of the equation in standard form is easy to be factorized;
- the Square-root Theorem if the equation is/can be expressed in the form $(x+s)^2 = t$;
- the quadratic formula if the quadratic expression of the equation in standard form is difficult or impossible to be factorized using integer coefficients.

Example 3.3.14 Solve the equation $x^2 - 9x - 22 = 0$.

Explanation The quadratic expression $x^2 - 9x - 22$ can be factorize easily. Thus we apply the factor method.

Solution
$$\begin{aligned} x^2 - 9x - 22 &= 0 \\ (x-11)(x+2) &= 0 \qquad \text{Factorization} \\ x - 11 = 0 \quad \text{or} &\quad x + 2 = 0 \qquad \text{Product Zero Principle} \\ x = 11 \quad \text{or} &\quad x = -2 \end{aligned}$$



The solutions are 11 and -2. □

Example 3.3.15 Solve the equation $x^2 + 10x + 25 = 8$.

Explanation The equation can be expressed in the form $(px+q) = s^2$ easily. Thus we apply the Square-root Theorem.

Solution
$$\begin{aligned} x^2 + 10x + 25 &= 8 \\ (x+5)^2 &= 8 \qquad \text{By Perfect Square Identity} \\ x + 5 = \sqrt{8} \quad \text{or} &\quad x + 5 = -\sqrt{8} \qquad \text{By Square-root Theorem} \\ x = \sqrt{8} - 5 \quad \text{or} &\quad x = -\sqrt{8} - 5 \end{aligned}$$

The solutions are $\sqrt{8} - 5$ and $-\sqrt{8} - 5$. □

Example 3.3.16 Solve the equation $6x^2 - 11x - 10 = 0$.

Explanation Although the quadratic expression $6x^2 - 11x - 10$ can be factorized using integer coefficients, it is time consuming. Below we apply the quadratic formula.

Solution Note that $(-11)^2 - 4 \cdot 6 \cdot (-10) = 121 + 240 = 361 > 0$.

By the quadratic formula, $x = \frac{-(-11) \pm \sqrt{361}}{2 \cdot 6} = \frac{11 \pm 19}{12}$.

That is, $x = \frac{11 + 19}{12} = \frac{5}{2}$ or $x = \frac{11 - 19}{12} = -\frac{2}{3}$.

The solutions are $\frac{5}{2}$ and $-\frac{2}{3}$. □

Example 3.3.17 Solve the equation $x^2 - 10x + 22 = 0$.

Explanation The quadratic expression $x^2 - 10x + 22$ can't be factorized using integer coefficients. This is because $(-10)^2 - 4 \cdot 1 \cdot 22 \neq k^2$ for all integers k.

Solution Note that $(-10)^2 - 4 \cdot 1 \cdot 22 = 100 - 88 = 12 > 0$.

By the quadratic formula, $x = \frac{-(-10) \pm \sqrt{12}}{2 \cdot 1} = \frac{10 \pm \sqrt{12}}{2}$.

The solutions are $\frac{10 + \sqrt{12}}{2}$ and $\frac{10 - \sqrt{12}}{2}$. □

Remark Since $\sqrt{12} = 2\sqrt{3}$, it follows that $\frac{10 \pm \sqrt{12}}{2} = \frac{10 \pm 2\sqrt{3}}{2} = 5 \pm \sqrt{3}$. Thus the solutions are $5 + \sqrt{3}$ and $5 - \sqrt{3}$.

Exercise 3.3.2

1. For each of the following quadratic equations, solve it by using the quadratic formula.

(a)	$x^2 + 10x + 25 = 0$	(b)	$x^2 - 10x + 21 = 0$
(c)	$x^2 - x - 12 = 0$	(d)	$x^2 + 3x - 18 = 0$
(e)	$x^2 + 8x + 20 = 0$	(f)	$x^2 + 17x + 60 = 0$
(g)	$16 - 8x + x^2 = 0$	(h)	$2x^2 + x - 3 = 0$
(i)	$2x^2 - 3x - 5 = 0$	(j)	$4x^2 - x - 5 = 0$
(k)	$x^2 + 6x + 9 = 0$	(l)	$16x^2 - 40x + 25 = 0$
(m)	$x^2 = 6x - 8$	(n)	$2x^2 + 5x = 3$
(o)	$2x^2 = 6x + 8$	(p)	$4x + 4x^2 = 1$
(q)	$x^2 + 6x = 16$	(r)	$2x^2 + 7 = 5x$
(s)	$3(x^2 + 1) = x + 5$	(t)	$(x + 2)(x - 3) = 6$
(u)	$(2x - 1)(x + 2) = 25$		

2. For each of the following quadratic equations, solve it by using the quadratic formula; leave the radical sign '$\sqrt{\ }$' in the answers if necessary.

   (a) $x^2 + 7x + 11 = 0$ (b) $6x^2 - 5x - 2 = 0$

   (c) $2x^2 = 1 - 4x$ (d) $x(3x - 2) = 13$

   (e) $4x(x - 4) = 11$ (f) $\sqrt{5}x = x^2 - 4$

3. For each of the following quadratic equations, solve it by using the quadratic formula; give the answers correct to 3 significant figures if the solutions are irrational numbers.

   (a) $x^2 + 4x + 2 = 0$ (b) $x^2 - 9x + 7 = 0$

   (c) $2x^2 = x - 7$ (d) $(x + 3)(x - 3) = 5x + 1$

3.4 Problems Leading to Quadratic Equations

To solve a problem leading to a quadratic equation with one unknown, the following procedures can be used as a guide:

(0) Read the problem carefully; understand the problem.

(1) **Figure out the unknown(s)**.

Use a variable (for example x) to represent an unknown quantity.

- (a) In some problems, there is only one unknown quantity; we can denote it by x (for example, see Example 3.4.4). Sometimes, the unknown quantity is already represent by x (for example, see Example 3.4.3).
- (b) In some problems, there are more than one unknown quantities. However, the unknown quantities are related to each other. If we denote one of them by x, then the other unknown quantity/quantities can be expressed in terms of x (for example, see Example 3.4.2).

(2) **Note the restriction(s) on the variable**, if there is any. For example, if x represents an unknown length, then $x > 0$.

(3) **Set up an equation** (in x) using the information given in the problem.

In setting up the equation, the following should be noted:

- (a) Both sides of the equations must involve quantities of the same type, for example, length, area etc.
- (b) The units of the quantities in both sides of the equation must be the same.
- (c) The values of both sides of the equation must be equal.

(4) **Solve the equation** using an appropriate method (for example, the factor method or the quadratic formula).

(5) **Check the solutions**. Use the restriction on the variable obtained in Step 2 to see if the solutions are acceptable; reject the one that does not satisfies the restriction.

(6) **Answer the problem**.

Below we will consider some problems leading to quadratic equations. For the first one (which is Problem 3.1.1 in Section 3.1), we give detail explanation following the above guideline.

Example 3.4.1 The length of a rectangular carpet is 1 m longer than its width. If the length of the diagonal of the carpet is 5 m, find the dimensions of the carpet.

Explanation We follow Steps 1–6 given in the guideline.

(1) There are two unknowns, namely, the length and width of the carpet. Since the length is 1 m longer than the width, if we denote one of the unknown by x, then the other unknown can be expressed in terms of x.

In the solution below, we denote the length by x (in meter). From the given information, the width is $(x - 1)$.

(2) Since x is length, it follows that $x > 0$. Moreover, since $(x - 1)$ is also length (of the shorter side), it follows that $x - 1 > 0$, that is, $x > 1$.

(3) Since the length of the diagonal is 5, we can set up an equation using the Pythagoras Theorem:

$$x^2 + (x - 1)^2 = 5^2$$

In the above equation, both sides are areas. The unit of the areas on both sides are square meter.

(4) To solve the quadratic equation, we can rewrite it in standard form and then solve it by the factor method or quadratic formula.

(5) Solving the equation, we get two possibilities for x; one of them is rejected since $x > 1$.

(6) To answer the question, we write down the length and width.

Solution Let the length of the carpet be $x\,m$. Then the width is $(x-1)\,m$.

Note that $x > 1$.

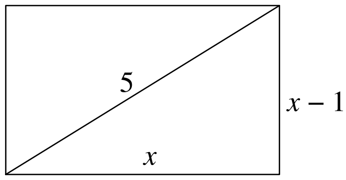


Figure 3.4.1

By the Pythagoras Theorem, we have

$$x^2 + (x-1)^2 = 5^2$$

Solving

$$\begin{aligned} x^2 + (x^2 - 2x + 1) &= 25 \\ 2x^2 - 2x - 24 &= 0 && \text{Rewrite in standard form} \\ x^2 - x - 12 &= 0 && \text{Multiply both sides by } \tfrac{1}{2} \\ (x-4)(x+3) &= 0 \\ x = 4 \quad \text{or} \quad & x = -3 \\ & \text{(rejected)} && \text{since } x > 1 \end{aligned}$$

When $x = 4$, $x - 1 = 3$.

The length of the carpet is $4\,m$ and the width is $3\,m$. □

Remark To answer the question, we can also write: "the dimensions are $4\,m \times 3\,m$."

If the denote the width by x, then the length is $(x+1)$. Note that $x > 0$. From the Pythagoras Theorem, we get $x^2 + (x+1)^2 = 5^2$. Solving, we get $x = 3$ or $x = -4$ (since $x > 0$).

Example 3.4.2 The product of two consecutive positive odd integers is 63. Find the two numbers.

Explanation If we denote one of the two numbers by x, then the other number can be expressed in terms of x.

Solution Let the smaller number be x. Then the larger number is $(x+2)$.

Note that $x > 0$ and x is odd.

Since the product of the two numbers is 63, it follows that

$$x(x+2) = 63$$

Solving

$$\begin{aligned} x^2 + 2x - 63 &= 0 \\ (x+9)(x-7) &= 0 \\ x = -9 \quad \text{or} \quad & x = 7 \\ \text{(rejected)} \end{aligned}$$

When $x = 7$, $x + 2 = 9$.

The two numbers are 7 and 9. □

Example 3.4.3 Figure 3.4.2 shows an L-shaped lawn. Suppose that the length of AB is $8\,m$, that the length of AF is $10\,m$, that the lengths of BC and EF are $x\,m$ and that the area of the lawn is $45\,m^2$. Find the value of x.

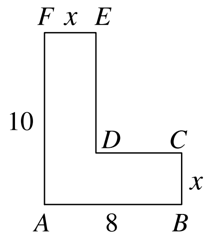
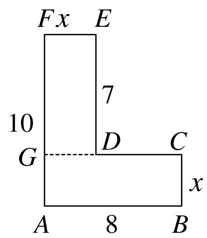


Figure 3.4.2

Solution Denote G to be the intersection point of the lines AF and CD.



Then
$$\begin{aligned} \text{length of } AG &= x\,m \\ \text{length of } FG &= (10-x)\,m \end{aligned}$$

Note that $0 < x < 10$.

Moreover,
$$\text{area of the L-shaped lawn} = \text{area of } ABCG + \text{area of } DEFG$$

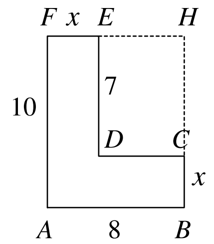
Note that
$$\begin{aligned} \text{area of } ABCG &= 8x\,m^2 \\ \text{area of } DEFG &= x\,(10-x)\,m^2 \end{aligned}$$

Since the area of the lawn is $45\,m^2$, it follows that
$$8x + x\,(10-x) = 45$$

Solving
$$\begin{aligned} 8x + 10x - x^2 &= 45 \\ 0 &= x^2 - 18x + 45 \\ 0 &= (x-3)(x-15) \\ x = 3 \quad &\text{or} \quad x = 15 \\ & \quad\;\; \text{(rejected)} \end{aligned}$$

The value of x is 3. □

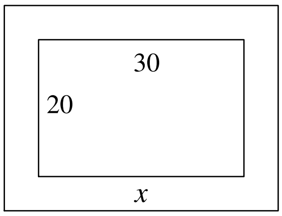
Remark Instead of introducing the point G and using sum of areas, we may consider the point H that lies on the intersection of Line BC and Line EF.



Using area of the L-shaped lawn = area of $ABHF$ − area of $CHED$,
we get $45 = 8 \cdot 10 - (8 - x)(10 - x)$.

Example 3.4.4 A rectangular picture is 30 cm long and 20 cm wide. It is surrounded by a frame with uniform width. If the area of the frame is the same as that of the picture. Find the width of the frame.

Solution Let the width of the frame be x cm.



Note that $x > 0$.

Moreover, length of larger rectangle = $(30 + 2x)$ cm
width of larger rectangle = $(20 + 2x)$ cm

Since

area of frame = area of larger rectangle − area of smaller rectangle

and the area of the picture is $(30 \cdot 20)$ cm^2, it follows that

$$30 \cdot 20 = (30 + 2x)(20 + 2x) - 30 \cdot 20.$$

Solving

$$\begin{aligned} 600 &= 600 + 60x + 40x + 4x^2 - 600 \\ 0 &= 4x^2 + 100x - 600 \\ 0 &= x^2 + 25x - 150 \\ 0 &= (x + 30)(x - 5) \end{aligned}$$

$x = -30$ (rejected) or $x = 5$

The width of the frame is 5 cm. □

Example 3.4.5 In Figure 3.4.3, $ABCD$ is a square of side $10\,cm$. Four congruent isosceles right-angled triangles are cut off from the square to form a regular octagon. Find the area of the octagon; give your answer correct to 3 significant figures.

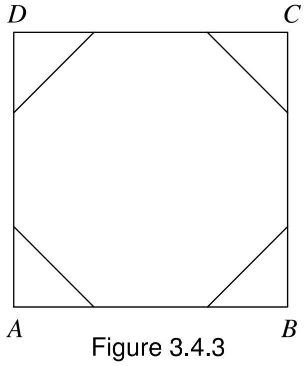
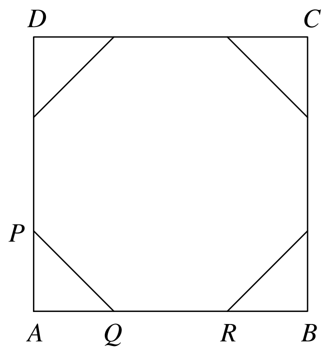


Figure 3.4.3

Explanation A regular octagon is a polygon with 8 sides of equal lengths.

- If we know the length of a side of the octagon, then we can find the area (see Solution 1).
- If we know any side of a triangle cut off, then we can find the area of the triangle; hence we can find the area of the octagon (see Solution 2).

Solution 1 Consider the points P, Q and R on the octagon as shown in the figure.



Let the length of a side of the octagon be $x\,cm$. Then $PQ = QR = x\,cm$.

Since the length of AB is $10\,cm$ and the triangles cut off are congruent, it follows that

$$10 = 2 \cdot AQ + x$$

Hence $AP = AQ = \dfrac{10 - x}{2}$.

Note that $0 < x < 10$.

Applying the Pythagoras Theorem to the isosceles right-angled triangle APQ, we get

$$\left(\frac{10 - x}{2}\right)^2 + \left(\frac{10 - x}{2}\right)^2 = x^2$$

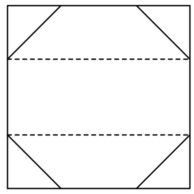
Solving

$$\begin{aligned} \frac{100 - 20x + x^2}{4} + \frac{100 - 20x + x^2}{4} &= x^2 \\ \frac{100 - 20x + x^2}{2} &= x^2 \\ 100 - 20x + x^2 &= 2x^2 \\ 0 &= x^2 + 20x - 100 \end{aligned}$$

(soln 1 cont'd) By the quadratic formula, $x = \dfrac{-20 \pm \sqrt{20^2 - 4(1)(-100)}}{2(1)} = \dfrac{-20 \pm \sqrt{800}}{2}$

That is, $x = \dfrac{-20 + \sqrt{800}}{2}$ or $x = \dfrac{-20 - \sqrt{800}}{2}$ (rejected)

By dividing the octagon into three regions, we see that

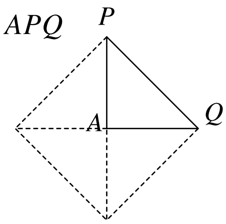


$$\begin{aligned}
\text{area of the octagon (in } cm^2) &= \frac{(10 + x) \cdot \frac{10 - x}{2}}{2} \times 2 + 10 \cdot x \\
&= \frac{1}{2}(100 - x^2) + 10x \\
&= \frac{1}{2}\left(100 - \frac{-20 + \sqrt{800}}{2}\right)^2 + 10 \cdot \frac{-20 + \sqrt{800}}{2} \\
&\approx \frac{1}{2}(100 - 4.14214)^2 + 10 \cdot 4.14214 \\
&\approx 82.8427
\end{aligned}$$

The area of the octagon, correct to 3 significant figure, is 82.8 cm^2. □

Remark Alternatively, we can use

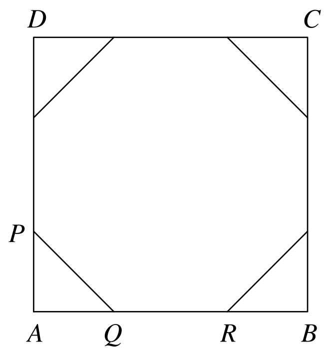
area of the octagon = area of square $ABCD - 4 \times$ area of triangle APQ



Note that the four congruent isosceles right-angled triangle can be arranged to form a square with side PQ. Hence

$$\begin{aligned}
\text{area of the octagon} &= \text{area of } ABCD - \text{area of square with side } PQ \\
&= (10^2 - x^2)\ cm^2
\end{aligned}$$

Solution 2 Consider the points P, Q and R on the octagon as shown in the figure.



Let $AQ = x\ cm$. Then $QR = (10 - 2x)\ cm$.

Note that $0 < x < 5$.

Applying the Pythagoras Theorem to the isosceles right-angled triangle APQ, we get

$$x^2 + x^2 = (10 - 2x)^2$$

$$\begin{aligned}
\text{Solving} \quad 2x^2 &= 100 - 40x + 4x^2 \\
0 &= 2x^2 - 40x + 100 \\
0 &= x^2 - 20x + 50
\end{aligned}$$

(soln 2 cont'd) By the quadratic formula,

$$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(1)(50)}}{2(1)} = \frac{20 \pm \sqrt{200}}{2}$$

that is,

$$x = \frac{20 + \sqrt{200}}{2} \text{ (rejected)} \quad \text{or} \quad x = \frac{20 - \sqrt{200}}{2}$$

$$\begin{aligned} \text{Hence} \quad \text{area of } \triangle APQ &= \frac{x \cdot x}{2}\, cm^2 \\ &= \frac{1}{2}\left(\frac{20 - \sqrt{200}}{2}\right)^2 cm^2 \\ &\approx 0.5 \times 2.92893^2\, cm^2 \\ &\approx 4.28932\, cm^2 \end{aligned}$$

$$\begin{aligned} \text{Therefore} \quad \text{area of the octagon} &\approx (10^2 - 4 \times 4.28932)\, cm^2 \\ &\approx 82.8427\, cm^2 \end{aligned}$$

The area of the octagon, correct to 3 significant figure, is 82.8 cm^2. □

Remark Alternatively, we may first consider $\triangle APQ$ to get

$$PQ = \sqrt{x^2 + x^2} = \sqrt{2}x$$

and then using $AB = AQ + QR + RB$ to get a linear equation in x:

$$10 = x + \sqrt{2}x + x$$

$$\begin{aligned} \text{Solving} \quad 10 &= 2x + \sqrt{2}x \\ 10 &= (2 + \sqrt{2})x \\ x &= \frac{10}{2 + \sqrt{2}} \quad \text{which is the same as } \frac{20 - \sqrt{200}}{2} \end{aligned}$$

Exercise 3.4

1. In Figure 3.4.4 , $ABCD$ is a rectangle. Suppose that the lengths of AB and BC are $(x + 5)\, cm$ and $x\, cm$ respectively and that the area of the rectangular region is 84 cm^2. Find the value of x.

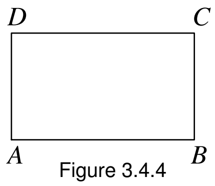
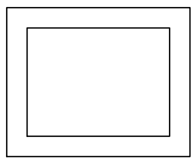


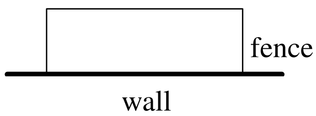
Figure 3.4.4

2. It is given that the number of diagonals of a convex polygon with n sides is $\frac{n(n-3)}{2}$. If there are 14 diagonals in a convex polygon, how many sides does the polygon has?

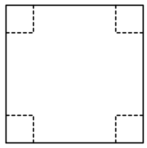
3. A ball is thrown vertically upwards. Its height above the ground after x seconds is $(1 + 9x - 2x^2)\,m$. When is the ball $5\,m$ above the ground?
4. If Mr Chan drives at a speed of $(x + 15)$ *km per hour*, then he can travel 100 *km* in $(x - 5)$ *hours*. Find the value of x.
5. The sum of a positive integer and its square is 72. Find the number.
6. The sum of squares of two consecutive positive odd numbers is 130. Find the two numbers.
7. The product of two consecutive positive numbers is more than 3 times their sum by 11. Find the two numbers.
8. The sum of two numbers is 8 and the sum of their squares is 34. Find the two numbers.
9. The sum of the squares of three consecutive positive integers is less than the square of their sum. Find the three numbers.
10. Tai Man is 3 years older than his sister Siu Man, and 21 years younger than his mother. Nine years ago, the product of the ages of Tai Man and Siu Man was the same as that of their mother. How old is Tai Man now?
11. The lengths of the three sides of a right-angled triangle are three consecutive even numbers. Find the length of the hypotenuse.
12. The base of a right-angled triangle is $7\,cm$ longer than its height and $2\,cm$ shorter than its hypotenuse. Find the length of the base of the triangle.
13. The area of a rectangular cardboard is $600\,cm^2$ and its perimeter is $1\,m$. Find the dimensions of the cardboard.
14. A rectangular photo with dimensions $21\,cm \times 16\,cm$ is surrounded by a border of uniform width. If the area of the border is $258\,cm^2$, find the width of the border.



15. A rectangular piece of land is enclosed by a wall and a fence of length $25\,m$ along the other three sides. Find the length and width of the piece of land if its area is $75\,m^2$.



16. Four identical squares of side $2\,cm$ are cut off from the corners of a square paper and the remaining portion is folded to form a box without cover. If the capacity of the box is $72\,cm^3$, find the length of the side of the original square paper.



17. In Figure 3.4.5, $ABCD$ is a square of side $9\,cm$, P and Q are points on AB and BC respectively such that the length of PB is twice as that of BQ and that the area of the triangle PQD is $23\,cm^2$. Find the length of PQ.

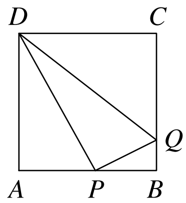
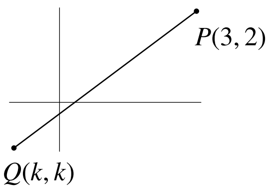


Figure 3.4.5

18. In the rectangular coordinate plane, P is the point $(3,2)$ and Q is the point (k,k) where k is a negative number. Suppose that the distance between P and Q is 5. Find the value of k.



19. In a two-digit number, the units digit is greater than the tens digit by 2 and the sum of squares of the two digits is greater than the two-digit number by 1. Find the two-digit number.

20. Mr Chan deposits $\$5,000$ in a bank at the beginning of each month. The interest rate is $r\%$ per annum where r is a positive inetger and interest is compounded monthly. If Mr Chan gets an amount of $\$10,050.1$ at the end of the second month. Find the value of r.

3.5 Roots and Coefficients of Quadratic Equations

Terminology 3.5.1 A solution to an equation is also called a ***root*** of the equation.

In this section, we will consider two things:

(1) The number of real root(s) that a quadratic equation has.

(2) The relation between the roots and the coefficients of a quadratic equation.

3.5.1 Nature of Roots of Quadratic Equations

Given a quadratic equation

$$ax^2 + bx + c = 0 \tag{3.5.1}$$

where a, b, c are real numbers with $a \neq 0$, there may be (exactly) two real roots, (exactly) one real root or no real root. According to Theorem 3.3.1 (or the quadratic formula), the number of roots can be determined by the value of $b^2 - 4ac$.

Terminology 3.5.2 For a quadratic equation $ax^2 + bx + c = 0$, where a, b, c are real numbers with $a \neq 0$, we call ***the discriminant of the equation***, and write Δ, to mean the number $b^2 - 4ac$.

Remark Δ, pronounced as '*delta*', is a Greek letter; it corresponds to the English letter D.

- For the case where $\Delta > 0$, Equation (3.5.1) has (exactly) two real roots which can be obtained from the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. To emphasize that the real roots are unequal, we say that

  (1) *the equation has two unequal* (or *distinct*) *real roots*.

- For the case where $\Delta = 0$, Equation (3.5.1) has exactly one real root.
  If we apply the quadratic formula, we will get $x = \frac{-b \pm 0}{2a}$. This means that the "two" roots are equal. For this reason, we say that

  (2*a*) *the equation has two equal real roots*.

  If we denote the unique real root of the equation by r, then the left-side of Equation (3.5.1) can be factorized as $a(x - r)^2$, that is, the quadratic equation can be written as

$$a(x - r)^2 = 0.$$

  For this reason, we say that

  (2*b*) *the equation has a double* (or *repeated*) *real root*.

- For the case where $\Delta < 0$, Equation (3.5.1) does not have any real root. Thus we say that

  (3*a*) *the equation has no real roots*.

  If we allow complex numbers as solutions, then the quadratic equation has (exactly) two complex roots that are not real numbers. For this reason, we say that

  (3*b*) *the equation has two unequal* (or *distinct*) *non-real roots*.

Terminology 3.5.3 To *determine the nature* of the roots of a quadratic equation means to determine whether the roots are real or non-real and to determine whether the roots are distinct or repeated.

Remark The nature of the roots is related to the number of real roots. Thus to determine the nature of the roots, we can simply state how many real roots the quadratic equation has.

Example 3.5.1 Determine the nature of the roots of the quadratic equation $2x^2 + 3x + 4 = 0$.

Solution Note that
$$\begin{aligned}\Delta &= 3^2 - 4 \cdot 2 \cdot 4 \\ &= -23 \\ &< 0\end{aligned}$$
The equation has no real roots. □

Remark We may also write "*The equation has two distinct non-real roots*".

Example 3.5.2 Determine the nature of the roots of the quadratic equation $3x^2 - 4x + 1 = 0$.

Solution Note that
$$\begin{aligned}\Delta &= (-4)^2 - 4 \cdot 3 \cdot 1 \\ &= 4 \\ &> 0\end{aligned}$$
The equation has two distinct real roots. □

Example 3.5.3 Determine the nature of the roots of the quadratic equation $4x^2 + 12x + 9 = 0$.

Solution Note that
$$\begin{aligned}\Delta &= 12^2 - 4 \cdot 4 \cdot 9 \\ &= 0\end{aligned}$$
The equation has only one real root (repeated). □

In each of the remaining examples in this section, the given quadratic equation involves an unknown constant. The (possible) value(s) of the constant can be determined if we know the nature of the roots.

Example 3.5.4 Suppose that the quadratic equation $2x^2 - 10x + k = 0$ has only one real root. Find the value of k.

Solution The equation has only one real root is equivalent to that $\Delta = 0$, that is,

$$\begin{aligned} (-10)^2 - 4 \cdot 2 \cdot k &= 0 \\ 100 - 8k &= 0 \end{aligned}$$

Hence $k = \frac{100}{8} = \frac{25}{2}$. □

Example 3.5.5 Suppose that the quadratic equation $x^2 + 3x - 2 = k$ has real root(s). Find the possible values of k.

Explanation The equation has real root(s) means that it has one real root or two real roots. The question asks for all possible values of k.

Solution First we rewrite the equation in standard form:

$$x^2 + 3x - 2 - k = 0$$

The equation has real root(s) is equivalent to that $\Delta \geq 0$, that is,

$$\begin{aligned} 3^2 - 4 \cdot 1 \cdot (-2 - k) &\geq 0 \\ 9 + 8 + 4k &\geq 0 \\ 4k &\geq -17 \end{aligned}$$

Hence $k \geq \frac{-17}{4}$. □

Example 3.5.6 Suppose that the quadratic equation $2x(x + 4) = 3 - k$ has no real roots. Find the possible values of k.

Solution First we rewrite the equation in standard form:

$$2x^2 + 8x + k - 3 = 0$$

The equation has no real roots is equivalent to that $\Delta < 0$, that is,

$$\begin{aligned} 8^2 - 4 \cdot 2 \cdot (k - 3) &< 0 \\ 64 - 8k + 24 &< 0 \\ 88 &< 8k \end{aligned}$$

Hence $k > 11$. □

Example 3.5.7 Consider the quadratic equation

$$x^2 - 2(k+1)x + 5k + 11 = 0$$

where k is a positive real number. Suppose that the equation has a repeated real root. Find the root of the equation.

Explanation Using the condition that the equation has only one real root, we can find the value of k (note that $k > 0$). Hence we can find the root.

Solution The equation has a repeated real root is equivalent to that $\Delta = 0$, that is,

$$\begin{aligned}
(-2(k+1))^2 - 4 \cdot 1 \cdot (5k+11) &= 0 \\
4(k^2 + 2k + 1) - 4(5k + 11) &= 0 \\
k^2 + 2k + 1 - 5k - 11 &= 0 \\
k^2 - 3k - 10 &= 0 \\
(k-5)(k+2) &= 0 \\
k = 5 \quad \text{or} \quad & k = -2 \\
& \text{(rejected since } k \text{ is positive)}
\end{aligned}$$

Hence $k = 5$ and the equation is

$$x^2 - 2(5+1)x + (5 \times 5 + 11) = 0.$$

Solving
$$\begin{aligned}
x^2 - 12x + 36 &= 0 \\
(x-6)^2 &= 0 \\
x - 6 &= 0 \\
x &= 6
\end{aligned}$$

The root of the equation is 6. □

Example 3.5.8 Consider the quadratic equation

$$(k^2 + 1)x^2 = 5x - 7$$

where k is a real number. Show that the equation has no real roots.

Solution First we rewrite the quadratic equation in standard form:

$$(k^2 + 1)x^2 - 5x + 7 = 0$$

Note that
$$\begin{aligned}
\Delta &= (-5)^2 - 4(k^2 + 1)(7) \\
&= 25 - 28k^2 - 28 \\
&= -28k^2 - 3 \\
&\leq -3
\end{aligned}$$

Since $\Delta < 0$, it follows that the quadratic equation has no real roots.

Exercise 3.5.1

1. For each of the following quadratic equations, find the value of the discriminant.

   (*a*) $x^2 + 5x + 3$ (*b*) $2x^2 + 3x + 4 = 0$

   (*c*) $4x^2 + 25 = 20x$ (*d*) $3(x^2 - 2) = 5x$

   (*e*) $9x^2 + 4 = 0$ (*f*) $4x^2 + 3x = 0$

   (*g*) $5x^2 - 6 = 0$

2. For each of the following quadratic equations, express the discriminant in terms of k

   (*a*) $3x^2 - 5x + k = 0$ (*b*) $(k + 1)x^2 - 2kx = 2 - k$

   (*c*) $2x(3x - 2) + 2 = k$ (*d*) $k(x^2 + 1) = 4 - kx$

3. For each of the following quadratic equations, determine the nature of the roots.

   (*a*) $x^2 + 4x - 1 = 0$ (*b*) $3x^2 - 4x + 2 = 0$

   (*c*) $x^2 - 6x + 9 = 0$ (*d*) $3x^2 + 2x + 1 = 0$

   (*e*) $3x^2 = 10 - x$ (*f*) $4x^2 = 12x - 9$

   (*g*) $2x^2 = 6x - 7$ (*h*) $3x^2 + 4 = 0$

   (*i*) $4x^2 - 5 = 0$ (*j*) $2x^2 = 3x$

4. Suppose that the quadratic equation $3x^2 - 4x + k = 0$ has only one real root. Find the value(s) of k.

5. Suppose that the quadratic equation $5x^2 + 4kx + 20 = 0$ has a repeated real root. Find the value(s) of k.

6. Suppose that the quadratic equation $x^2 + kx + k + 3 = 0$ has a double real root. Find the value(s) of k.

7. Suppose that the quadratic equation $2kx(x + 1) = 2 - k - 5x^2$ has two equal real roots. Find the value(s) of k.

8. Suppose that the quadratic equation $kx^2 + 3x - 2 = 0$ has two distinct real roots. Find the possible values of k.

9. Suppose that the quadratic equation $kx^2 - 3x + 5 = 0$ has two unequal real roots. Find the possible values of k.

10. Suppose that the quadratic equation $kx^2 + 3x - 2 = 0$ has no real roots. Find the possible values of k.

11. Suppose that the quadratic equation $2x^2 + 3x + 1 = 2k$ has two unequal non-real roots. Find the possible values of k.

12. Suppose that the quadratic equation $2x^2 + 3x + 1 = 2k$ has real root(s). Find the possible values of k.

13. Suppose that the equation $x^2 + kx + 2k + 15 = 0$ has only one real root.

    (*a*) Find the values of k.

    (*b*) For each value of k, find the root of the equation.

14. Let p and q be real numbers. Show that the quadratic equation $(x - p)(x - q) = pq$ has real roots(s).

15. Let m be a real number. Show that the quadratic equation $x^2 - (m + 1)x + m^2 - 2 = 0$ has no real roots.

3.5.2 Relation between Roots and Coefficients

Before discussing the relation between roots and coefficients of a quadratic equation, we consider an example.

Example 3.5.9 Suppose that α and β are the roots of the quadratic equation

$$x^2 - 7x + 12 = 0$$

Find the value of each of the following expressions.

(a) $\alpha^2 + \beta^2$

(b) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

Explanation The given equation has two distinct real roots. To find $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$, it doesn't matter whether we denote the larger root by α or β; this is because the values of $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$ are unchanged if the roles of α and β are interchanged.

Solution

$$\begin{aligned} x^2 - 7x + 12 &= 0 \\ (x-3)(x-4) &= 0 \\ x = 3 \quad &\text{or} \quad x = 4 \end{aligned}$$

The roots of the equations are 3 and 4. Thus $(\alpha, \beta) = (3, 4)$ or $(\alpha, \beta) = (4, 3)$.

(a)
$$\begin{aligned} \alpha^2 + \beta^2 &= 3^2 + 4^2 \\ &= 25 \end{aligned}$$

(b)
$$\begin{aligned} \frac{1}{\alpha} + \frac{1}{\beta} &= \frac{1}{3} + \frac{1}{4} \\ &= \frac{7}{12} \end{aligned}$$

□

Note that the expressions $\alpha^2 + \beta^2$ and $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ can be expressed in terms of $\alpha + \beta$ and $\alpha\beta$:

$$\begin{aligned} \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ \frac{1}{\alpha} + \frac{1}{\beta} &= \frac{\beta + \alpha}{\alpha\beta} \end{aligned}$$

Thus, if we know how to find the values of $\alpha + \beta$ and $\alpha\beta$, then we can find $\alpha^2 + \beta^2$ and $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$. Below we give a general result (Theorem 3.5.1) which tells us how to find the sum and product of roots using the coefficients of the quadratic expression. The derivation of the formulas is described as follows.

Consider the following quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

where a, b, c are real numbers with $a \neq 0$. Suppose that the equation has two distinct real roots. Then the two roots can be found using the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. If we denote the two roots by α and β, then

$$\begin{aligned} \alpha + \beta &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b - b}{2a} \\ &= -\frac{b}{a} \end{aligned} \quad \text{and} \quad \begin{aligned} \alpha\beta &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{(2a)^2} \\ &= \frac{b^2 - (b^2 - 4ac)}{4a^2} \\ &= \frac{4ac}{4a^2} \\ &= \frac{c}{a} \end{aligned}$$

The above results are summarized in the following theorem.

Theorem 3.5.1 Consider a quadratic equation in standard form:

$$ax^2 + bx + c = 0$$

where a, b, c are real numbers with $a \neq 0$. Suppose that the equation has two distinct real roots. Then

- Sum of the roots $= -\dfrac{b}{a}$
- Product of the roots $= \dfrac{c}{a}$

Remark The theorem is also valid if there is only one real root or no real roots.

- For the case where there is only one real root, the real root (denoted by α), is counted twice. Thus the sum of roots is 2α and the product of roots is α^2.
- For the case where there is no real roots, there are two distinct non-real roots. Thus, we can still consider their sum and product.

We apply Theorem 3.5.1 to re-do Example 3.5.9.

Example 3.5.10 Suppose that α and β are the roots of the quadratic equation

$$x^2 - 7x + 12 = 0$$

For each of the following expressions, find its value using Theorem 3.5.1.

(a) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

(b) $\alpha^2 + \beta^2$

Explanation The idea is to express the given expressions in terms of $\alpha + \beta$ and $\alpha\beta$.

Solution By Theorem 3.5.1,

$$\alpha + \beta = -\frac{-7}{1} = 7 \qquad \text{and} \qquad \alpha\beta = \frac{12}{1} = 12$$

$$(a) \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{7}{12}$$

$$\begin{aligned}(b) \quad \alpha^2 + \beta^2 &= (\alpha^2 + 2\alpha\beta + \beta^2) - 2\alpha\beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= 7^2 - 2 \cdot 12 \\ &= 25\end{aligned}$$

□

Example 3.5.11 Suppose that α and β are the roots of the quadratic equation

$$2x^2 - 5x - 1 = 0$$

Find the value of each of the following expressions.

(a) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(b) $(\alpha - \beta)^2$

Explanation We will use two methods to find the answers.

- Since Δ is not a perfect square, the solutions α and β involve the radical sign. Calculating the expressions by direct substitution is tedious.
- To apply Theorem 3.5.1, we have to express $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$ and $(\alpha - \beta)^2$ in terms of $\alpha + \beta$ and $\alpha\beta$.

Solution 1 By the quadratic formula, $x = \dfrac{-(-5) \pm \sqrt{(-5)^2 - 4 \cdot 2 \cdot (-1)}}{2 \cdot 2} = \dfrac{5 \pm \sqrt{33}}{4}$

(soln 1 cont'd) (*a*)

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\frac{5+\sqrt{33}}{4}}{\frac{5-\sqrt{33}}{4}}+\frac{\frac{5-\sqrt{33}}{4}}{\frac{5+\sqrt{33}}{4}}$$

$$= \frac{(5+\sqrt{33})^2}{(5-\sqrt{33})(5+\sqrt{33})}+\frac{(5-\sqrt{33})^2}{(5+\sqrt{33})(5-\sqrt{33})}$$

$$= \frac{25+10\sqrt{33}+33}{25-33}+\frac{25-10\sqrt{33}+33}{25-33}$$

$$= \frac{116}{-8}$$

$$= -\frac{29}{2}$$

(*b*)

$$(\alpha-\beta)^2 = \left(\frac{5+\sqrt{33}}{4}-\frac{5-\sqrt{33}}{4}\right)^2$$

$$= \left(\frac{2\sqrt{33}}{4}\right)^2$$

$$= \frac{33}{4}$$

□

Solution 2 By Theorem 3.5.1, $\alpha+\beta=-\frac{-5}{2}=\frac{5}{2}$ and $\alpha\beta=\frac{-1}{2}$

(*a*)

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha} = \frac{\beta^2+\alpha^2}{\alpha\beta}$$

$$= \frac{\alpha^2+2\alpha\beta+\beta^2-2\alpha\beta}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$= \frac{\left(\frac{5}{2}\right)^2-2\cdot\frac{-1}{2}}{\frac{-1}{2}}$$

$$= -\frac{29}{2}$$

(*b*)

$$(\alpha-\beta)^2 = \alpha^2-2\alpha\beta+\beta^2$$

$$= \alpha^2+2\alpha\beta+\beta^2-4\alpha\beta$$

$$= (\alpha+\beta)^2-4\alpha\beta$$

$$= \left(\frac{5}{2}\right)^2-4\cdot\frac{-1}{2}$$

$$= \frac{33}{4}$$

□

In Example 3.5.10 and Example 3.5.11, we can use Theorem 3.5.1 to find the values of the given expressions. This is because the expressions can be written in terms of $\alpha+\beta$ and $\alpha\beta$. Expressions that can be written in this way must be *symmetric* in the sense that the expressions are unchanged if the roles of α and β are interchanged. For example, the expressions $\alpha^2+\beta^2$ and $(\alpha-\beta)^2$ are symmetric:

$$\begin{aligned} \alpha^2+\beta^2 &\equiv \beta^2+\alpha^2 \\ (\alpha-\beta)^2 &\equiv (\beta-\alpha)^2 \end{aligned}$$

Similarly, the expressions $(\alpha+2)(\beta+2)$ and $(\alpha^2\beta+\beta^2\alpha)$ are also symmetric:

$$\begin{aligned} (\alpha+2)(\beta+2) &\equiv (\beta+2)(\alpha+2) \\ \alpha^2\beta+\beta^2\alpha &\equiv \beta^2\alpha+\alpha^2\beta \end{aligned}$$

and they can be written in terms of $\alpha+\beta$ and $\alpha\beta$ (see Exercise 3.5.2).

In the next example, we can't find the value of the given expression using Theorem 3.5.1 since $(\alpha-2)(\beta+1)$ is not symmetric:

$$\text{in general} \qquad (\alpha-2)(\beta+1) \neq (\beta-2)(\alpha+1)$$

Example 3.5.12 Suppose that α and β are the roots of the quadratic equation

$$2x^2 = x+1$$

and that $\alpha>\beta$. Find the value of $(\alpha-2)(\beta+1)$.

Solution First, rewrite the quadratic equation in standard form:

$$2x^2-x-1=0$$

$$\begin{aligned} \text{Solving} \qquad (2x+1)(x-1) &= 0 \\ 2x+1=0 \quad &\text{or} \quad x-1=0 \\ x=-\frac{1}{2} \quad &\text{or} \quad x=1 \end{aligned}$$

Since $\alpha>\beta$, it follows that

$$\alpha=1 \qquad \text{and} \qquad \beta=-\frac{1}{2}$$

$$\begin{aligned} \text{Hence} \qquad (\alpha-2)(\beta+1) &= (1-2)\cdot\left(-\frac{1}{2}+1\right) \\ &= -\frac{1}{2} \end{aligned}$$

□

In the next example, although the expression $(\alpha-\beta)$ is not symmetric, it can be expressed in terms of $(\alpha-\beta)^2$, which is symmetric. This is because

$$\alpha-\beta=\sqrt{(\alpha-\beta)^2} \qquad \text{since } \alpha>\beta.$$

Example 3.5.13 Suppose that α and β are the roots of the quadratic equation

$$x^2 + x - 3 = 0$$

and that $\alpha > \beta$. Find the value of $(\alpha - \beta)$.

Explanation We can find the values of α and β directly. Alternatively, we can find the value of $(\alpha - \beta)^2$ using Theorem 3.5.1 and the apply $\alpha - \beta = \sqrt{(\alpha - \beta)^2}$ since $\alpha > \beta$.

Solution 1 By the quadratic formula,

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1 \cdot (-3)}}{2 \cdot 1} = \frac{-1 \pm \sqrt{13}}{2}$$

Since $\alpha > \beta$, it follows that

$$\alpha = \frac{-1 + \sqrt{13}}{2} \quad \text{and} \quad \beta = \frac{-1 - \sqrt{13}}{2}$$

Hence
$$\begin{aligned} \alpha - \beta &= \frac{-1 + \sqrt{13}}{2} - \frac{-1 - \sqrt{13}}{2} \\ &= \sqrt{13} \end{aligned}$$

□

Solution 2 By Theorem 3.5.1,

$$\alpha + \beta = -\frac{1}{1} = -1 \quad \text{and} \quad \alpha\beta = \frac{-3}{1} = -3$$

Note that
$$\begin{aligned} (\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta \\ &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= (-1)^2 - 4 \cdot (-3) \\ &= 13 \end{aligned}$$

Since $\alpha > \beta$, it follows that
$$\begin{aligned} \alpha - \beta &= \sqrt{(\alpha - \beta)^2} \\ &= \sqrt{13} \end{aligned}$$

□

Example 3.5.14 Suppose a is a non-zero real number and the sum of the roots of the quadratic equation $ax^2 + (a + 2)x - 3 = 0$ is equal to 3. Find the value of a.

Solution By Theorem 3.5.1, sum of roots $= \dfrac{-(a + 2)}{a}$

Thus, $\dfrac{-(a + 2)}{a} = 3$

Solving
$$\begin{aligned} -a - 2 &= 3a \\ -2 &= 4a \end{aligned}$$

Therefore $a = \dfrac{-1}{2}$

□

Example 3.5.15 Suppose k is a real number and one of the root of the quadratic equation $2x^2 + 3x - k = 0$ is equal to 4. Find the other root and the value of k.

Explanation There are two ways to get the answers:

- Use substitution to find the value of k. Hence find the other root.
- Use sum and product of roots to obtain two equations involving k and the other root.

Solution 1 Since 4 is a root of the equation, it follows that

$$2 \cdot 4^2 + 3 \cdot 4 - k = 0$$

Hence $k = 44$.

The equation is $2x^2 + 3x - 44 = 0$

Solving

$$\begin{aligned} 2x^2 - 8x + 11x - 44 &= 0 \qquad -8 + 11 = 3, \quad (-8) \cdot 11 = 2 \cdot (-44) \\ 2x(x-4) + 11(x-4) &= 0 \\ (x-4)(2x-11) &= 0 \\ x - 4 = 0 \quad &\text{or} \quad 2x + 11 = 0 \\ x = 4 \quad &\text{or} \quad x = -\frac{11}{2} \end{aligned}$$

The other root is $\frac{11}{2}$. □

Solution 2 Denote the other root by β. By Theorem 3.5.1,

$$4 + \beta = -\frac{3}{2} \qquad \text{and} \qquad 4\beta = \frac{-k}{2}$$

From the first equality, we get

$$\begin{aligned} \beta &= -\frac{3}{2} - 4 \\ &= -\frac{11}{2} \end{aligned}$$

Therefore, from the second equality, we get $4 \cdot \left(-\frac{11}{2}\right) = \frac{-k}{2}$

Thus $k = 44$. □

Exercise 3.5.2

1. For each of the following quadratic equations, find the sum of the roots and product of roots.

   (*a*) $x^2 - 9x + 7 = 0$ (*b*) $2x^2 + 5x - 15 = 0$

   (*c*) $5x^2 - 7x = 6$ (*d*) $3x(x+2) = 5 - x$

2. Rewrite each of the following expressions in terms of $\alpha + \beta$ and $\alpha\beta$.

   (*a*) $(\alpha + 2)(\beta + 2)$ (*b*) $\alpha^2\beta + \beta^2\alpha$ (*c*) $\frac{1}{\alpha} + \frac{1}{\beta}$

   (*d*) $\alpha^2 + \beta^2$ (*e*) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ (*f*) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

   (*g*) $(\alpha - \beta)^2$ (*h*) $\frac{1}{\alpha - 1} + \frac{1}{\beta - 1}$ (*i*) $\alpha^3 + \beta^3$

3. Suppose that α and β are the roots of the quadratic equation $2x^2 - x - 3 = 0$. Without solving the equation, find the value of each of the following expressions.

   (*a*) $2\alpha + 2\beta$ (*b*) $\alpha^2\beta^2$ (*c*) $3 - \alpha - \beta$

   (*d*) $(\alpha + 1)(\beta + 1)$ (*e*) $\frac{1}{\alpha} + \frac{1}{\beta}$

4. Suppose that α and β are the roots of the quadratic equation $3x^2 = 6 - 5x$. Without solving the equation, find the value of each of the following expressions.

   (*a*) $\alpha^2 + \beta^2$ (*b*) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (*c*) $\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right)$

5. Suppose that α and β are the roots of the quadratic equation $x^2 + 5x + k = 0$, where k is a real number. Rewrite each of the following expressions in terms of k.

   (*a*) $(\alpha - \beta)^2$ (*b*) $\frac{1}{\alpha - 1} + \frac{1}{\beta - 1}$ (*c*) $\alpha^3 + \beta^3$

6. Suppose that α and β are the roots of the quadratic equation $x^2 + px + q = 0$, where p and q are real numbers. Rewrite each of the following expressions in terms of p and q.

   (*a*) $\alpha^2 + \beta^2$ (*b*) $(\alpha - 3)(\beta - 3)$

7. Suppose that α and β are the roots of the quadratic equation $x^2 = 7x + 11 = 0$ and that $\alpha > \beta$. Find the value of $(2\alpha - \beta)$.

8. Consider the quadratic equation $7x^2 + kx - 5 = 0$, where k is a real number. Suppose that the sum of the roots of the equation is equal to 5. Find the value of k.

9. Consider quadratic equation $2x^2 + (3 - 4b)x - 5 = 0$, where b is a real number. Suppose that the roots of the equation are the negatives of each other. Find the value of b.

10. Consider the quadratic equation $3x^2 + 7x + k = 0$, where k is a real number. Suppose that the product of the roots of the equation is equal to -2. Find the value of k.

11. Consider the quadratic equation $ax^2 + 7x + 8 - 9a = 0$, where a be a non-zero real number. Suppose that the roots of the equation are the reciprocals of each other. Find the value of a.

12. Consider the quadratic equation $2x^2 - 7x + c = 0$, where c is a real number. Suppose that 3 is a root of the equation. Find the other root and the value of c.

13. Consider the quadratic equation $2x^2 + 3x + k = 0$, where k is a real number. Suppose that one root of the equation is equal to twice the other root. Find the two roots and the value of k.

14. Consider the quadratic equation $x^2 - 4x + k = 0$, where k is a real number. Suppose that the difference of the roots of the equation is equal to 2. Find the two roots and the value of k.

3.6 Forming Quadratic Equations

In Section 3.2 and Section 3.3, we have discussed how to find the roots of a quadratic equation. In this section, we will consider the reverse problem: given two real numbers α and β, how can we find a quadratic equation whose roots are α and β? *Forming an equation* is the reverse process of *solving an equation*. The following example (diagram) illustrates the relationship between the two process:

Solving Equation ↓

$$\begin{aligned} x^2 - 4x - 21 &= 0 \\ (x-7)(x+3) &= 0 \\ x - 7 = 0 \quad &\text{or} \quad x + 3 = 0 \\ x = 7 \quad &\text{or} \quad x = -3 \end{aligned}$$

↑ Forming Equation

In general, we have the following result.

Theorem 3.6.1 Let α and β be distinct real numbers. Then

$$(x-\alpha)(x-\beta) = 0 \tag{3.6.1}$$

is a quadratic equation in x whose roots are α and β.

To get a quadratic equation (in x) in standard form whose roots are α and β, we can simply expand the left-side of Equation (3.6.1) to get

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0 \tag{3.6.2}$$

Note that any non-zero multiple of Equation(3.6.2) is a quadratic equation with roots α and β. For example, $2x^2 - 2(\alpha+\beta)x + 2\alpha\beta = 0$ is such an equation. Usually, we would like to give a quadratic equation in "simple" form $ax^2 + bx + c = 0$ such that

- a, b and c are integers, if possible;
- if a, b and c are integers, they are *relatively prime*, that is, 1 and -1 are the only common factors of a, b and c.

Example 3.6.1 Find a quadratic equation in x whose roots are -3 and $\frac{5}{2}$. Express your answer in standard form such that the coefficients are relatively prime integers.

Explanation To get a quadratic equation with the given roots, we apply Theorem 3.6.1. To get a quadratic in standard form, we expand the left-side. To get integer coefficients, we multiply the equation by a suitable integer.

Solution A quadratic equation with roots -3 and $\frac{5}{2}$ is

$$\left(x-(-3)\right)\left(x-\frac{5}{2}\right)=0$$

That is,

$$\begin{aligned} (x+3)\left(x-\frac{5}{2}\right) &= 0 \\ x^2+\frac{5}{2}x-\frac{15}{2} &= 0 \quad \text{Expand left-side} \end{aligned}$$

Multiply the above equation by 2, we get a required equation:

$$2x^2+5x-15=0$$

□

Remark $-2x^2-5x+15=0$ is also a possible answer.

Example 3.6.2 Find a quadratic equation in x whose roots are $\frac{2+\sqrt{5}}{3}$ and $\frac{2-\sqrt{5}}{3}$. Express your answer in standard form such that the coefficients are relatively prime integers.

Solution A quadratic equation with roots $\frac{2+\sqrt{5}}{3}$ and $\frac{2-\sqrt{5}}{3}$ is

$$\left(x-\frac{2+\sqrt{5}}{3}\right)\left(x-\frac{2-\sqrt{5}}{3}\right)=0$$

That is,

$$\begin{aligned} \left(\left(x-\frac{2}{3}\right)-\frac{\sqrt{5}}{3}\right)\left(\left(x-\frac{2}{3}\right)+\frac{\sqrt{5}}{3}\right) &= 0 \\ \left(x-\frac{2}{3}\right)^2-\left(\frac{\sqrt{5}}{3}\right)^2 &= 0 \quad \text{By Identity for Difference of Squares} \\ x^2-\frac{4}{3}x+\frac{4}{9}-\frac{5}{9} &= 0 \\ x^2-\frac{4}{3}x-\frac{1}{9} &= 0 \end{aligned}$$

Multiply the above equation by 9, we get a required equation:

$$9x^2-12x-1=0$$

□

For the case where the root is a double root, we have the following result which can be obtained from Theorem 3.6.1 by putting $\beta = \alpha$.

Theorem 3.6.2 Let α be a real number. Then

$$(x-\alpha)^2 = 0$$

is a quadratic equation in x that has α as the only root.

Example 3.6.3 Find a quadratic equation in x that has a double root $\frac{2}{7}$. Express your answer in standard form such that the coefficients are relatively prime integers.

Explanation A quadratic equation has a double root means that it has only one real root.

Solution A quadratic equation with only one root $\frac{3}{7}$ is

$$\left(x-\frac{3}{7}\right)^2 = 0$$

Expanding, we get $x^2 - \frac{6}{7}x + \frac{9}{49} = 0$

Multiply the above equation by 49, we get a required equation:

$$49x^2 - 42x + 9 = 0$$

□

To find a quadratic equation whose roots are given, instead of using Equation (3.6.1), we may also use Equation (3.6.2).

Theorem 3.6.3 Let α and β be distinct real numbers. Denote $s = \alpha + \beta$ and $p = \alpha\beta$. Then

$$x^2 - sx + p = 0$$

is a quadratic equation whose roots are α and β.

Example 3.6.4 Find a quadratic equation in x whose roots are equal to three times the roots of the equation $x^2 - 2x - 3 = 0$ respectively. Express your answer in standard form such that the coefficients are relatively prime integers.

Explanation We can find such an equation by any one of the following two methods.

- Find the roots of the given equation and hence that of a required equation;
- Apply Theorem 3.6.3.

Solution 1 Solving

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ (x-3)(x+1) &= 0 \\ x = 3 \quad &\text{or} \quad x + 1 = 0 \\ x = 3 \quad &\text{or} \quad x = -1 \end{aligned}$$

The roots of $x^2 - 2x - 3 = 0$ are 3 and -1.

(soln 1 cont'd) We want to find a quadratic whose roots are 9 and −3.

The following is such an equation:

$$(x-9)(x-(-3))=0$$

$x^2-6x-27=0$ is a required equation. □

Solution 2 Denote the roots of the equation $x^2-2x-3=0$ by α and β. Then

$$\begin{aligned}\alpha+\beta &= -\frac{-2}{1} &\quad\text{and}\quad \alpha\beta &= \frac{-3}{1}\\ &= 2 & &= -3\end{aligned} \tag{3.6.3}$$

We want to find a quadratic equation whose roots are 3α and 3β.

The following is such an equation:

$$x^2-sx+p=0$$

where $s=3\alpha+3\beta$ and $p=3\alpha\cdot 3\beta$.

By (3.6.3),

$$\begin{aligned}s &= 3(\alpha+\beta) &\quad\text{and}\quad p &= 9\alpha\beta\\ &= 3\cdot 2 & &= 9\cdot(-3)\\ &= 6 & &= -27\end{aligned}$$

$x^2-6x-27=0$ is a required equation. □

Example 3.6.5 Suppose α and β are the roots of the quadratic equation $2x^2+6x+1=0$. Find a quadratic equation in x whose roots are $(2-\alpha)$ and $(2-\beta)$. Express your answer in standard form such that the coefficients are relatively prime integers.

Explanation Since the discriminant of the given quadratic equation is not a perfect square, it follows that α and β involve the radical sign. Below we apply Theorem 3.6.3.

Solution Since α and β are the roots of the quadratic equation $2x^2+6x-1=0$, it follows that

$$\begin{aligned}\alpha+\beta &= -\frac{6}{2} &\quad\text{and}\quad \alpha\beta &= \frac{-1}{2}\\ &= -3 & &\end{aligned} \tag{3.6.4}$$

We want to find a quadratic equation whose roots are $(2-\alpha)$ and $(2-\beta)$.

The following is such an equation:

$$x^2-sx+p=0 \tag{3.6.5}$$

where $s=(2-\alpha)+(2-\beta)$ and $p=(2-\alpha)\cdot(2-\beta)$.

By (3.6.4),

$$s = 4 - \alpha - \beta = 4 - (\alpha + \beta) = 4 - (-3) = 7$$

and

$$p = 4 - 2\alpha - 2\beta + \alpha\beta = 4 - 2(\alpha + \beta) + \alpha\beta = 4 - 2 \cdot (-3) + \left(\frac{-1}{2}\right) = -\frac{19}{2}$$

Equation (3.6.5) is $x^2 - 7x - \frac{19}{2} = 0$.

$2x^2 - 14x - 19 = 0$ is a required equation. □

Example 3.6.6 Suppose α and β are the roots of the quadratic equation $x^2 = 5x - 2$. Find a quadratic equation in x whose roots are $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$ and $(\alpha^2 + \beta^2)$. Express your answer in standard form such that the coefficients are relatively prime integers.

Explanation Using Theorem 3.5.1, we can find the values of $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$ and $(\alpha^2 + \beta^2)$. To find a required equation, we may apply Theorem 3.6.1 or Theorem 3.6.3.

Solution Rewrite the given equation in standard form: $x^2 - 5x + 2 = 0$

Since α and β are the roots of the equation, it follows that

$$\alpha + \beta = -\frac{-5}{1} = 5 \quad \text{and} \quad \alpha\beta = \frac{2}{1} = 2$$

Hence

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{5}{2}$$

and

$$\alpha^2 + \beta^2 = (\alpha^2 + 2\alpha\beta + \beta^2) - 2\alpha\beta = (\alpha + \beta)^2 - 2\alpha\beta = 5^2 - 2 \cdot 2 = 45$$

A quadratic equation with roots $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$ and $(\alpha^2 + \beta^2)$ is

$$x^2 - sx + p = 0 \tag{3.6.6}$$

where

$$s = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + (\alpha^2 + \beta^2) = \frac{5}{2} + 45 = \frac{95}{2}$$

and

$$p = \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) \cdot (\alpha^2 + \beta^2) = \frac{5}{2} \cdot 45 = \frac{225}{2}$$

Equation (3.6.6) is $x^2 - \frac{95}{2}x + \frac{225}{2} = 0$.

$x^2 - 95x + 225 = 0$ is a required equation. □

Exercise 3.6

1. For each of the following, find a quadratic equation in x whose roots are the given real numbers. Express your answer in standard form such that the coefficients are relatively prime integers.

   (*a*) $2,\ 5$ (*b*) $-3,\ 7$ (*c*) $0,\ 6$

   (*d*) $-1,\ \frac{8}{3}$ (*e*) $-3, 3$ (*f*) $-\frac{2}{3}, \frac{3}{2}$

   (*g*) $\sqrt{3}+1, \sqrt{3}-1$ (*h*) $\frac{3+\sqrt{5}}{4}, \frac{3-\sqrt{5}}{4}$

2. For each of the following, find a quadratic equation in x that has the given real number as the only root. Express your answer in standard form such that the coefficients are relatively prime integers.

   (*a*) -4 (*b*) 0 (*c*) $\frac{7}{3}$

3. For each of the following, find a quadratic equation in x that has the given real number as its double root. Express your answer in standard form.

   (*a*) $\sqrt{2}$ (*b*) $\frac{1-\sqrt{7}}{3}$

4. (*a*) Suppose that α and β are the roots of the quadratic equation $x^2 - 4x + 3 = 0$ and that $\alpha > \beta$. Find the values of α and β.

   (*b*) Find a quadratic equation in x whose roots are $(\alpha^2 + \beta^2)$ and $(\alpha^2 - \beta^2)$. Express your answer in standard form such that the coefficients are integers.

5. Suppose that α and β are the roots of the quadratic equation $x^2 - 4x - 5 = 0$. For each of the following, find a quadratic equation in x whose roots are the given values; express your answer in standard form such that the coefficients are relatively prime integers.

   (*a*) $2\alpha,\ 2\beta$ (*b*) $\frac{1}{\alpha}, \frac{1}{\beta}$ (*c*) $\alpha^2,\ \beta^2$

   (*d*) $\alpha-3,\ \beta-3$ (*e*) $\alpha-\beta,\ \beta-\alpha$ (*f*) $\frac{1}{\alpha+1}, \frac{1}{\beta+1}$

6. Find a quadratic equation in x whose roots are less than three times the roots of $3x^2 + 4x - 1 = 0$ by 1 respectively. Express your answer in standard form such that the coefficients are relatively prime integers.