Contents

Chapter 2

Linear Equations and Lines

In this chapter, we will not consider complex numbers. Thus, a number always means a real number.

- The study of equations belongs to a branch of mathematics called *algebra*.
- The study of geometric objects (for example, lines and circles) belongs to a branch of mathematics called *geometry*.

By using coordinate systems, we can study equations using geometric objects and vice versa.

(1) In Section 2.1, we give a review on *rectangular coordinate plane*.

(2) In Section 2.2, we give a review on and *slopes of non-vertical lines* in the rectangular coordinate plane.

(3) In Section 2.3 we introduce the concept of the *graph* of a linear equation with two unknowns and we show that the graph is always a straight line.

(4) In Section 2.4, we consider different ways to represent lines on the rectangular coordinate plane.

(5) In Section 2.5, intersection of lines on the rectangular coordinate plane.

First, we give a review on solving equations with one unknown.

2.0 Equations with One Unknown

In junior forms, students have seen that in solving a practical problem, we may introduce an unknown and obtain an equality involving the unknown. For example, students have encountered something like $x + 3 = 9 - 2x$, which is called a linear equation with one unknown.

Terminology 2.0.1 By an ***equation with one unknown***, we mean an equality involving an unknown real number and some known or given real numbers.

Each of the following is an equation with one unknown.

(1) $x + 3 = 9 - 2x$, where x is an unknown.

(2) $w^2 = \pi w$, where w is an unknown.

(3) $x^2 = ax$, where a is a constant (that is, a given real number) and x is an unknown.

Sometimes, it may be helpful if we state the symbol for the unknown. For example, we may say that Equation (3) is *an equation with one unknown* x; in this case, it is understood that a is a constant.

Terminology 2.0.2 By a ***solution*** to an equation with one known x, we mean a real number such that when we substitute x by that real number, the equality is satisfied.

To check whether a real number is a solution to an equation with one known, we can use direct substitution to see whether the numbers obtained on the left-side and right-side of the equality sign are equal or not. For example, in Equation (1),

- if we substitute $x = 5$, we get

$$\text{L.S.} = 5 + 3 = 8, \qquad \text{R.S.} = 9 - 2 \cdot 5 = -1;$$

  thus 5 is not a solution to Equation (1);

- if we substitute $x = 2$, we get

$$\text{L.S.} = 2 + 3 = 5, \qquad \text{R.S.} = 9 - 2 \cdot 2 = 5;$$

  thus 2 is a solution to Equation (1).

Terminology 2.0.3 To ***solve*** an equation with one unknown means to find all solutions to the equation.

Given an equation with one unknown, it is impossible to solve the equation by direct substitution; this is because there are infinitely many real numbers. Instead, to solve the equation, we use other equations that are "simpler" and *equivalent to* to original equation.

Terminology 2.0.4 By saying that an equation with one known ***is equivalent to*** another equation with one unknown, we mean that the two equations have the same solutions, that is, every solution to the first equation is a solution to the second equation and vice versa.

Equation (1) can be solved in the following way:

$$
\begin{array}{lrcl}
(1) & x+3 & = & 9-2x \\
(1a) & x+2x & = & 9-3 \\
(1b) & 3x & = & 6 \\
(1c) & x & = & 2
\end{array}
$$

We start with the original Equation (1) and use properties of real numbers to obtain Equation ($1a$) which is "simpler" and is equivalent to Equation (1). Repeating this process, we get Equation ($1b$) which is "simpler" and is equivalent to Equation ($1a$) and then we get Equation ($1c$) which is in the form "x = a number" and is equivalent to Equation ($1b$). Note that Equation ($1c$) is equivalent to the original equation and so 2 is the (only) solution to Equation (1).

In solving Equation (1), some students may skip Equation ($1a$) and/or Equation ($1b$). It is alright to do that if you feel comfortable. In fact, if you use computers to solve the equation, you will get the answer without the intermediate steps. Although we can use computers to perform arithmetic operations, to solve equations, to plot graphs, to perform set operations etc., you are advised to do "simple drills". The aim of doing exercises is *not* to train students get correct answers quickly *(you can't compete with computers)*. When you do exercises, you should note the *logic* behind the calculations.

Several Uses of Equality Sign The equality sign '=' can be used in different ways as the following examples illustrate.

(I) $1 + 2 = 3$.

(II) $x^2 + 1 = 5$.

(III) Denote $a = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$.

(IV) In the equation given in (II), substitute $x = 2$ in the left-side of the equality sign.

The equality sign in (I) has the usual meaning: the numbers (or objects) on the left-side and right-side of the equality sign are equal.

The equality sign in (II) is an equality in an equation. It is true when $x = 2$ (for example) and it is not true when $x = 1$ (for example). Instead of using the equality sign, some authors use '=='. Using such a notation, the equation in (II) can be written as

(IIa) $x^2 + 1 == 5$.

The use of the equality sign in (III) has been discussed in Chapter **??**. The sentence in (III) means that the number obtained by finding the sum of the first ten positive integers is denoted by a. The symbol '=' assigns a name to a number. The name is written on the left side and the number on the right side. Instead of using the equality sign, some authors use the symbol ':='. Using such a notation, the sentence in (III) can be written as

(IIIa) Denote $a := 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$.

The equality sign in (IV) has a different meaning. The phrase "*substitute* $x = 2$" means "replace x by 2". Instead of the equality sign, some authors prefer writing in words:

(IVa) In the equation given in (II), substitute x by 2 in the left-side of the equality sign.

However, in this course, we will not use the symbols ':=', and '=='. Readers can determine the meaning of '=' from the context.

2.1 Rectangular Coordinate Plane

In Chapter **??** Section **??**, we introduce the *real number line* which describes a one-to-one correspondence between real numbers and points on a line. In this section, we will consider the *rectangular coordinate plane* which describes a one-to-one correspondence between ordered pairs of real numbers and points on a plane. Below we give the details.

- First we take a horizontal line and a vertical line on the plane. Their point of intersection is called the *origin*. We label points on the horizontal line by real numbers, with 0 at the origin and positive real numbers on the right of the origin. Similarly, with the same scale as that for the horizontal line, we label points on the vertical line by real numbers, with 0 at the origin and positive real numbers above the origin. The horizontal and vertical lines, with points labeled by real numbers, are called the *horizontal axis* and *vertical axis* respectively. In Figure 2.1.1 the arrows on the horizontal and vertical axes indicate the directions in which the numbers increase.

  Remark Since the directions are understood, the arrows can be omitted.

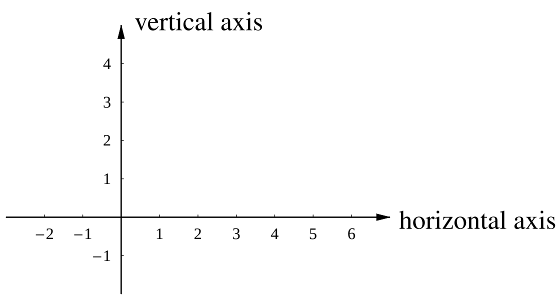


Figure 2.1.1

- For each point P on the plane, we can label it by an *ordered pair* of real numbers, that is, a pair of real numbers enclosed by parenthesis. To this end, we draw perpendiculars from P to the horizontal and vertical axes. The first perpendicular meets the horizontal axis at a point that can be represented by a real number, denoted by a. Similarly, the second perpendicular meets the vertical axis at a point that can be represented by a real number, denoted by b.

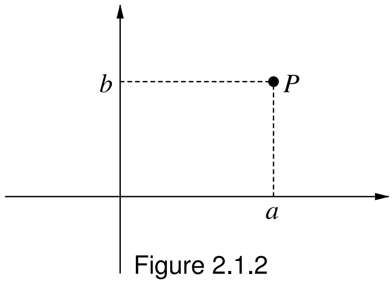


Figure 2.1.2

We identify the point P as the ordered pair (a, b) and we write $P = (a, b)$ or $P(a, b)$. The real numbers a and b are called the *first coordinate* and *second coordinate* of P respectively. The plane described in this way is called the *rectangular coordinate plane* or the *Cartesian plane*.

Remark Usually, an arbitrary point P on the rectangular coordinate plane is denoted by (x, y). For this reason, the first coordinate and second coordinate of P are usually called the *x-coordinate* and *y-coordinate* of P respectively. For the same reason, the horizontal axis and vertical axis are called the x-axis and y-axis respectively.

Sometimes, for convenience, we use different scales for the horizontal and vertical axes. For example, in order to display the point $P(1, 20)$ using a figure whose height is not too large, different scales for the axes are used (see Figure 2.1.3 (a)). However, in this case, geometric objects are distorted. In Figure 2.1.3 (b), the square with vertices at $(0, 0)$, $(3, 0)$, $(3, 3)$ and $(0, 3)$ looks like a rectangle with base much larger than height.

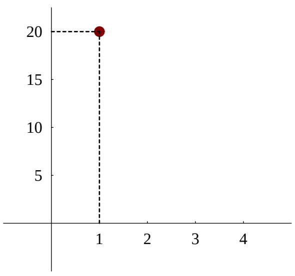


Figure 2.1.3 (a)

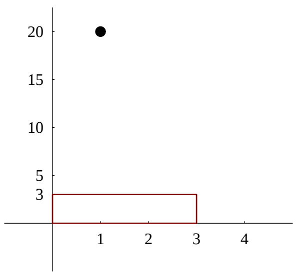


Figure 2.1.3 (b)

Notation 2.1.1 We denote $\mathbb{R}^2$ to be the set of all ordered pairs (x, y) where x and y are real numbers.

Geometrically, the set $\mathbb{R}^2$ is just the rectangular coordinate plane. Suppose $P = (a, b)$ and $Q = (c, d)$ are points on the rectangular coordinate plane. We write $(a, b) = (c, d)$ to mean that P and Q are the same point and we write $(a, b) \neq (c, d)$ to mean that P and Q are different points.

Equality of Ordered Pairs Let a, b, c, d be real numbers. We write

- $(a, b) = (c, d)$ to mean $a = c$ and $b = d$;
- $(a, b) \neq (c, d)$ to mean $a \neq c$ or $b \neq d$.

Explanation $(a, b) \neq (c, d)$ means "it is not true that $a = c$ and $b = d$", that is, "at least one of the two equalities is not true".

With the rectangular coordinate plane identified as $\mathbb{R}^2$, points on the plane are elements of $\mathbb{R}^2$ and geometrical objects (for example, lines, circles, circular disks, rectangles, rectangular regions) on the plane are subsets of $\mathbb{R}^2$. The following phrases about points have the same meanings:

(1a) a point (lying) on (or in) the rectangular coordinate plane

(1b) a point belonging to the rectangular coordinate plane

(1c) an ordered pair of real numbers

(1d) an element of $\mathbb{R}^2$

A (straight) line on (or in) the rectangular coordinate plane is a subset of $\mathbb{R}^2$ satisfying some conditions *(the conditions are very complicated and thus will not be discussed in this course)*.

Figure 2.1.4 (a) shows (part of) a line on the rectangular coordinate plane.

Figure 2.1.4 (a)

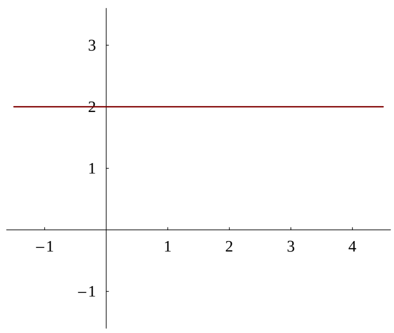


Figure 2.1.4 (b)

It is impossible to shows the whole line; we use two arrows to indicate that the line extends indefinitely in the directions of the arrows. To display geometric objects, we choose suitable ranges of values of x and y. In Figure 2.1.4 (a) and Figure 2.1.4 (b), we choose $-1.5 \le x \le 4.5$ and $-1.5 \le y \le 3.5$. In Figure 2.1.4 (b), the two arrows for the line are omitted. The figure shows the part of the line in the *window* determined by the chosen ranges of values of x and y.

The line shown in In Figure 2.1.4 (a) or (b) is the horizontal line 2 units above the x-axis. As a subset of $\mathbb{R}^2$, the line is the following set:

$$\{(x, y) \in \mathbb{R}^2 : y = 2\}$$

that is, the set of all points on the rectangular coordinate plane whose y coordinates are equal to 2.

Given two distinct points A and B, there is one and only one line that passes through both A and B. That line is called *the line determined by A and B* and is denoted by $\overleftrightarrow{AB}$ or $\text{Line } AB$.

The following phrases (or notation) about a point P and a line ℓ on the rectangular coordinate plane have the same meanings:

(2a) ℓ passes through P

(2b) P lies on ℓ

(2c) P belongs to ℓ

(2d) $P \in \ell$

Let A and B be two distinct points. Denote ℓ to be the line determined by A and B. The line segment with endpoints A and B is the subset of ℓ consisting of all the points that *lie between* A and B (including the two endpoints). *Readers can use their intuition to interpret the meaning of "lie between"*. We write $\overline{AB}$ or $\text{Segment } AB$ to denote the line segment with endpoints A and B.

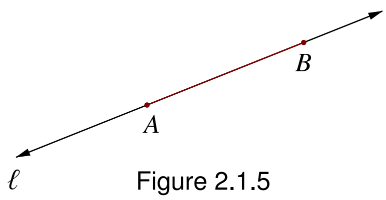


Figure 2.1.5

Terminologies and Notations

- Let A, B and C be three distinct points. We say that A, B and C are *collinear* to mean that there exists a line that passes through the points A, B and C, or equivalently, that the line determined by any two of the three points passes through the third point.

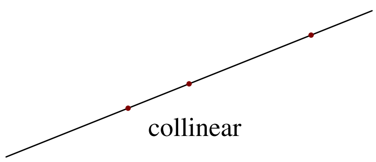


Figure 2.1.6 (a)

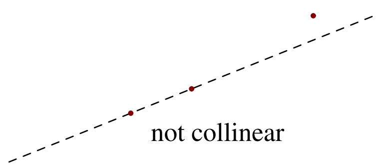


Figure 2.1.6 (b)

- Let ℓ_1 and ℓ_2 be two distinct lines. We say that ℓ_1 and ℓ_2 are *parallel*, and write $\ell_1 \parallel \ell_2$, to mean that ℓ_1 and ℓ_2 are coplanar and the intersection of ℓ_1 and ℓ_2 is empty.

  Explain Two lines are *coplanar* means that they lie on the same plane.

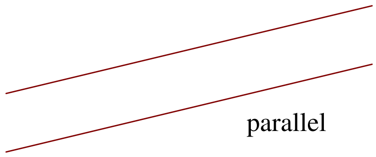


Figure 2.1.7 (a)

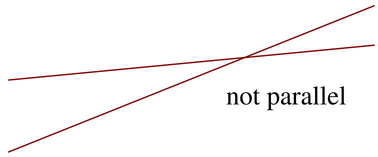


Figure 2.1.7 (b)

  Remark In this chapter, from Section 2.2 through Section 2.5, we will only consider lines on the rectangular coordinate plane. Thus, in this case, all lines are coplanar and so $\ell_1 \parallel \ell_2$ means $\ell_1 \cap \ell_2 = \{\,\}$.

- Let ℓ_1 and ℓ_2 be two distinct lines. We say that ℓ_1 and ℓ_2 are *perpendicular*, and write $\ell_1 \perp \ell_2$, to mean that adjacent angles determined by ℓ_1 and ℓ_2 are congruent (both of them are right angles).

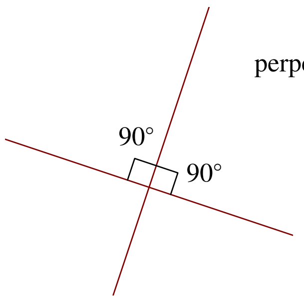


Figure 2.1.8 (a)

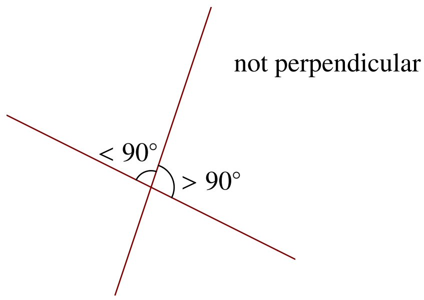


Figure 2.1.8 (b)

Exercise 2.1

1. For each of the points A, B, C, D and E shown in Figure 2.1.9, write down its x-coordinate and y-coordinate.

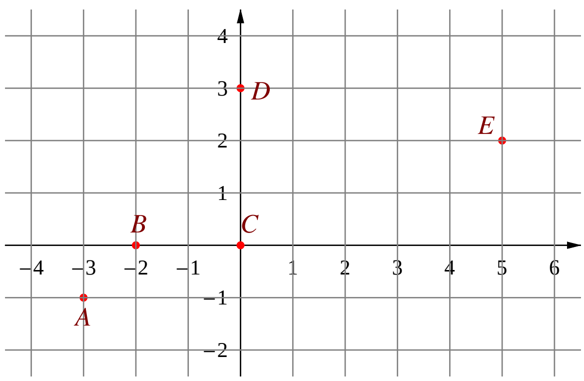


Figure 2.1.9

2. Denote $P = (-3, 3)$, $Q = (4, 0)$, $R = (0, 2)$ and $S = (5, 4)$.
   - (a) In (the window of) the rectangular coordinate plane shown in Figure 2.1.10, locate the points P, Q, R and S and sketch Segment PQ and Segment RS.
   - (b) Are Line PQ and Line RS parallel?

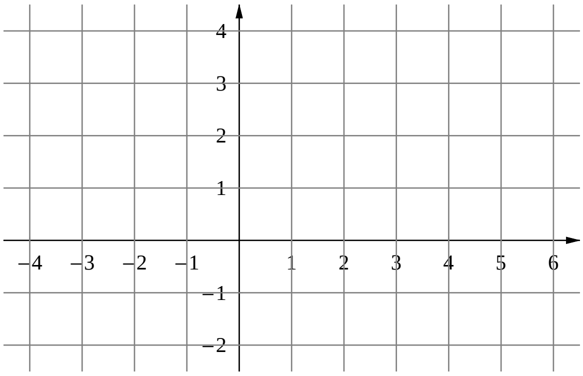


Figure 2.1.10

2.2 Slopes of Lines

In this section, we give a review of the concept of slopes of lines. First, we consider slopes of line segments.

Definition 2.2.1 Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two distinct points on the rectangular coordinate plane such that Segment PQ is not vertical. We call ***the slope of Segment*** PQ, and write m_{PQ}, to mean the real number $\frac{y_2 - y_1}{x_2 - x_1}$.

Explanation Segment PQ is not vertical means that $x_1 \neq x_2$. Thus in the expression $\frac{y_2 - y_1}{x_2 - x_1}$, the denominator is not equal to 0.
In finding m_{PQ}, the role of P and Q can be interchanged. This is because

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}$$

Remark If Segment PQ is vertical, that is, if $x_1 = x_2$, then the expression $\frac{y_2 - y_1}{x_2 - x_1}$ is meaningless. We say that the slope Segment PQ is *undefined*.

A non-vertical line segment can be sloping up, horizontal or sloping down (see Figures 2.2.1 (a), (b) and (c)). These three cases correspond to that the slope of the line segment is positive, zero or negative.

(1) $m_{PQ} > 0$ if and only if $\overline{PQ}$ is sloping up;

(2) $m_{PQ} = 0$ if and only if $\overline{PQ}$ is horizontal;

(3) $m_{PQ} < 0$ if and only if $\overline{PQ}$ is sloping down.

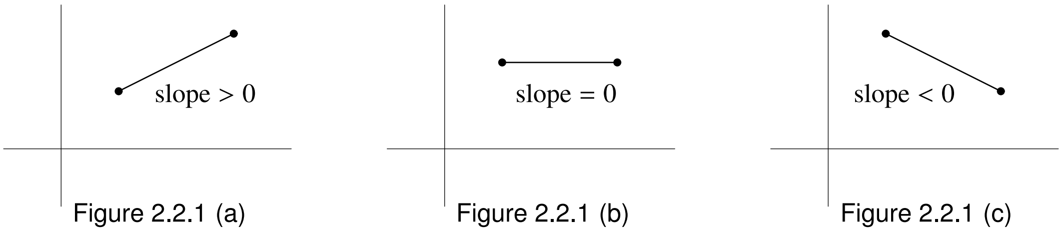
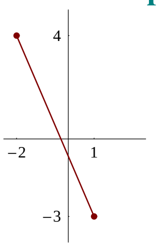


Figure 2.2.1 (a) Figure 2.2.1 (b) Figure 2.2.1 (c)

Example 2.2.1 Denote $P = (1, -3)$ and $Q = (-2, 4)$. Determine whether the line segment with endpoints P and Q is sloping up, horizontal or sloping down.



Solution By definition, $m_{PQ} = \dfrac{4-(-3)}{-2-1} = \dfrac{7}{-3}$

Since $m_{PQ} < 0$, it follows that Segment PQ is sloping down. □

Next, we consider slopes of non-vertical lines. Figure 2.2.2 shows a line (denoted by ℓ) passing through the points $A(1, 1)$, $B(3, 2)$, $C(-3, -1)$ and $D(-7, -3)$.

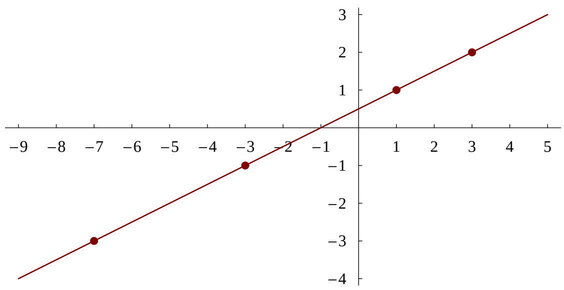


Figure 2.2.2

Note that the slopes of Segment AB and Segment CD are equal.

$$m_{AB} = \frac{2-1}{3-1} = \frac{1}{2}, \qquad m_{CD} = \frac{-3-(-1)}{-7-(-3)} = \frac{-2}{-4} = \frac{1}{2}$$

In fact, if we take any two points P and Q belonging to ℓ and calculate the slope of Segment PQ, the value m_{PQ} is always equal to $\frac{1}{2}$. This is a special case of the following general result.

Theorem 2.2.1 Let ℓ be a non-vertical line on the rectangular coordinate plane. Let P and Q be two distinct points belonging to ℓ and let R and S be two distinct points belonging to ℓ. Then the slope of Segment PQ and the slope of Segment RS are equal, that is, $m_{PQ} = m_{RS}$.

Explanation To prove the result, we can consider three cases: (i) the line is sloping up; (ii) the line is horizontal; (iii) the line is sloping down. For case (ii), the result is obvious. For cases (i) and (iii), the result can be proved using the Corresponding Angles Postulate (1) and similar triangles.

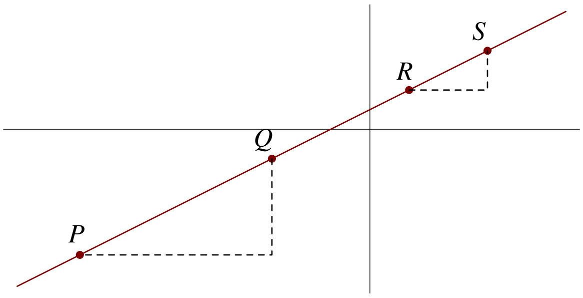


Figure 2.2.3

Corresponding Angles Postulate

(1) If a transversal intersects two parallel lines, then corresponding angles are congruent.

(2) If two lines and a transversal form corresponding angles that are congruent, then the two lines are parallel.

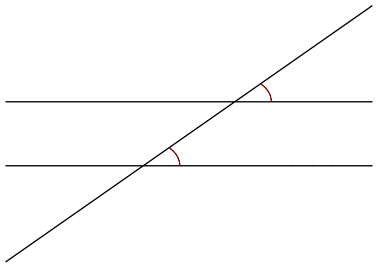


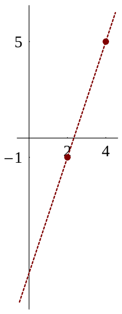
Figure 2.2.4

Definition 2.2.2 Let ℓ be a non-vertical line on the rectangular coordinate plane. We call ***the slope of*** ℓ, and write m_ℓ, to mean the real number m_{PQ}, where P and Q are two distinct points belonging to ℓ.

Explanation By Theorem 2.2.1, the value m_{PQ} is independent of the choices of P and Q. We say that the slope of ℓ is *well-defined*.

Remark The slope of a vertical line is undefined.

Example 2.2.2 Denote ℓ to be the line passing through the points $P(2,-1)$ and $Q(4,5)$. Find the slope of ℓ.



Solution By definition,

$$m_\ell = m_{PQ} = \frac{5-(-1)}{4-2} = \frac{6}{2} = 3$$

□

We can use slopes to determine whether two lines are parallel (perpendicular) or not.

Slope Test for Parallel Lines Let ℓ_1 and ℓ_2 be two distinct non-vertical lines on the rectangular coordinate plane. Then $\ell_1 \parallel \ell_2$ if and only if $m_{\ell_1} = m_{\ell_2}$.

Explanation The conclusion can be divided into two parts:

(1) If $\ell_1 \parallel \ell_2$, then $m_{\ell_1} = m_{\ell_2}$.

(2) If $m_{\ell_1} = m_{\ell_2}$, then $\ell_1 \parallel \ell_2$.

To prove (1) and (2), we can use similar triangles and apply the Corresponding Angles Postulate (1) and (2) respectively.

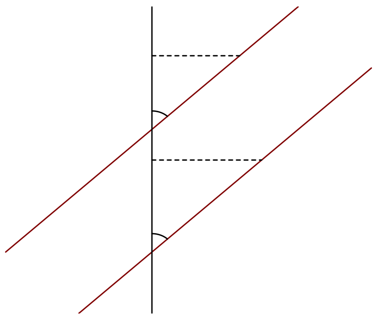


Figure 2.2.5

Remark If ℓ_1 and ℓ_2 are two distinct vertical lines, then ℓ_1 and ℓ_2 are parallel.
If one of the lines ℓ_1 and ℓ_2 is vertical and the other is not vertical, then ℓ_1 and ℓ_2 are not parallel.
If ℓ_1 and ℓ_2 are the same line, then $m_{\ell_1} = m_{\ell_2}$ (but we don't say that ℓ_1 and ℓ_2 are parallel, we say that ℓ_1 and ℓ_2 coincide).

Slope Test for Perpendicular Lines Let ℓ_1 and ℓ_2 be non-vertical lines on the rectangular coordinate plane. Then $\ell_1 \perp \ell_2$ if and only if $m_{\ell_1} \cdot m_{\ell_2} = -1$.

Explanation The conclusion can be divided into two parts:

(1) If $\ell_1 \perp \ell_2$, then $m_{\ell_1} \cdot m_{\ell_2} = -1$.

(2) If $m_{\ell_1} \cdot m_{\ell_2} = -1$, then $\ell_1 \perp \ell_2$.

To prove (1), we construct two right-angled triangles as shown in Figure 2.2.6. By the assumption that $\ell_1 \perp \ell_2$, we have $\gamma = 90°$ from which we get $\alpha = \beta$ (hence the two right-angled triangles are similar).

To prove (2), we first show that the intersection of ℓ_1 and ℓ_2 is non-empty (hence we can construct two right-angled triangles as shown in Figure 2.2.6). Then using the assumption $m_{\ell_1} \cdot m_{\ell_2} = -1$, we can show that the two right-angled triangles are similar (hence $\alpha = \beta$ and so $\gamma = 90°$).

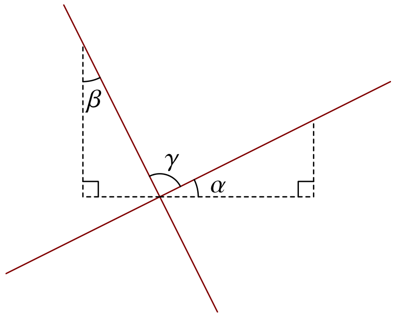


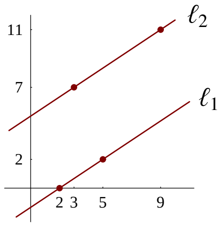
Figure 2.2.6

Remark If one of the lines ℓ_1 and ℓ_2 is vertical and the other is horizontal, then ℓ_1 and ℓ_2 are perpendicular.

If one of the lines ℓ_1 and ℓ_2 is vertical and the other is not horizontal, then ℓ_1 and ℓ_2 are not perpendicular.

Example 2.2.3 Denote $A = (2, 0)$, $B = (5, 2)$, $C = (3, 7)$ and $D = (9, 11)$. Denote ℓ_1 to be the line passing through A and B and denote ℓ_2 to be the line passing through C and D. Show that ℓ_1 and ℓ_2 are parallel.

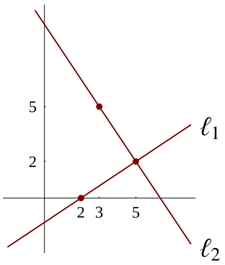
Solution By the definition of slope,



$$m_{\ell_1} = m_{AB} = \frac{2-0}{5-2} = \frac{2}{3} \quad \text{and} \quad m_{\ell_2} = m_{CD} = \frac{11-7}{9-3} = \frac{4}{6} = \frac{2}{3}$$

Since $m_{\ell_1} = m_{\ell_2}$, it follows that ℓ_1 and ℓ_2 are parallel. □

Example 2.2.4 Denote $A = (2, 0)$, $B = (5, 2)$ and $C = (3, 5)$. Denote ℓ_1 to be the line passing through A and B and denote ℓ_2 to be the line passing through B and C. Show that ℓ_1 and ℓ_2 are perpendicular.



Solution By the definition of slope,

$$m_{\ell_1} = m_{AB} = \frac{2-0}{5-2} = \frac{2}{3} \qquad m_{\ell_2} = m_{BC} = \frac{5-2}{3-5} = \frac{3}{-2}$$

Since $m_{\ell_1} \cdot m_{\ell_2} = \frac{2}{3} \cdot \frac{3}{-2} = -1$, it follows that ℓ_1 and ℓ_2 are perpendicular. □

We can also use slopes to determine whether three points are collinear or not.

Slope Test for Collinear Points Let P, Q and R be three distinct points on the rectangular coordinate plane such that Segment PQ and Segment QR are not vertical. Then the points P, Q and R are collinear if and only if $m_{PQ} = m_{QR}$.

Explanation The conclusion can be divided into two parts:

(1) If the points P, Q and R are collinear, then $m_{PQ} = m_{QR}$.

(2) If $m_{PQ} = m_{QR}$, then the points P, Q and R are collinear.

To prove (1), we can apply Theorem 2.2.1. To prove (2), we can use the Corresponding Angles Postulate and similar triangles to show that R is the point of the intersection of Line PQ and the vertical line passing through R.

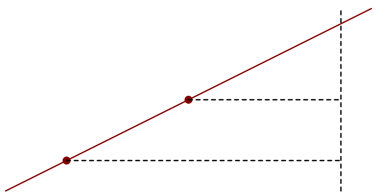
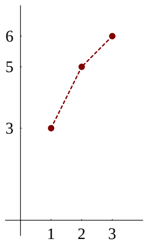


Figure 2.2.7

Remark If PQ and QR are vertical, then the points P, Q and R are collinear.

Example 2.2.5 Denote $P = (1, 3)$, $Q = (2, 5)$ and $R = (3, 6)$. Determine whether the points P, Q and R are collinear or not.



Solution The slopes of Segment PQ and Segment QR are

$$m_{PQ} = \frac{5-3}{2-1} = 2, \qquad m_{QR} = \frac{6-5}{3-2} = 1$$

Since $m_{PQ} \neq m_{QR}$, it follows that the points P, Q and R are not collinear. □

Exercise 2.2

1. For each of the following, determine whether the line segment with endpoints P and Q is vertical or not; and if the line segment is not vertical, find its slope.
   - (a) $P = (-5, 7)$, $Q = (3, 2)$
   - (b) $P = (3, 7)$, $Q = (-2, 7)$
   - (c) $P = (2, 3)$, $Q = (2, 6)$
2. For each of the following, determine whether the points P, Q and R are collinear or not.
   - (a) $P = (1, 5)$, $Q = (2, 3)$, $R = (4, -1)$
   - (b) $P = (-1, -2)$, $Q = (1, 3)$, $R = (3, 9)$
   - (c) $P = (-2, 3)$, $Q = (-2, 0)$, $R = (-2, 5)$
3. Suppose a is a real number and the points $(3, 0)$, $(a, 1-a)$ and $(2, 3)$ are collinear. Find the value of a.
4. For each of the following, determine whether the line passing through P and Q is vertical or not; and if the line is not vertical, find its slope and determine whether it is sloping up, horizontal or sloping down.
   - (a) $P = (-3, 4)$, $Q = (3, 4)$
   - (b) $P = (1, 4)$, $Q = (1, 2)$
   - (c) $P = (3, 8)$, $Q = (11, 4)$
5. For each of the following, determine whether Line AB and Line CD are parallel or not.
   - (a) $A = (1, 2)$, $B = (2, 4)$, $C = (3, 7)$, $D = (5, 11)$
   - (b) $A = (1, 3)$, $B = (2, -4)$, $C = (2, -1)$, $D = (3, 2)$
   - (c) $A = (1, 3)$, $B = (1, 4)$, $C = (2, 3)$, $D = (3, 4)$
   - (d) $A = (2, 3)$, $B = (2, 0)$, $C = (-1, 1)$, $D = (-1, 2)$
6. For each of the following, determine whether Line AB and Line CD are perpendicular or not.
   - (a) $A = (3, -2)$, $B = (5, 1)$, $C = (0, 5)$, $D = (2, 8)$
   - (b) $A = (1, 0)$, $B = (0, -2)$, $C = (2, 3)$, $D = (6, 1)$
   - (c) $A = (1, 1)$, $B = (-3, 1)$, $C = (1, 2)$, $D = (3, 7)$
   - (d) $A = (0, 1)$, $B = (2, 1)$, $C = (-1, 0)$, $D = (-1, 2)$

2.3 Graphs of Linear Equations with Two Unknowns

In junior forms, readers have seen that in solving a practical problem, we may introduce two unknowns and obtain equation(s) involving the unknowns. For example, students have encountered something like $2x + 3y = 4$, which is called a *linear equation with two unknowns*. In this section, we will consider graphs of such equations.

Terminology 2.3.1 By ***an equation with two unknowns***, we mean an equality involving two unknown real numbers and some known or given real numbers.

Each of the following is an equation with two unknowns.

(1) $3y = 3 - 4x$

(2) $m^2 + n^2 = 1$

In Equation (1), the unknowns are x and y and in Equation (2), the unknowns are m and n. In an equation with two unknowns, the order of the unknown is important; this is because a solution of such an equation is an ordered pair of real numbers. Thus, whenever possible, we will use x and y to denote the unknowns with the convention that x is the first unknown.

Terminology 2.3.2 By ***a solution to an equation with two unknowns x and y***, we mean an ordered pair of real numbers, denoted by (x_0, y_0), such that when we substitute $(x, y) = (x_0, y_0)$, the equality is satisfied.

Explanation Substitute $(x, y) = (x_0, y_0)$ means substitute $x = x_0$ and $y = y_0$.

Example 2.3.1 Consider the following equation with two unknowns

$$x^2 + 2y = 6 \tag{2.3.1}$$

For each of the following ordered pair of real numbers, determine whether it is a solution to Equation (2.3.1) or not.

(*a*) (2, 1) (*b*) (3, −1)

Solution (*a*) Putting $(x, y) = (2, 1)$ into Equation (2.3.1), we get

$$\text{L.S.} = 2^2 + 2 \cdot 1 = 6 = \text{R.S.}$$

Thus (2, 1) is a solution to the equation.

(*b*) Putting $(x, y) = (3, -1)$ into Equation (2.3.1), we get

$$\text{L.S.} = 3^2 + 2 \cdot (-1) = 7 \neq \text{R.S.}$$

Thus $(3, -1)$ is not a solution to the equation. □

It should be pointed out that the equation given in Example 2.3.1 has infinitely many solutions. This is because if we substitute x by any given real number, we can always find the corresponding value of y such that the equality is satisfied. It should also be pointed out that an equation with two unknowns may have no solution. For example, the equation $x^2 + y^2 + 1 = 0$ does not have any solution. This is because no matter how we substitute (x, y) by an ordered pair of real numbers (x_0, y_0), the number obtained on left-side is at least 1 (and hence cannot be equal to 0).

Terminology 2.3.3 By saying that an equation with two unknowns ***is equivalent to*** another equation with two unknowns, we mean that the two equations have the same solutions, that is, every solution to the first equation is a solution to the second equation and vice versa.

Example 2.3.2 Consider the following equations with two unknowns.

(1) $3x^2 - 7 = 11 - 6y$

(2) $3x^2 + 6y = 18$

(3) $x^2 + 2y = 6$

From properties of real numbers, we see that Equation (1) is equivalent to Equation (2) which in turn is equivalent to Equation (3).

Remark To consider solutions to any one of the above equations, we may use the last one, for example. Thus by Example 2.3.1, the ordered pair $(2, 1)$ is a solution to all the above equations and the ordered pair $(3, -1)$ is not a solution to any one of the above equations.

Terminology 2.3.4 An equation that can be written in the form

$$ax + by + c = 0 \tag{2.3.2}$$

where a, b, c are real numbers with a, b not both 0, is called a ***linear equation with two unknowns***.

Explanation '*a, b not both* 0' means '*not (a = 0 and b = 0)*', that is, "*at least one of the numbers a and b is not equal to* 0".

Remark In the expression $ax + by + c$, there are three terms, namely, ax, by and c which are called the *x-term*, the *y-term* and the *constant term*. The numbers a, b and c are called the *coefficients* of the x-term, y-term and constant term respectively.

When a linear equation with two unknowns is written in the form given in (2.3.2), the equation is said to be in *standard form* or *general linear form*. For example,

(1) $2x + 3y + 4 = 0$

is a linear equation with two unknowns in standard form. The following linear equations with two unknowns are also considered to be in standard form

(2) $2x - 5y + 6 = 0$

(3) $x - y = 0$

This is because the values of a, b and c can be recognized immediately. The following table gives the values of a, b and c for the linear equations with two unknowns in (1), (2) and (3).

Linear Equation with 2 unknowns	a	b	c
$2x + 3y + 4 = 0$	2	3	4
$2x - 5y + 6 = 0$	2	−5	6
$x - y = 0$	1	−1	0

The equation given in the next example is also a linear equation with two unknowns. This is because it can be written in the form $ax + by + c = 0$.

Example 2.3.3 Rewrite the equation $x + 2y = 2(y - x + 1)$ in the form $ax + by + c = 0$ and write down the values of a, b and c.

Explanation The question asks for a linear equation with two unknowns in standard form that is *equivalent* to the given equation.

Solution

$$\begin{aligned} x + 2y &= 2(y - x + 1) \\ x + 2y &= 2y - 2x + 2 \\ x + 2y - 2y + 2x - 2 &= 0 \\ 3x - 2 &= 0 \end{aligned}$$

$a = 3,\ b = 0$ and $c = -2$. □

Remark Any non-zero multiple of the equation $3x - 2 = 0$ is a linear equation with two unknowns that is equivalent to the given equations.

In the above solution, the equation $3x - 2 = 0$ is considered to be an equation with two unknowns. However, in different situations, it can be considered to be an equation with one unknown or three unknowns etc. The meaning of the equation can be seen from the context. If there is ambiguity, we may specify the unknowns (see Example 2.3.4).

The concept of solutions to equations with two unknowns described in Terminology 2.3.2 also applies to linear equations with two unknowns. For clarity, we state the definition of solution again for the special case where the equation is linear.

Definition 2.3.5 Consider the linear equation with two unknowns

$$ax + by + c = 0 \tag{2.3.3}$$

where a, b and c are real numbers with a, b not both 0. By a ***solution*** to Equation (2.3.3), we mean an ordered pair of real numbers, denoted by (x_0, y_0), such that $ax_0 + by_0 + c = 0$.

Example 2.3.4 Consider the following linear equation with two unknowns x and y

$$2y + 6 = 0 \tag{2.3.4}$$

For each of the following ordered pair of real numbers, determine whether it is a solution to Equation (2.3.4) or not.

(a) $(1, -2)$ (b) $(2, -3)$

Explanation As a linear equation with two unknowns x and y, Equation (2.3.4) means the equation $0x + 2y + 6 = 0$.

Solution (a) Putting $(x, y) = (1, -2)$ into Equation (2.3.4), we get

$$\text{L.S.} = 2 \cdot (-2) + 6 = 2 \neq \text{R.S.}$$

Thus $(1, -2)$ is a not solution to the equation.

(b) Putting $(x, y) = (2, -3)$ into Equation (2.3.4), we get

$$\text{L.S.} = 2 \cdot (-3) + 6 = 0 = \text{R.S.}$$

Thus $(2, -3)$ is a solution to the equation. □

For the linear equation with two unknowns given in Example 2.3.4, it is easily seen that an ordered pair of real numbers (x_0, y_0) is a solution if and only if $y_0 = -3$. Since an ordered pair of real numbers is identified as a point in the rectangular coordinate plane, the collection of all the solutions to Equation (2.3.4) is the subset of the rectangular coordinate plane consisting of all the points whose y-coordinates are equal to -3, that is, the horizontal line 3 units below the x-axis. This set of points is called the *graph* of Equation (2.3.4).

Definition 2.3.6 We call ***the graph of an equation with two unknowns***, to mean the set of all the solutions to the equation.

For a linear equation $ax + by + c = 0$ with two unknowns, where a, b, c are real numbers with a, b not both 0, its graph is the the following subset of the rectangular coordinate plane:

$$\{(r, s) \in \mathbb{R}^2 : ar + bs + c = 0\}$$

Later, we will see that it is always a straight line.

To check whether a point (x_0, y_0) belongs to the graph of a linear equation with two unknowns, we can use direct substitution to see whether the equality is satisfied or not. For the equation

$$x + 2y = 6$$

it is easily seen that $(4, 1)$ is a solution to the equation whereas $(3, 2)$ is not a solution to the equation; hence the points $(4, 1)$ belongs to the graph of the equation whereas the point $(3, 2)$ does not belong to the graph of the equation. Note that if we substitute y by any given real number, we can solve for the corresponding value of x. For example, Table 2.3.1 is obtained by putting y equal to $-1, 0, 1, 2, 3, 4$ respectively, and then solving for the corresponding value of x.

x	8	6	4	2	0	−2
y	−1	0	1	2	3	4

Table 2.3.1

From Table 2.3.1, we see that the points $(8, -1)$, $(6, 0)$, $(4, 1)$, $(2, 2)$, $(0, 3)$ and $(-2, 4)$ belong to the graph of the equation $x + 2y = 6$. These points are shown in Figure 2.3.2.

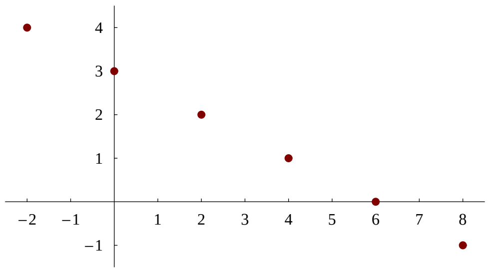


Figure 2.3.2

Note that the points shown in Figure 2.3.2 lie on a straight line. It is reasonable to guess that the graph of the equation $x + 2y = 6$ is a line. In fact, we have the result that

the graph of every linear equation with two unknowns is a line.

Details are given in the following theorem.

Theorem 2.3.1 Consider the following linear equation with two unknowns:

$$ax + by + c = 0 \tag{2.3.5}$$

where a, b, c are real numbers with a, b not both 0. The graph of Equation (2.3.5) is described as follows:

(a) If $a = 0$, then the graph is the horizontal line passing through the point $(0, -\frac{c}{b})$.

(b) If $b = 0$, then the graph is the vertical line passing through the point $(-\frac{c}{a}, 0)$.

(c) If $a \neq 0$ and $b \neq 0$, then the graph is the line passing through the points $(-\frac{c}{a}, 0)$ and $(0, -\frac{c}{b})$.

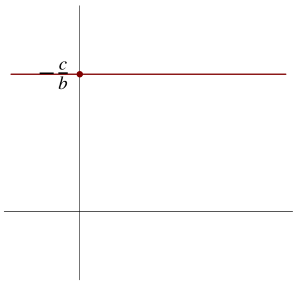


Figure 2.3.3 (a)

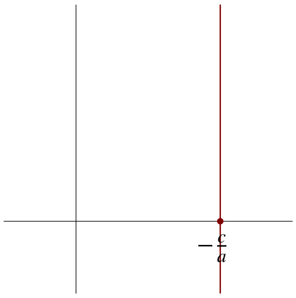


Figure 2.3.3 (b)

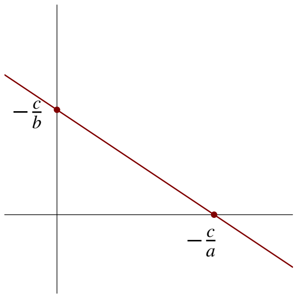


Figure 2.3.3 (c)

Explanation The condition 'a, b not both 0' implies that if one of the numbers a and b is 0, then the other is not 0.

Proof (a) If $a = 0$, then Equation (2.3.5) reduces to $by + c = 0$ which is equivalent to $y = -\frac{c}{b}$ since $b \neq 0$. Hence the graph of Equation (2.3.5) is the subset of $\mathbb{R}^2$ consisting of all ordered pairs whose y-coordinates are equal to $-\frac{c}{b}$, that is, the horizontal line passing through the point $(0, -\frac{c}{b})$.

(b) The proof for (b) is similar to that for (a).

(c) Denote $\mathcal{G}$ to be the graph of Equation (2.3.5) and denote $\mathcal{L}$ to be the line passing through the points $(-\frac{c}{a}, 0)$ and $(0, -\frac{c}{b})$. We want to show that $\mathcal{G} = \mathcal{L}$. For this, we will show that

(i) every element of $\mathcal{G}$ is an element of $\mathcal{L}$;

(ii) every element of $\mathcal{L}$ is an element of $\mathcal{G}$.

To prove (i) and (ii), first we establish two equalities.

(pf cont'd) Denote $X = (-\frac{c}{a}, 0)$ and denote $Y = (0, -\frac{c}{b})$. By the definition of slope,

$$m_{XY} = \frac{-\frac{c}{b} - 0}{0 - \frac{-c}{a}} = \frac{-\frac{c}{b}}{\frac{c}{a}} = -\frac{a}{b} \tag{2.3.6}$$

For every point $P(r, s)$ on the rectangular coordinate plane with $P \neq Y$, we have

$$m_{YP} = \frac{s - \left(-\frac{c}{b}\right)}{r - 0} = \frac{sb + c}{rb} \tag{2.3.7}$$

(i) Suppose $P(r, s) \in \mathcal{G}$. By Definition 2.3.6, the ordered pair (r, s) is a solution to Equation (2.3.5), hence

$$ar + bs + c = 0 \tag{2.3.8}$$

To show that $P(r, s) \in \mathcal{L}$, we consider two cases:
(α) $P = X$ or $P = Y$ (β) $P \neq X$ and $P \neq Y$

(α) In this case, since the line $\mathcal{L}$ passes through X and Y, it follows that $P \in \mathcal{L}$.

(β) In this case,

$$\begin{aligned} m_{YP} &= \frac{sb + c}{rb} && \text{By (2.3.7)} \\ &= \frac{-ar}{rb} && \text{By (2.3.8)} \\ &= -\frac{a}{b} \\ &= m_{XY} && \text{By (2.3.6)} \end{aligned}$$

Therefore, the points X, Y and P are collinear and so $P \in \mathcal{L}$.

(ii) Suppose $P(r, s) \in \mathcal{L}$. We want to show that $P(r, s) \in \mathcal{G}$. For this, we consider two cases: (α) $P = Y$ (β) $P \neq Y$

(α) By direct substitution, we see that $(0, -\frac{c}{b})$ is a solution to Equation (2.3.5). Thus $Y \in \mathcal{G}$ and so $P \in \mathcal{G}$.

(β) Since the points X, Y and P belongs to $\mathcal{L}$, it follows that $m_{XY} = m_{YP}$. Hence by (2.3.6) and (2.3.7), we get

$$-\frac{a}{b} = \frac{sb + c}{rb}$$

which yields $-ar = sb + c$, that is,

$$ar + bs + c = 0$$

Therefore, the ordered pair (r, s) is a solution Equation 2.3.5 and so $P \in \mathcal{G}$.

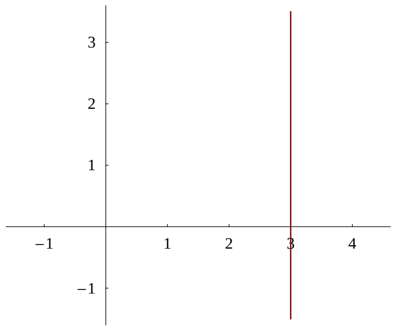
□

Example 2.3.5 Sketch the graph of following linear equation with two unknowns

$$2x - 6 = 0 \tag{2.3.9}$$

Explanation The given equation is $2x + 0y - 6 = 0$. It can be written as $x = 3$. To sketch the graph, we choose appropriate ranges of values of x and y.

Solution The graph of Equation (2.3.9) consists of all the points whose x-coordinates are equal to 3. It is the vertical line shown in the following figure.



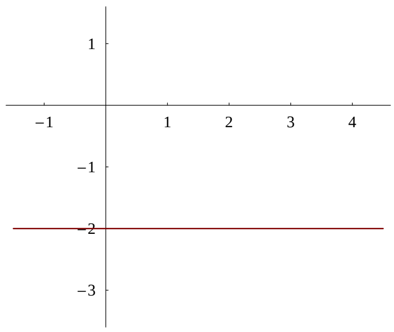
□

Example 2.3.6 Sketch the graph of following linear equation with two unknowns

$$5y + 10 = 0 \tag{2.3.10}$$

Explanation The given equation is $0x + 5y + 10 = 0$. It can be written as $y = -2$.

Solution The graph of Equation (2.3.10) consists of all the points whose y-coordinates are equal to -2. It is the horizontal line shown in the following figure.



□

Example 2.3.7 Sketch the graph of following linear equation with two unknowns

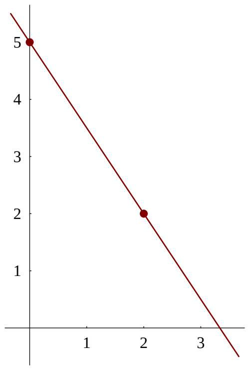
$$3x + 2y - 10 = 0 \tag{2.3.11}$$

Explanation A line is uniquely determined by two points. To find two points on the line, we can substitute x by two real numbers and solve for the corresponding values of y. The values of x in the solution are chosen so that the values of y are integers.

Solution In Equation (2.3.11), putting $x = 0$, we get $y = 5$;

putting $x = 2$, we get $y = 2$.

The points $(0, 5)$ and $(2, 2)$ belong to the graph of Equation (2.3.11). The graph is the line shown in the following figure.



□

To specify a line, one way is to tell which two points belong to the line. For example, we can say

(1) *the line* that passes through the points $(0, 5)$ and $(2, 2)$.

From Example 2.3.7, we see that the line described in (1) is the graph of the equation $3x+2y-10 = 0$. Thus, to specify the line, we can also say

(2) *the line* represented by (or given by) the equation $3x + 2y - 10 = 0$.

Theorem 2.3.2 Denote ℓ to be the line given by

$$ax + by + c = 0 \tag{2.3.12}$$

where a, b, c are real numbers with $b \neq 0$. Then the slope of the line ℓ is equal to $-\frac{a}{b}$, that is, $m_\ell = -\frac{a}{b}$.

Explanation To prove the result, one way is to apply Theorem 2.3.1; for that, we have to consider the two cases (a) and (c) in the theorem. Below we obtain the slope of ℓ by finding two points on the line first.

Proof In Equation (2.3.12), putting $x = 0$, we get $y = \frac{-c}{b}$;

putting $x = 1$, we get $y = \frac{-c-a}{b}$.

Hence the points $P(0, \frac{-c}{b})$ and $Q(1, \frac{-c-a}{b})$ belong to ℓ.

Therefore
$$\begin{aligned} m_\ell &= m_{PQ} \\ &= \frac{\frac{-c-a}{b} - \frac{-c}{b}}{1-0} \\ &= \frac{-a}{b} \end{aligned}$$
□

Remark If $b = 0$, then the line ℓ is vertical and so its slope is undefined.

Example 2.3.8 Find the slope of the line given by

$$2x = 3y - 4 \tag{2.3.13}$$

Explanation To find the slope a line, we can find two points on the line first (that is, repeat the calculation in the proof of Theorem 2.3.2). In the solution, we use Theorem 2.3.2 directly.

Solution First we rewrite Equation (2.3.13) in standard form $2x - 3y + 4 = 0$.

By Theorem 2.3.2, Slope of $\ell = -\frac{2}{-3} = \frac{2}{3}$. □

Remark Although Theorem 2.3.2 provides an easy way to find the slope of a line represented by a linear equation in standard form, many students memorize the formula $m_\ell = -\frac{a}{b}$ incorrectly. Some students forget the '−' sign; some interchange the numerator and denominator. If you can't remember the formula, you may use slope-intercept form (see Section 2.4.2).

Next we introduce a concept that will be used in the next section as well as in later chapters. The line (denoted by ℓ) shown in Figure 2.3.4 intersects the x-axis at the point $X(3,0)$ and intersects the y-axis at the point $Y(0,2)$. Some authors call the points X and Y to be the x-intercept and y-intercept of the line ℓ. Since a point on the x-axis (respectively y-axis) is determined by its x-coordinate (respectively y-coordinate), we will call the *numbers* 3 and 2 to be the x-intercept and y-intercept of the line ℓ.

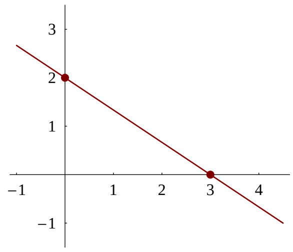


Figure 2.3.4

A line may intersect the x-axis (or the y-axis) at exactly one point, at no point or at infinitely many points.

- The line shown in Figure 2.3.4 intersects the x-axis at exactly one point and intersects the y-axis at exactly one point.
- The line shown in Figure 2.3.5 (a) is a horizontal line which is different from the x-axis. It intersects the y-axis at exactly one point but it doesn't intersect the x-axis.
- The line shown in Figure 2.3.5 (b) is a vertical line which is different from the y-axis. It intersects the x-axis at exactly one point but it doesn't intersect the y-axis.
- The line shown in Figure 2.3.5 (c) is the x-axis. It intersects the x-axis at infinitely many points and it intersects the y-axis at exactly one point (the origin).
- The line shown in Figure 2.3.5 (d) is the y-axis. It intersects the x-axis at exactly one point (the origin) and it intersects the y-axis at infinitely many points.

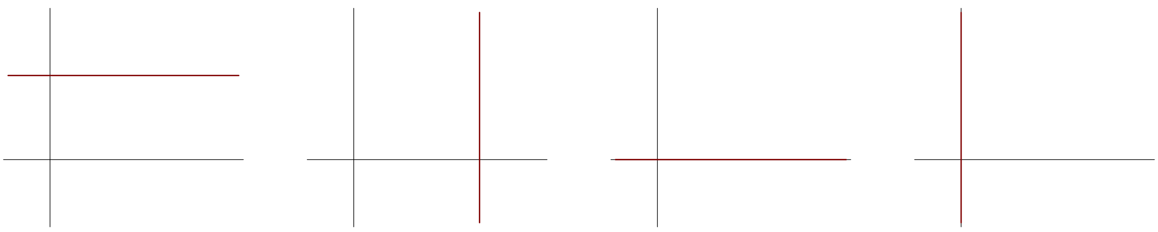


Figure 2.3.5 (a) Figure 2.3.5 (b) Figure 2.3.5 (c) Figure 2.3.5 (d)

In mathematics, a line is also considered to be a curve (*a line is a curve with zero curvature everywhere*). The following gives the definition of an x-intercept (respectively a y-intercept) of a curve, provided the curve intersects the x-axis (respectively the y-axis).

Definition 2.3.7 Let $\mathcal{C}$ be a curve on the rectangular plane. We call

- ***an x-intercept of*** $\mathcal{C}$ to mean the x-coordinate of a point that belongs to the intersection of $\mathcal{C}$ and the x-axis;
- ***a y-intercept of*** $\mathcal{C}$ to mean the y-coordinate of a point that belongs to the intersection of $\mathcal{C}$ and the y-axis.

Remark If the intersection of $\mathcal{C}$ and the x-axis (respectively the y-axis) is empty, we say that $\mathcal{C}$ has no x-intercept (respectively y-intercept).

Example 2.3.9 The circle shown in Figure 2.3.6 has center at the origin and radius 1.

- The x-intercepts of the circle are -1 and 1.
- The y-intercepts of the circle are -1 and 1.

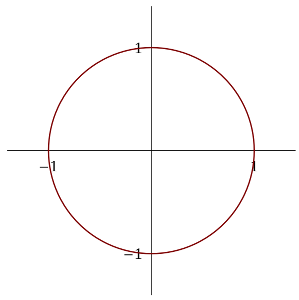


Figure 2.3.6

Example 2.3.10 The circle shown in Figure 2.3.7 has center at $(0, 2)$ and radius 2.

- The x-intercept of the circle is 0.
- The y-intercepts of the circle are 0 and 4.

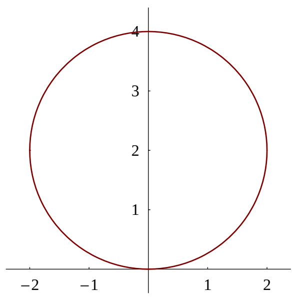


Figure 2.3.7

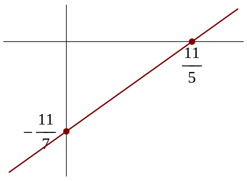
For a line given by a linear equation with two unknowns,

- if the line is horizontal or vertical, it is easy to find its x-intercept(s) and y-intercept(s).
- if the line is not vertical nor horizontal, to find its unique x-intercept (respectively y-intercept), we can substitute $y = 0$ (respectively $x = 0$) into the equation that represents the line.

Example 2.3.11 Denote ℓ to be the line given by

$$5x - 7y - 11 = 0$$

Find the x-intercept and y-intercept of ℓ.



Solution Putting $y = 0$ into the given equation, we get $5x - 11 = 0$.

The x-intercept of ℓ is $\frac{11}{5}$.

Putting $x = 0$ into the given equation, we get $-7y - 11 = 0$.

The y-intercept of ℓ is $-\frac{11}{7}$. □

Remark For a non-vertical and non-horizontal line that is represented by a linear equation with two unknowns in standard form, we can use Theorem 2.3.1 to write down its x-intercept and y-intercept. However, it is easy to find the intercepts by substitution (*you can even do that in your head*); there is no need to memorize the formulas for the intercepts.

Exercise 2.3

1. For each of the following linear equations in two unknowns, rewrite it in the form $ax + by + c = 0$; and write down the values of a, b and c.

   (*a*) $x - 2y + 3 = 0$ (*b*) $x + y = 3$
   (*c*) $y = 2 + 3x$ (*d*) $y = 3$
   (*e*) $3y + 4 = 0$ (*f*) $4x - 1 = 0$

2. For each of the following ordered pairs of real numbers, determined whether it is a solution to the equation $2x + 5y + 7 = 0$.

   (*a*) $(1, 1)$ (*b*) $(4, -3)$ (*c*) $(-11, 3)$ (*d*) $(2, -2)$

3. For each of the following linear equations with two unknowns x and y, determine whether the ordered pair $(1, 2)$ is a solution to the given equation.

   (*a*) $2x + y = 3$ (*b*) $3x - y - 1 = 0$
   (*c*) $2(x - y) + 3 = 4 - 3x$ (*d*) $x = 1$
   (*e*) $x = 2$ (*f*) $y = 2$

4. For each of the following, find the value of a such that the given ordered pair of real numbers is a solution to the given linear equation with two unknowns x and y.

   (*a*) $2x - 5y + 4 = 0,\quad (2a, 1 + a)$
   (*b*) $ax + (1 - a)y = 13,\quad (2, -3)$
   (*c*) $2x - ay = 3 - a^2,\quad (2 - a, a)$

5. For each of the following linear equations with two unknowns x and y, sketch its graph.

   (a) $3x - 2y + 6 = 0$ (b) $5x + 7y = 12$

   (c) $y + 3 = 0$ (d) $2x - 3y = 3(1 - y)$

6. For each of the following linear equations with two unknowns x and y, determine whether the line represented by the equation is vertical or not; and if the line is not vertical, find its slope using Theorem 2.3.2.

   (a) $3x - 2y + 6 = 0$ (b) $5x + 7y = 12$

   (c) $y + 3 = 0$ (d) $2x + 3 = 0$

   (e) $2 - 3y = 4 + 5x$ (f) $2x - 3y = 3(1 - y)$

7. For each of the following linear equations with two unknowns, find the slope of the graph of the equation by finding two points belonging to the graph.

   (a) $x + 2y - 5 = 0$ (b) $5y - x = 4$

   (c) $x = y$ (d) $2x + y = 2(3 - y + x)$

8. For each of the following linear equations with two unknowns, find the x-intercept and y-intercept, if any, of the line represented by the equation.

   (a) $4x + 5y - 8 = 0$ (b) $2(3 - y) = 3(2 + x)$

   (c) $x + y = 0$ (d) $2x + y = 2(3 - y + x)$

2.4 Equations of Lines

In the last section, we have seen that given a linear equation $ax + by + c = 0$ with two unknowns, its graph is a line. Example 2.3.5, Example 2.3.6 and Example 2.3.7 illustrate how to sketch the graph for the three cases: (i) $b = 0$, (ii) $a = 0$, (iii) $a \neq 0$ and $b \neq 0$, respectively. In this section, we will consider the reverse problem. Given a line on the rectangular coordinate plane, how can we find a linear equation with two unknowns that represents the line? First, we consider an example.

Example 2.4.1 Find a linear equation with two unknowns that represents the line passing through the points $P(1, 2)$ and $Q(7, 4)$.

Solution We want to find real numbers a, b, c with a, b not both 0 such that the points P and Q belong to the graph of the equation $ax + by + c = 0$.

Since the line passing through the points P and Q is not horizontal, it follows that $a \neq 0$. By multiplying the equation by a suitable number, we may assume that $a = 1$.

(soln cont'd) Since $P(1,2)$ and $Q(7,4)$ belong to the graph of $x + by + c = 0$, it follows that

$$\begin{aligned} 1 + b \cdot 2 + c &= 0 \\ 7 + b \cdot 4 + c &= 0 \end{aligned}$$

From the two equations, we get $6 + 2b = 0$. Hence $b = -3$.

Back substitution gives $1 + (-3) \cdot 2 + c = 0$. Hence $c = 5$.

$x - 3y + 5 = 0$ is a required equation. □

Remark Since the line is not vertical, it follows that $b \neq 0$. Taking $b = 1$ and repeating the calculations in the above solution, we get another equation $-\frac{1}{3}x + y - \frac{5}{3} = 0$ satisfying the requirement. In fact, any non-zero multiple of the equation $x - 3y + 5 = 0$ is a suitable answer.

Terminology 2.4.1 Let ℓ be a line on the rectangular coordinate plane. We call ***an equation of ℓ*** to mean a linear equation with two unknowns whose graph is ℓ.

Example 2.4.2 From Example 2.4.1, we see that $x - 3y + 5 = 0$ is an equation of the line passing through the points $(1,2)$ and $(7,4)$.

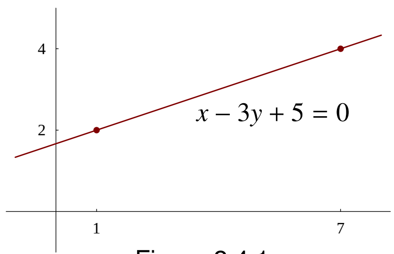
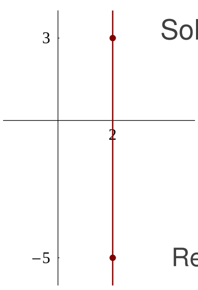


Figure 2.4.1

Remark Some authors call $x - 3y + 5 = 0$ to be <u>*the*</u> equation of the line passing through the points $(1,2)$ and $(7,4)$. This is because any linear equation with two unknowns whose graph is the given line must be a non-zero multiple of $x - 3y + 5 = 0$.

Example 2.4.3 Denote ℓ to be the line determined by the points $(2,3)$ and $(2,-5)$. Find an equation, in standard form, of ℓ.

Explanation Every vertical line can be represented by an equation in the form $x = k$.



Solution Note that the line ℓ is vertical.

Each of the following is an equation of ℓ

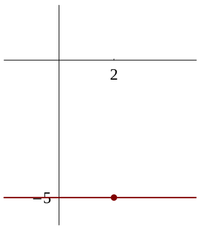
$$\begin{aligned} x &= 2 && \text{equation of vertical line} \\ x - 2 &= 0 && \text{(standard form)} \end{aligned}$$

□

Remark The equation '$x - 2 = 0$' means '$x + 0y - 2 = 0$'.

Example 2.4.4 Find an equation, in standard form, of the horizontal line passing through the point $(2, -5)$.

Explanation Every horizontal line can be represented by an equation in the form $y = k$.



Solution Each of the following is an equation of the horizontal line passing through the point $(2, -5)$.

$$\begin{aligned} y &= -5 && \text{equation of vertical line} \\ y + 5 &= 0 && \text{(standard form)} \end{aligned}$$

□

Remark The equation '$y + 5 = 0$' means '$0x + y + 5 = 0$'.

Given a vertical or horizontal line, we can use inspection to find an equation of the line. Given a non-vertical line, sometimes we can use inspection to find an equation of the line. For example, suppose ℓ is the line passing through the points $(1, 1)$ and $(2, 2)$, then it is easy to see that $x = y$ is an equation of ℓ. However, in many cases, it may not be easy to use inspection to find an equation of a line passing through two given points (for example, consider the points P and Q in Example 2.4.1).

To find an equation of a line determined by two points, besides using the method in Example 2.4.1, we can also use *two-point form*. More generally, we can use "*special forms*" to obtain equations of lines with "*special given conditions*".

(1) In Section 2.4.1, we will discuss *equations in point-slope form*.

(2) In Section 2.4.2, we will discuss *equations in slope-intercept form*.

(3) In Section 2.4.3, we will discuss the *equations in two-point form*.

(4) In Section 2.4.4, we will discuss *equations in intercepts form*.

2.4.1 Equations in Point-slope Form

To specify a non-vertical line, besides telling two points on the line, we can also tell the slope of the line together with a point on the line. For example,

- if ℓ is the line having slope equal to $\frac{2}{3}$ and passing through the point $(-1, 1)$, then we can find another point on ℓ and hence determine the line ℓ (see Figure 2.4.2).

In other words, *a given slope and a given point determine a non-vertical line*.

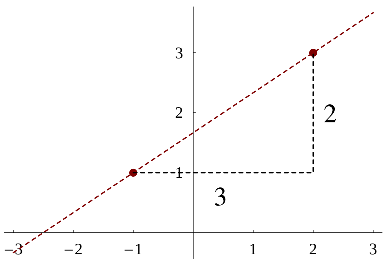


Figure 2.4.2

Theorem 2.4.1 Let m be a real number and let (x_1, y_1) be a point belonging to the rectangular coordinate plane. Then the graph of the equation

$$y - y_1 = m(x - x_1) \tag{2.4.1}$$

is the line having slope equal to m and passing through the point (x_1, y_1).

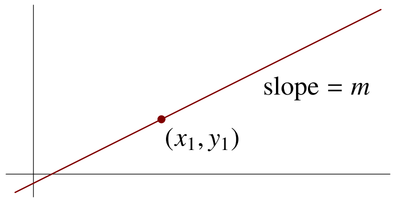


Figure 2.4.3

Proof Note that Equation (2.4.1) is a linear equation with two unknowns. Moreover, it is equivalent to the following equation:

$$mx_1 - y + (y_1 - mx_1) = 0 \tag{2.4.2}$$

By Theorem 2.3.1, the graph of Equation (2.4.1) is a line. We want to show that (i) the slope of ℓ is equal to m, (ii) the point (x_1, y_1) belongs to ℓ, where ℓ is the line represented by Equation (2.4.1) or Equation (2.4.2).

(i) Equation (2.4.2) is in standard form. By Theorem 2.3.2,

$$m_\ell = -\frac{-m}{1} = m$$

(ii) Substituting $(x, y) = (x_1, y_1)$ into Equation (2.4.1), we get

$$\text{L.S.} = y_1 - y_1 = 0, \qquad \text{R.S.} = m(x_1 - x_1) = m \cdot 0 = 0$$

Therefore, the point (x_1, y_1) belongs to ℓ. □

Terminology 2.4.2 When a linear equation with two unknowns is written in the form given in (2.4.1), we say that the equation is in ***point-slope form***.

Remark Some authors consider the following to be an equation in point-slope form:

$$\frac{y - y_1}{x - x_1} = m \tag{2.4.3}$$

To be exact, Equation (2.4.3) *does not* represent a line— the point (x_1, y_1) does not belong to the graph of the equation. Equation (2.4.3) is equivalent to

$$y - y_1 = m(x - x_1) \quad \text{and} \quad (x, y) \neq (x_1, y_1)$$

The graph of Equation (2.4.3) is a line minus a point. However, it is easy to obtain Equation (2.4.1) using Equation (2.4.3).

Theorem 2.4.1 can be used in two ways:

(1) For a (non-vertical) line represented by an equation in point-slope form, we can write down its slope and a point belonging to the line immediately (see Example 2.4.5).

Remark Linear equations with two unknowns are usually written in standard. It is not convenient to convert standard form to point-slope form. To find a point belonging to a line represented by linear equations with two unknowns in standard form, we can use substitution. To find the slope, we can convert standard form to slope-intercept form (see Section 2.4.2).

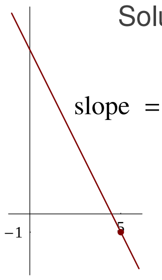
(2) For a (non-vertical) line with given slope and passing through a given point, we can find an equation, in point-slope form and hence in standard form, of the line (see Example 2.4.6).

Example 2.4.5 Denote ℓ to be the line given by

$$y - 3 = 7(x - 2)$$

Then the slope of ℓ is equal to 7 and the point $(2, 3)$ belongs to ℓ.

Example 2.4.6 Find an equation, in standard form, of the line with slope equal to -2 and passing through the point $(5, -1)$.



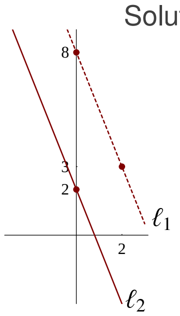
Solution Each of the following is an equation of the line with slope equal to -2 and passing through the point $(5, -1)$.

$$\begin{aligned} y - (-1) &= -2(x - 5) && \text{(point-slope form)} \\ y + 1 &= -2x + 10 && \text{rewrite equation} \\ 2x + y - 9 &= 0 && \text{(standard form)} \end{aligned}$$

□

Example 2.4.7 Find an equation, in standard form, of the line passing through the point $(0, 2)$ and parallel to the line passing through the points $(2, 3)$ and $(0, 8)$.

Explanation We can find the slope of the required line and then apply point-slope form.



Solution Denote ℓ_1 to be the line passing through the points $A(2, 3)$ and $B(0, 8)$ and denote ℓ_2 to be the line passing through the point $(0, 2)$ and parallel to ℓ_1.

By the definition of slope,

$$m_{\ell_1} = m_{AB} = \frac{8-3}{0-2} = -\frac{5}{2}$$

Since $\ell_1 \parallel \ell_2$, it follows that $m_{\ell_2} = m_{\ell_1} = -\frac{5}{2}$.

(soln cont'd) Each of following is an equation of ℓ_2.

$$\begin{aligned} y-2 &= -\frac{5}{2}(x-0) && \text{(point-slope form)} \\ 2y-4 &= -5x && \text{rewrite equation} \\ 5x+2y-4 &= 0 && \text{(standard form)} \end{aligned}$$ □

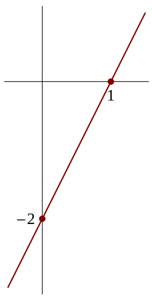
Remark The point $(0,2)$ is the point of intersection of ℓ_2 and the y-axis, that is, the number 2 is the y-intercept of ℓ_2. To find an equation of ℓ_2, we can also use *slope-intercept form* (see Section 2.4.2).

Example 2.4.8 Find an equation, in standard form, of the line passing through the points $A(1,0)$ and $B(0,-2)$.

Explanation To apply the point-slope form, we can use any one of the points A and B. In the solution, we use the point A.

Solution Denote ℓ to be the line passing through A and B.

By the definition of slope,



$$m_\ell = m_{AB} = \frac{-2-0}{0-1} = \frac{-2}{-1} = 2$$

Each of the following is an equation of ℓ.

$$\begin{aligned} y-0 &= 2(x-1) && \text{(point-slope form)} \\ y &= 2x-2 && \text{rewrite equation} \\ 2x-y-2 &= 0 && \text{(standard form)} \end{aligned}$$ □

Remark We can also use *two-point form* to obtain an equation of ℓ (see Section 2.4.3).

Note that the two points are special: A is the point of intersection of ℓ and the x-axis and B is the point of intersection of ℓ and the y-axis. We can also use *intercepts-form* to obtain an equation of ℓ (see Section 2.4.4).

Exercise 2.4.1

1. For each of the following, sketch the line with the given slope m and passing through the given point P.

   (a) $m = \frac{3}{5}$, $P = (2,1)$ (b) $m = 2$, $P = (2,1)$

   (c) $m = -\frac{1}{3}$, $P = (1,4)$ (d) $m = 0$, $P = (1,4)$

2. For each of the following, find an equation, in standard form, of the line with the given slope m and passing through the given point P.

   (a) $m = \frac{3}{2}$, $P = (0,1)$ (b) $m = -\frac{1}{2}$, $P = (2,1)$

   (c) $m = -2$, $P = (5,0)$ (d) $m = 3$, $P = (4,5)$

   (e) $m = \frac{7}{11}$, $P = (0,0)$ (f) $m = 0$, $P = (12,34)$

3. Denote ℓ to be the line with slope equal to 2 and passing through the point $(3,4)$.

   (a) Find an equation, in point-slope form, of ℓ.

   (b) Find another equation, in point-slope form, of ℓ.

   How many equations in point-slope form of ℓ are there?

4. For each of the following, find an equation, in standard form, of the line passing through the given points P and Q.

   (a) $P = (3,1)$ $Q = (4,3)$ (b) $P = (2,-1)$ $Q = (5,-3)$

   (c) $P = (0,1)$ $Q = (4,1)$ (d) $P = (3,-2)$ $Q = (0,0)$

5. Find an equation, in standard form, of the line having slope equal to -3 and x-intercept equal to 5.

6. Find an equation, in standard form, of the line having slope equal to $\frac{5}{2}$ and y-intercept equal to -3.

7. Find an equation, in standard form, of the line passing through the point $(-2,3)$ and parallel to the line passing through the points $(2,5)$ and $(5,4)$.

8. Find an equation, in standard form, of the line passing through the point $(4,0)$ and parallel to the line with x-intercept equal to -2 and y-intercept equal to 3.

9. Find an equation, in standard form, of the line passing through the point $(1,3)$ and perpendicular to the line passing through the points $(-1,3)$ and $(2,7)$.

2.4.2 Equations in Slope-Intercept Form

In the last section, we mentioned that *a given slope and a given point determine a non-vertical line*. Note that every non-vertical line intersects the y-axis consists at one and only one point which is determined by the y-intercept of the line. Thus, *a given slope and a given y-intercept determine a non-vertical line*.

Theorem 2.4.2 Let m and b be real numbers. Then the graph of the equation

$$y = mx + b \tag{2.4.4}$$

is the line having slope equal to m and y-intercept equal to b.

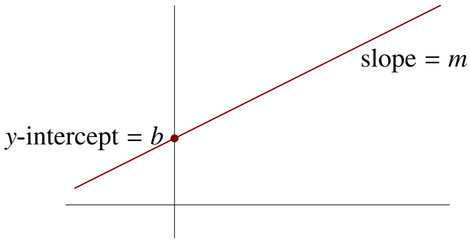


Figure 2.4.4

Proof Equation (2.4.4) is equivalent to the following equation:

$$y - b = m(x - 0) \tag{2.4.5}$$

By Theorem 2.4.1, the graph of Equation (2.4.5), and hence that of Equation (2.4.4), is the line having slope m and passing through the point $(0, b)$. Hence the required result follows. □

Terminology 2.4.3 When a linear equation with two unknowns is written in the form given in (2.4.4), we say that the equation is in ***slope-intercept form***.

Remark In the term '*slope-intercept form*', intercept refers to the y-intercept.

Theorem 2.4.2 can be used in two ways:

(1) For a (non-vertical) line represented by an equation in slope-intercept form, we can write down its slope and its intersection with the y-axis immediately (see Example 2.4.9).

Remark Slope-intercept form is especially useful for finding slopes of non-vertical lines given by equations in standard form (see Example 2.4.11).

(2) For a (non-vertical) line with given slope and passing through a given point on the y-axis, we can find an equation, in slope-intercept form and hence in standard form, of the line (see Example 2.4.10).

Example 2.4.9 Denote ℓ to be the line given by

$$y = 2x - 3$$

(a) Find the slope of ℓ.

(b) Find the point of intersection of ℓ and the y-axis.

Solution Note that $y = 2x + (-3)$ is the equation in slope-intercept form of ℓ.

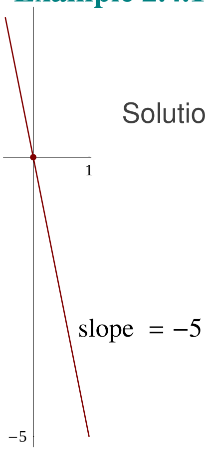
(a) The slope of ℓ is equal to 2.

(b) The y-intercept of ℓ is equal to -3.

Hence the point of intersection of ℓ and the y-axis is $(0, -3)$.

□

Example 2.4.10 Find an equation, in standard form, of the line having slope equal to -5 and passing through the origin.



Solution Denote ℓ to be the line having slope equal to -5 and passing through the origin.
Note that the y-intercept of ℓ is equal to 0.
Each of the following is an equation of ℓ.

$$\begin{aligned} y &= -5x + 0 && \text{(slope-intercept form)} \\ 5x + y &= 0 && \text{(standard form)} \end{aligned}$$

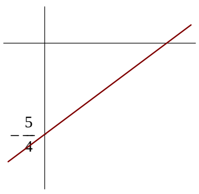
□

Example 2.4.11 Find the slope of the line given by the equation

$$3x - 4y - 5 = 0$$

Explanation To find the slope, we can use Theorem 2.3.2. Below, we use slope-intercept form.

Solution Rewrite the given equation in slope-intercept form.



$$\begin{aligned} 3x - 4y - 5 &= 0 \\ 3x - 5 &= 4y \\ \frac{3}{4}x - \frac{5}{4} &= y \end{aligned}$$

It doesn't matter whether y is on the left or right side

The required slope is equal to $\frac{3}{4}$.

□

Exercise 2.4.2

1. For each of the following, find an equation, in standard form, of the line with the given slope m and given y-intercept.

   (a) $m = -2$, y-intercept $= 3$ (b) $m = \frac{5}{3}$, y-intercept $= -2$

   (c) $m = 1$, y-intercept $= 0$ (d) $m = 0$, y-intercept $= 3$

2. For each of the following, find an equation, in standard form, of the line with the given slope m and passing the given point P.

   (a) $m = 5$, $P = (0, 7)$ (b) $m = -2$, $P = (0, 0)$

3. Denote ℓ to be the line with slope equal to 2 and passing through the point $(0, 3)$. Find the equation, in slope-intercept form, of ℓ.

   Can you find another equation in slope-intercept form of the line ℓ? Do you see why we say "an equation in point-slope form" but we say "the equation in slope-intercept form"?

4. For each of the following, find the slope of the line represented by the given linear equations with two unknowns.

   (a) $y = 3x + 4$ (b) $y = 5 - x$

   (c) $y + 7 = 0$ (d) $y = \frac{3 + x}{2}$

5. For each of the following, rewrite the given equation in slope-intercept form and hence the slope of the line represented by the equation.

   (a) $2x + 7y - 8 = 0$ (b) $x - 2y + 3 = 0$

   (c) $5x - 2y = 0$ (d) $7y - 2x = 3$

   (e) $x = 4 - 9y$ (f) $2(3 - x) = 4(5 + 6y)$

2.4.3 Equations in Two-point Form

Theorem 2.4.3 Let (x_1, y_1) and (x_2, y_2) be two distinct points belonging to the rectangular coordinate plane such that $x_1 \neq x_2$. Then the graph of the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \tag{2.4.6}$$

is the line that passes through the points (x_1, y_1) and (x_2, y_2).

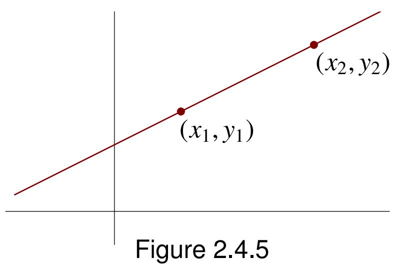


Figure 2.4.5

Explanation The condition $x_1 \neq x_2$ for the points (x_1, y_1) and (x_2, y_2) means that the line segment joining the two points is not vertical.

Proof Note that Equation (2.4.6) is a linear equation with two unknowns. By Theorem 2.3.1, the graph of Equation (2.4.6) is a line.

- Substituting $(x, y) = (x_1, y_1)$ into Equation (2.4.6), we get

$$\text{L.S.} = y_1 - y_1 = 0, \qquad \text{R.S.} = \frac{y_2 - y_1}{x_2 - x_1}(x_1 - x_1) = 0$$

  Thus, the point (x_1, y_1) belongs to the graph of Equation (2.4.6).

- Substituting $(x, y) = (x_2, y_2)$ into Equation (2.4.6), we get

$$\text{L.S.} = y_2 - y_1, \qquad \text{R.S.} = \frac{y_2 - y_1}{x_2 - x_1}(x_2 - x_1) = y_2 - y_1$$

  Thus, the point (x_2, y_2) belongs to the graph of Equation (2.4.6).

Therefore, the graph of Equation (2.4.6) is the line that passes through the points (x_1, y_1) and (x_2, y_2). □

Terminology 2.4.4 When a linear equation with two unknowns is written in the form given in (2.4.6), we say that the equation is in ***two-point form***.

Remark Some authors consider the following to be an equation in point-slope form:

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.4.7}$$

To be exact, Equation (2.4.7) *does not* represent a line (its graph is a line minus a point, see the remark to Terminology 2.4.2).

Theorem 2.4.3 can be used in two ways:

(1) For a line represented by an equation in two-point form, we can write down two points belonging to the line immediately (see Example 2.4.12).

Remark Linear equations with two unknowns are seldom written in two-point form.

(2) For a non-vertical line passing through two given points, we can find an equation, in two-point form and hence in standard form, of the line (see Example 2.4.13).

Example 2.4.12 Denote ℓ to be the line given by

$$y - 2 = \frac{6-2}{7-1}(x - 1)$$

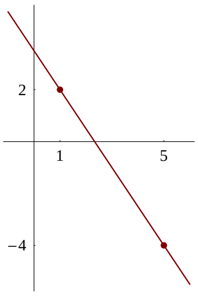
Then the points $(1, 2)$ and $(7, 6)$ belong to ℓ.

Example 2.4.13 Find an equation, in standard form, of the line that passes through the points $(1, 2)$ and $(5, -4)$.

Explanation In applying the two-point form, the order of the points is not important. In the solution, we take $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (5, -4)$.

Besides using two-point form, we can also use point-slope form (see Solution 2).

Solution 1 Each of the following is an equation of the line that passes through the points $(1, 2)$ and $(5, -4)$.



$$
\begin{aligned}
y - 2 &= \frac{-4-2}{5-1}(x-1) && \text{(two-point form)}\\
y - 2 &= -\frac{3}{2}(x-1) && \text{rewrite equation}\\
2y - 4 &= -3x + 3 && \text{rewrite equation}\\
3x + 2y - 7 &= 0 && \text{(standard form)}
\end{aligned}
$$

□

Solution 2 Denote ℓ to be the line that passes through the points $(1, 2)$ and $(5, -4)$. The slope of ℓ is

$$m_\ell = \frac{-4-2}{5-1} = -\frac{3}{2}$$

Each of the following is an equation of ℓ.

$$
\begin{aligned}
y - 2 &= -\frac{3}{2}(x-1) && \text{(point-slope form)}\\
2y - 4 &= -3x + 3 && \text{rewrite equation}\\
3x + 2y - 7 &= 0 && \text{(standard form)}
\end{aligned}
$$

□

Exercise 2.4.3

1. For each of the following, find an equation, in two-point form, of the line that passes through the given points P and Q and hence find an equation, in standard form, of the line.

   (a) $P = (3, 1)$ $Q = (4, 3)$ (b) $P = (2, -1)$ $Q = (5, -3)$

   (c) $P = (0, 1)$ $Q = (4, 1)$ (d) $P = (3, -2)$ $Q = (0, 0)$

2. Denote ℓ to be the line that passes through the points $P(1, 2)$ and $Q(3, -4)$.

   (a) Write down an equation, in two-point form, of ℓ.

   (b) By finding a point on ℓ different from P and Q, write down another equation, in two-point form, of ℓ.

   How many equations in two-point form of ℓ are there?

2.4.4 Equations in Intercepts Form

Suppose ℓ is a line that is not vertical nor horizontal. Then ℓ intersects the x-axis at one and only one point, denoted by $X(\alpha, 0)$, and intersects the y-axis at one and only one point, denoted by $Y(0, \beta)$. There are two possibilities: (1) $X = Y$, (2) $X \neq Y$

Note that if $X = Y$, then $\alpha = 0$ and $\beta = 0$; and if $X \neq Y$, then $\alpha \neq 0$ and $\beta \neq 0$.

(1) In this case, the line ℓ passes through the origin. Without additional information, we can't find an equation of ℓ.

(2) In this case, there is an equation of ℓ in a "simple form" as described in Theorem 2.4.3.

Theorem 2.4.4 Let α and β be real number(s) with $\alpha \neq 0$ and $\beta \neq 0$. Then the graph of the equation

$$\frac{x}{\alpha} + \frac{y}{\beta} = 1 \tag{2.4.8}$$

is the line having x-intercept equal to α and y-intercept equal to β.

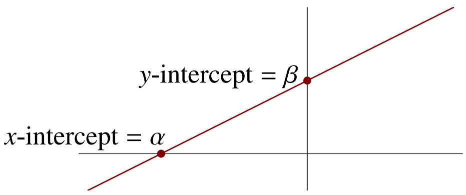


Figure 2.4.6

Proof Note that Equation (2.4.8) is a linear equation with two unknowns. By Theorem 2.3.1, the graph of Equation (2.4.8) is a line.

- Putting $(x, y) = (\alpha, 0)$ into Equation (2.4.8), we get

$$\text{L.S.H.} = \frac{\alpha}{\alpha} + \frac{0}{\beta} = 1 = \text{R.S.}$$

Thus the point $(\alpha, 0)$ belongs to the graph of Equation (2.4.8).

- Putting $(x, y) = (0, \beta)$ into Equation (2.4.8), we get

$$\text{L.S.H.} = \frac{0}{\alpha} + \frac{\beta}{\beta} = 1 = \text{R.S.}$$

Thus the point $(0, \beta)$ belongs to the graph of Equation (2.4.8).

Therefore, the graph of Equation (2.4.8) is the line having x-intercept equal to α and y-intercept equal to β. □

Terminology 2.4.5 When a linear equation with two unknowns is written in the form given in (2.4.8), we say that the equation is in ***intercepts form***.

Theorem 2.4.4 can be used in two ways:

(1) For a line represented by an equation in intercepts form, we know where the line intersects the x-axis and y-axis (see Example 2.4.14 and Example 2.4.16 solution 1).

(2) For a non-vertical and non-horizontal line passing through a given point on the x-axis and another given point on the y-axis, we can find an equation, in intercepts form and hence in standard form, of the line (see Example 2.4.15 solution 1).

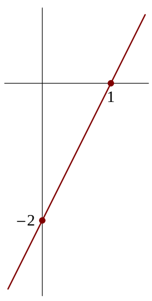
Example 2.4.14 Denote ℓ to be the line given by

$$\frac{x}{5} + \frac{y}{4} = 1$$

Then the x-intercept and y-intercept of ℓ are equal to 5 and 4 respectively, that is, the line ℓ intersects the x-axis at the point $(5, 0)$ and intersects the y-axis at the point $(0, 4)$.

Example 2.4.15 Find an equation, in standard form, of the line that passes through the points $(1, 0)$ and $(0, -2)$.

Solution 1 Denote ℓ to be the line that passes through the points $(1, 0)$ and $(0, -2)$. Note that the x-intercept and y-intercept of ℓ are equal to 1 and -2 respectively. Each of the following is an equation of ℓ.



$$\begin{aligned} \frac{x}{1} + \frac{y}{-2} &= 1 && \text{(intercepts-form)} \\ 2x - y &= 2 && \text{rewrite equation} \\ 2x - y - 2 &= 0 && \text{(standard form)} \end{aligned}$$

□

Solution 2 Denote ℓ to be the line passing through the points $(1, 0)$ and $(0, -2)$. The slope of ℓ is

$$m_\ell = \frac{-2 - 0}{0 - 1} = 2$$

Each of the following is an equation of ℓ.

$$\begin{aligned} y &= 2x + (-2) && \text{(slope-intercept form)} \\ 2x - y - 2 &= 0 && \text{(standard form)} \end{aligned}$$

□

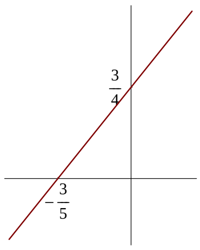
Remark We can also use point-slope form or two-point form to find an equation of the line.

Example 2.4.16 Denote ℓ to be the line given by

$$5x - 4y + 3 = 0$$

Find the x-intercept and y-intercept of ℓ.

Solution 1 Rewrite the given equation in intercepts form:



$$\begin{aligned} 5x - 4y + 3 &= 0 \\ 5x - 4y &= -3 \\ \frac{5x}{-3} - \frac{4y}{-3} &= 1 \\ \frac{x}{-\frac{3}{5}} + \frac{y}{\frac{3}{4}} &= 1 \end{aligned}$$

The x-intercept of ℓ is equal to $-\frac{3}{5}$ and the y-intercept of ℓ is equal to $\frac{3}{4}$. □

Solution 2 Putting $x = 0$ into the given equation, we get

$$\begin{aligned} 5 \cdot 0 - 4y + 3 &= 0 \\ 3 &= 4y \end{aligned}$$

The y-intercept of ℓ is equal to $\frac{3}{4}$.

Putting $y = 0$ into the given equation, we get

$$\begin{aligned} 5x - 4 \cdot 0 + 3 &= 0 \\ 5x &= -3 \end{aligned}$$

The x-intercept of ℓ is equal to $-\frac{3}{5}$. □

Exercise 2.4.4

1. For each of the following, rewrite the given equation in intercepts form and hence find the x-intercept and y-intercept of the line represented by the equation.

   (a) $2x + 7y = 1$ (b) $x - 2y = 3$

   (c) $2x + 3y - 5 = 0$ (d) $3x - 8y + 7 = 0$

2. For each of the following, use substitution to find, if any, the x-intercept and y-intercept of the line represented by the equation.

   (a) $2x + 7y = 1$ (b) $x - 2y = 3$

   (c) $2x + 3y - 5 = 0$ (d) $3x - 8y + 7 = 0$

   (e) $2x + y = 0$ (f) $2y + 3 = 0$

   Can you use the method in Question 1 to find the intercepts of the lines represented by the equations in (e) and (f)?

3. Denote ℓ to be the line with x-intercept equal to 7 and y-intercept equal to 5. Find an equation in intercepts form of ℓ.
   Can you find another equation in intercepts form of the line ℓ? Do you see why we say "the equation in intercepts form"?
4. For each of the following, find the equation, in intercepts form, of the line that passes through the given points P and Q and hence find an equation, in standard form, of the line.
   (a) $P = (2, 0)$, $Q = (0, 6)$ (b) $P = (-3, 0)$, $Q = (0, -3)$
   (c) $P = (0, 4)$, $Q = (1, 0)$
5. For each of the following, find an equation, in two-point form, of the line that passes through the given points P and Q and hence find an equation, in standard form, of the line.
   (a) $P = (2, 0)$, $Q = (0, 6)$ (b) $P = (-3, 0)$, $Q = (0, -3)$
   (c) $P = (0, 4)$, $Q = (1, 0)$
6. For each of the following, find the equation, in slope-intercept form, of the line that passes through the given points P and Q and hence find an equation, in standard form, of the line.
   (a) $P = (2, 0)$, $Q = (0, 6)$ (b) $P = (-3, 0)$, $Q = (0, -3)$
   (c) $P = (0, 4)$, $Q = (1, 0)$
7. For each of the following, find an equation, in point-slope form, of the line that passes through the given points P and Q and hence find an equation, in standard form, of the line.
   (a) $P = (2, 0)$, $Q = (0, 6)$ (b) $P = (-3, 0)$, $Q = (0, -3)$
   (c) $P = (0, 4)$, $Q = (1, 0)$

2.4.5 Miscellaneous Examples on Equations of Lines

Example 2.4.17 Figure 2.4.7 shows three points P, Q and R and (part of) a line ℓ which passes through the points P, Q and R.

(a) Write down the coordinates of the points P and R.

(b) Find the slope of the line ℓ.

(c) Find an equation in standard form of the line ℓ.

(d) Find the coordinates of the point Q.

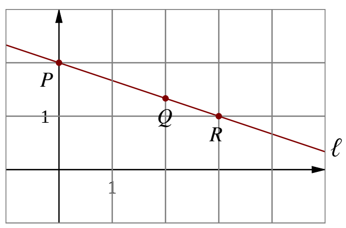


Figure 2.4.7

Explanation A line is infinitely long. We can't display the whole line. Thus the figure shows only *part of* the line. Usually, '*part of*' is omitted because it is understood.

Part (a) asks for the ordered pairs of real numbers that represents the point P and R.

Solution (a) $P = (0, 2)$ and $R = (4, 1)$.

(b)
$$\text{Slope of } \ell = m_{PR} = \frac{1-2}{4-0} = -\frac{1}{4}$$

(c) Note that the y-intercept of ℓ is equal to 2. Hence by (a), the equation in slope-intercept form of ℓ is

$$y = -\frac{1}{4}x + 2$$

$x + 4y - 8 = 0$ is an equation in standard form of ℓ.

(d) Note that the x-coordinate of Q is equal to 2.

Putting $x = 2$ into the equation in slope-intercept form of ℓ, we get

$$y = -\frac{1}{4} \cdot 2 + 2 = \frac{3}{2}$$

Hence $Q = (2, \frac{3}{2})$. □

Example 2.4.18 Denote ℓ to be the line that passes through the points $A(1, -2)$ and $B(3, 8)$. For each of the following points, determine whether it belongs to ℓ or not.

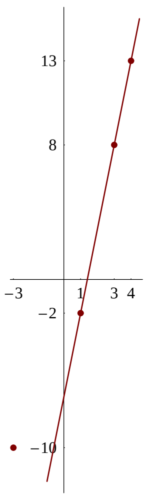
(a) $P(-3, -10)$ (b) $Q(4, 13)$

Explanation To determine whether P and Q lie on ℓ, we may use any one of the following methods.

(1) Find an equation of ℓ and then use substitution. To find an equation of ℓ, in the solution, we use point-slope form.

(2) Apply Test for Collinear Points.

Solution 1 By definition, $m_\ell = m_{AB} = \dfrac{8-(-2)}{3-1} = 5$



The following is an equation in point-slope form of ℓ.

$$y - 8 = 5(x - 3) \tag{2.4.9}$$

(a) Substitute $(x, y) = (-3, -10)$ into Equation (2.4.9), we get

$$\text{L.S.} = -10 - 8 = -18, \quad \text{R.S.} = 5 \cdot (-3 - 3) = -30$$

Therefore, the point $P(-3, -10)$ does not belong to ℓ.

(b) Substitute $(x, y) = (4, 13)$ into Equation (2.4.9), we get

$$\text{L.S.} = 13 - 8 = 5, \quad \text{R.S.} = 5 \cdot (4 - 3) = 5$$

Therefore, the point $Q(4, 13)$ belongs to ℓ. □

Solution 2 By definition, $m_{AB} = \dfrac{8-(-2)}{3-1} = 5.$

(a) Note that $m_{BP} = \dfrac{-10-8}{-3-3} = -3.$

Since $m_{AB} \neq m_{BP}$, it follows that the points A, B and P are not collinear and so P does not belong to ℓ.

(b) Note that $m_{BQ} = \dfrac{13-8}{4-3} = 5.$

Since $m_{AB} = m_{BQ}$, it follows that the points A, B and Q are collinear and so Q belongs to ℓ. □

Example 2.4.19 Denote ℓ_1 to be the line given by the equation $4x - 6y + 13 = 0$ and denote ℓ_2 to be the line that passes through the point $P(4, -2)$ and is perpendicular to ℓ_1.

(a) Find the slope of ℓ_1.

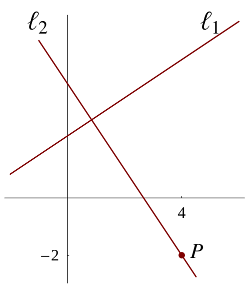
(b) Find an equation, in standard form, of ℓ_2.

Solution (a) Rewrite the given equation of ℓ_1 in slope-intercept form:

$$\begin{aligned} 4x - 6y + 13 &= 0 \\ 4x + 13 &= 6y \\ \frac{2}{3}x + \frac{13}{6} &= y \end{aligned}$$

The slope of ℓ_1 is equal to $\frac{2}{3}$.

(b) Since $\ell_1 \perp \ell_2$, it follows that



$$\begin{aligned} m_{\ell_1} \cdot m_{\ell_2} &= -1 \\ \frac{2}{3} \cdot m_{\ell_2} &= -1 && \text{By (a)} \\ m_{\ell_2} &= -\frac{3}{2} \end{aligned}$$

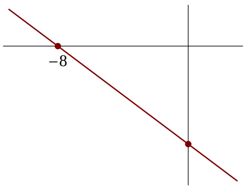
Each of the following is an equation of ℓ_2.

$$\begin{aligned} y - (-2) &= -\frac{3}{2}(x-4) && \text{(point-slope form)} \\ 2y + 4 &= -3x + 12 && \text{rewrite equation} \\ 3x + 2y - 8 &= 0 && \text{(standard form)} \end{aligned}$$

□

Example 2.4.20 Denote ℓ to be the line with slope equal to $-\frac{3}{4}$ and x-intercept equal to -8. Find the y-intercept of ℓ.

Solution 1 Note that ℓ passes through the point $(-8, 0)$.



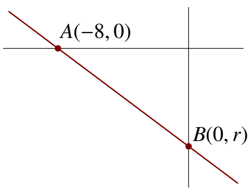
Each of the following is an equation of ℓ.

$$\begin{aligned} y - 0 &= -\frac{3}{4}(x - (-8)) \qquad \text{(point-slope form)} \\ 4y &= -3x - 24 \end{aligned}$$

Putting $x = 0$ into the above equation, we get $4y = -24$.

The y-intercept of ℓ is equal to -6. □

Solution 2 Denote r to be the y-intercept of ℓ. Denote $A = (-8, 0)$ and denote $B = (0, r)$. Since both A and B belongs to ℓ, it follows that



$$\begin{aligned} m_\ell &= m_{AB} \\ -\frac{3}{4} &= \frac{r - 0}{0 - (-8)} \\ -\frac{3}{4} &= \frac{r}{8} \\ -6 &= r \end{aligned}$$

The y-intercept of ℓ is equal to -6. □

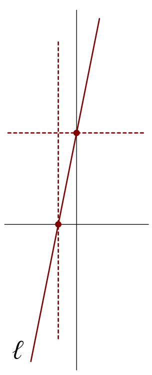
Example 2.4.21 Denote ℓ to be the line given by the equation $5x - y + 2 = 0$.

(a) Find an equation of the horizontal line that intersects ℓ at the y-axis.

(b) Find an equation of the vertical line that intersects ℓ at the x-axis.

Explanation First we find where the given line intersects the x-axis and the y-axis.

Solution Rewrite the given equation of ℓ in intercepts form:



$$\begin{aligned} 5x - y + 2 &= 0 \\ 5x - y &= -2 \\ \frac{5x}{-2} - \frac{y}{-2} &= 1 \\ \frac{x}{-\frac{2}{5}} + \frac{y}{2} &= 1 \end{aligned}$$

The x-intercept and y-intercept of ℓ are equal to $-\frac{2}{5}$ and 2 respectively.

(a) $y = 2$ is an equation of the horizontal line that intersects ℓ at the y-axis.

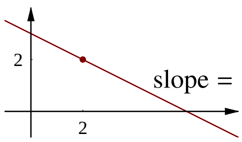
(b) $x = -\frac{5}{2}$ is an equation of the vertical line that intersects ℓ at the x-axis. □

Remark To find where ℓ intersects the x-axis and the y-axis, we can also put $y = 0$ and $x = 0$ respectively in the given equation of ℓ.

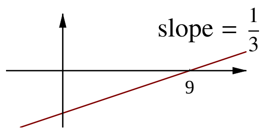
Exercise 2.4.5

1. For each of the following, find an equation in standard form of the line shown in the given figure.

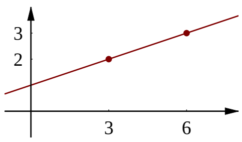
   (*a*)



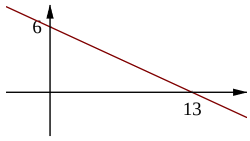
   (*b*)



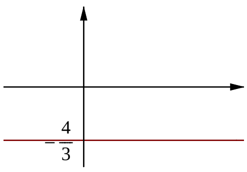
   (*c*)



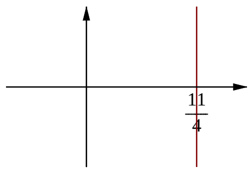
   (*d*)



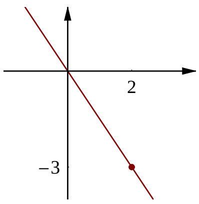
   (*e*)



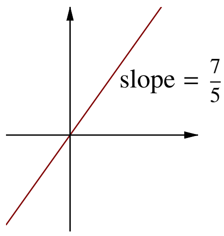
   (*f*)



   (*g*)



   (*h*)



2. Figure 2.4.8 shows two points A and B and a line ℓ.
   (*a*) Write down the coordinates the points A and B.
   (*b*) Find the slope of ℓ.
   (*c*) Find an equation, in standard form, of ℓ.
   (*d*) Find the point of intersection of ℓ and the x-axis.

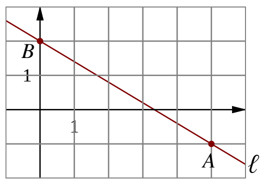


Figure 2.4.8

3. Figure 2.4.9 shows three points A, B and the origin and two lines ℓ_1 and ℓ_2 which are parallel.
   (*a*) Write down the coordinates of the points A and B.
   (*b*) Find the slope of ℓ_1.
   (*c*) Find an equation, in standard form, of ℓ_2.

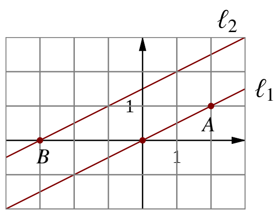


Figure 2.4.9

4. Denote ℓ to be the line that passes through the points $(3,0)$ and $(0,-1)$.
   (*a*) For each of the following points, determine whether it belongs to ℓ or not.
      (i) $P(1,\frac{1}{2})$ (ii) $Q(2,-\frac{1}{3})$ (iii) $R(-2,-3)$
   (*b*) Suppose $S(a,4)$ is a point belonging to ℓ. Find the value of a.

5. Denote ℓ_1 to be the line given by the equation $5x + 7y - 8 = 0$, denote ℓ_2 to be the line parallel to ℓ_1 and passing through the point $(1, -2)$ and denote ℓ_3 to be the line with y-intercept equal to $-\frac{1}{2}$ and perpendicular to ℓ_1.
   - (a) Find the slope of ℓ_1.
   - (b) Find an equation, in standard form, of ℓ_2.
   - (c) Find an equation, in standard form, of ℓ_3.
6. Denote ℓ_1 to be the line with x-intercept and y-intercept equal to -7 and 6 respectively. Denote ℓ_2 to be the line that passes through the point $(1, -2)$ and is perpendicular to ℓ_1.
   - (a) Find an equation, in standard form, of ℓ_1.
   - (b) Find an equation, in standard form, of ℓ_2.
7. Denote ℓ_1 to be the line given by the equation $5x - 7y + 2 = 0$. Denote ℓ_2 to be the line having slope equal to $-\frac{1}{3}$ and intersecting ℓ_1 at the point $P(a, 6)$.
   - (a) Find the value of a.
   - (b) Find an equation, in standard form, of ℓ_2.
8. Denote ℓ_1 to be the line given by the equation $7x + 11y - 14 = 0$ and denote ℓ_2 to be the line that is perpendicular to ℓ_1 and intersects ℓ_1 at the x-axis. Find an equation in standard form of ℓ_2.
9. Denote ℓ to be the line given by the equation $ax + 3y = 6$, where a is a real number. Suppose that the slope of ℓ is equal to -4.
   - (a) Find the value of a.
   - (b) Find the x-intercept and y-intercept of ℓ.
10. Denote ℓ to be the line given by the equation $ax+by-15 = 0$, where a and b are real numbers. Suppose that the x-intercept and the y-intercept of ℓ are equal to -3 and 5 respectively.
    - (a) Find the values of a and b.
    - (b) Find the slope of ℓ.
11. Denote ℓ_1 and ℓ_2 to be the line given by the following equations respectively:
    $$2x + 3y + 4 = 0, \qquad x - 2y + c = 0$$
    where c is a real number. Suppose that ℓ_1 and ℓ_2 intersect at the y-axis. Find the value of c.
12. Denote ℓ_1 and ℓ_2 to be the line given by the following equations respectively:
    $$4x + 3y + 1 = 0, \qquad ax + by = 0$$
    where a and b are non-zero real numbers.
    - (a) Suppose that the lines ℓ_1 and ℓ_2 are parallel. Find the value of $a : b$.
    - (b) Suppose that the lines ℓ_1 and ℓ_2 are perpendicular. Find the value of $a : b$.

2.5 Intersection of Lines

Terminology 2.5.1

- By a ***system of equations*** with two unknowns, we mean a collection of equations with two unknowns.
- By a ***solution*** to a system of equations with two unknowns, we mean an ordered pair of real numbers which is a solution to each of the equations in the system.
- To ***solve*** a system of equations with two unknowns means to find all solutions to the system of equations.

In junior forms, we have seen how to find solve a system of two linear equations with two unknowns using the *elimination method* and *substitution method*. Since a linear equation with two unknowns represents a line on the rectangular coordinate plane, solving a system of two linear equations with two unknowns is the same as finding the point(s) of intersection of the two lines represented by the equations. Thus

- to solve a system of two linear equations with two unknowns, we can use *graphical method*.
- to determine whether two lines on the rectangular coordinate plane intersect, we can consider equations that represents the lines.

In the following example, we use the elimination method, substitution method and graphical method to solve a system of two linear equations with two unknowns.

Example 2.5.1 Solve the following system of equations:

$$2x + 3y - 7 = 0 \quad (2.5.1)$$

$$3x - 4y - 2 = 0 \quad (2.5.2)$$

Solution 1 (*Elimination Method*) Multiplying Equations (2.5.1) and (2.5.2) by 4 and 3 respectively, we get

$$8x + 12y - 28 = 0 \quad (2.5.3)$$

$$9x - 12y - 6 = 0 \quad (2.5.4)$$

Adding Equation (2.5.3) and Equation (2.5.4), we get

$$17x - 34 = 0$$

which yields $x = 2$. Substituting $x = 2$ into Equation (2.5.1), we get

$$2 \cdot 2 + 3y - 7 = 0$$

which yields $y = 1$.

The solution to the given system is $(2, 1)$.

Solution 2 (*Substitution Method*) From Equation (2.5.1), we get

$$x = \frac{7-3y}{2} \tag{2.5.5}$$

Substituting this into Equation (2.5.2), we get

$$\begin{aligned} 3 \cdot \frac{7-3y}{2} - 4y - 2 &= 0 \\ 21 - 9y - 8y - 4 &= 0 \\ 17 &= 17y \\ 1 &= y \end{aligned}$$

Substitution $y = 1$ into Equation (2.5.5), we get

$$x = \frac{7 - 3 \cdot 1}{2} = 2$$

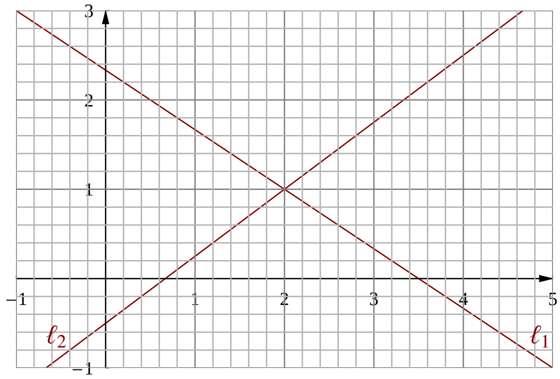
The solution to the given system is (2, 1). □

Solution 3 (*Graphical Method*) Denote ℓ_1 and ℓ_2 to be the line represented by Equation (2.5.1) and Equation (2.5.2) respectively.

By direct substitution, we see that

- the x-intercept and y-intercept of ℓ_1 are equal to $\frac{7}{2}$ and $\frac{7}{3}$ respectively;
- the x-intercept and y-intercept of ℓ_2 are equal to $\frac{2}{3}$ and $-\frac{1}{2}$ respectively.

The lines ℓ_1 and ℓ_2 are shown in the following figure:



From the graph, we see that ℓ_1 and ℓ_2 intersect at the point (2, 1).

The solution to the given system is (2, 1). □

Remark The graphical method may not be accurate, especially when the solution involves real numbers that are not integers.

Given two coplanar lines, there are three possibilities for their intersection:

(1) the two lines coincide (that is, they are the same line): in this case, there are infinitely many points of intersection;

(2) the two lines are parallel: in this case, there is no point of intersect;

(3) the two lines do not coincide and are not parallel: in this case, there is exactly one point of intersection.

For two non-vertical lines on the rectangular coordinate plane, we can determine whether they coincide or parallel or not using the equations in slope-intersect form of the lines.

Example 2.5.2 Denote ℓ_1 and ℓ_2 to be the line given by Equation (2.5.6) and Equation (2.5.7) respectively.

$$y = \frac{1}{2}x + 2 \tag{2.5.6}$$

$$y = \frac{1}{2}x - 1 \tag{2.5.7}$$

(*a*) Sketch the lines ℓ_1 and ℓ_2.

(*b*) Are the lines ℓ_1 and ℓ_2 parallel? Why?

Solution (*a*) First, for each of the lines ℓ_1 and ℓ_2, we find two points belonging to the line by substitution.

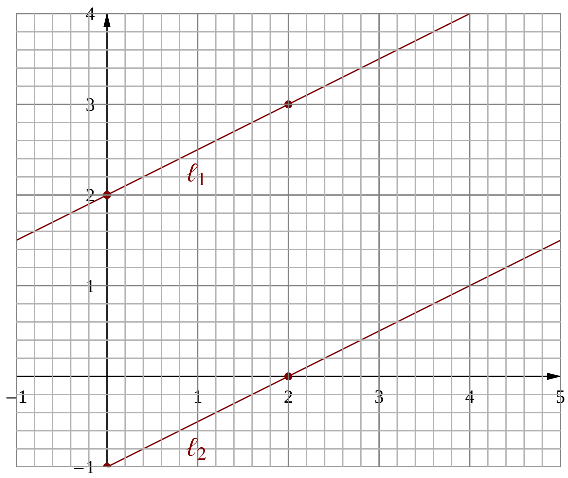
ℓ_1: $y = \frac{1}{2}x + 2$

x	y
0	2
2	3

ℓ_2: $y = \frac{1}{2}x - 1$

x	y
0	−1
2	0

The lines ℓ_1 and ℓ_2 are shown in the following figure:



(soln cont'd) (b) The lines ℓ_1 and ℓ_2 are parallel. This is because the slopes of ℓ_1 and ℓ_2 are equal and ℓ_1 and ℓ_2 are distinct lines. □

Remark It is easy to see that the system of equations (2.5.6) and (2.5.7) has no solution. From this, we can conclude that the lines ℓ_1 and ℓ_2 are parallel.

For two non-vertical lines in the rectangular coordinate plane, we can use their slopes and intercepts to determine the number of points of their intersection.

Theorem 2.5.1 Denote ℓ_1 and ℓ_2 to be the lines represented by Equation (2.5.8) and Equation (2.5.9) respectively.

$$y = m_1x + b_1 \tag{2.5.8}$$

$$y = m_2x + b_2 \tag{2.5.9}$$

where m_1, m_2, b_1, b_2 are real numbers. Then

(a) the lines ℓ_1 and ℓ_2 coincide if and only if $m_1 = m_2$ and $b_2 = b_2$.

(b) the lines ℓ_1 and ℓ_2 are parallel if and only if $m_1 = m_2$ and $b_2 \neq b_2$.

(c) the lines ℓ_1 and ℓ_2 do not coincide and are not parallel if and only if $m_1 \neq m_2$.

Remark The three cases in (a), (b) and (c) correspond to that ℓ_1 and ℓ_2 intersect at infinitely many points, at no point, and at exactly one point respectively.

Example 2.5.3 Denote ℓ_1 and ℓ_2 to be the line given by Equation (2.5.10) and Equation (2.5.11) respectively.

$$3x + 6y - 18 = 0 \tag{2.5.10}$$

$$4x + 8y - 24 = 0 \tag{2.5.11}$$

(a) Sketch the lines ℓ_1 and ℓ_2.

(b) How many points are there in the intersection of ℓ_1 and ℓ_2?

Solution (a) First, for each of the lines ℓ_1 and ℓ_2, we find two points belonging to the line by substitution.

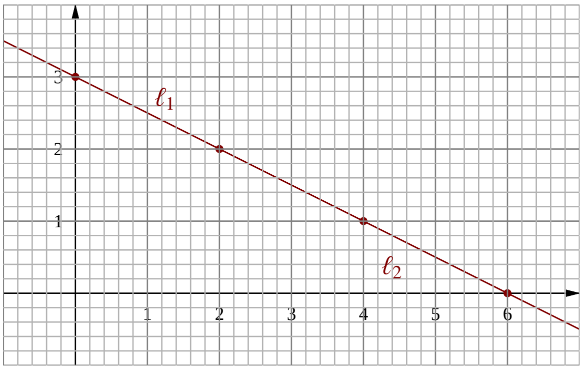
ℓ_1: $3x + 6y - 18 = 0$

x	y
0	3
2	2

ℓ_2: $4x + 8y - 24 = 0$

x	y
6	0
4	1

(soln cont'd) The lines ℓ_1 and ℓ_2 are shown in the following figure:



(b) Since both ℓ_1 and ℓ_2 pass through the points $(0, 3)$ and $(6, 0)$, it follows that the lines ℓ_1 and ℓ_2 coincide. Hence there are infinitely many points in the intersection of ℓ_1 and ℓ_2. □

For two lines represented by equations in standard form, to determine the number of points of intersection of the lines, there are results similar to that given in Theorem 2.5.1. The tests for the three cases can be described using ratios of the coefficients. Instead of stating the general result, we give three examples to illustrate how to apply Theorem 2.5.1. Note that the equations given in Example 2.5.4 are the same as that given in Example 2.5.3.

Example 2.5.4 Denote ℓ_1 and ℓ_2 to be the lines represented by the following equations respectively:

$$\begin{aligned} 3x + 6y - 18 &= 0 \\ 4x + 8y - 24 &= 0 \end{aligned}$$

Determine the number of points of intersection of ℓ_1 and ℓ_2.

Solution The equations in slope-intercept form of ℓ_1 and ℓ_2 are

$$\begin{aligned} y &= -\frac{3}{6}x + \frac{18}{6} \\ y &= -\frac{4}{8}x + \frac{24}{4} \end{aligned}$$

respectively. Note that

$$-\frac{3}{6} = -\frac{4}{8} \quad \text{and} \quad \frac{18}{6} = \frac{24}{4}$$

By Theorem 2.5.1, the lines ℓ_1 and ℓ_2 coincide.
The lines ℓ_1 and ℓ_2 have infinitely many points of intersection. □

Remark We can also determine the number of points of intersection of the lines by solving the system of linear equations. Note that the first equation is equivalent to the second equation. Hence the system has infinitely many solutions.

Example 2.5.5 Denote ℓ_1 and ℓ_2 to be the lines represented by the following equations respectively:

$$\begin{aligned} 2x - 3y + 5 &= 0 \\ 4x - 6y + 9 &= 0 \end{aligned}$$

Determine the number of points of intersection of ℓ_1 and ℓ_2.

Solution The equations in slope-intercept form of ℓ_1 and ℓ_2 are

$$\begin{aligned} y &= \frac{2}{3}x + \frac{5}{3} \\ y &= \frac{4}{6}x + \frac{9}{6} \end{aligned}$$

respectively. Note that

$$\frac{2}{3} = \frac{4}{6} \qquad \text{and} \qquad \frac{5}{3} \neq \frac{9}{6}$$

By Theorem 2.5.1, the lines ℓ_1 and ℓ_2 are parallel.
The lines ℓ_1 and ℓ_2 have no point of intersection. □

Remark By comparing two times the first equation and the second equation, we see that the system of the equations has no solution.

Example 2.5.6 Denote ℓ_1 and ℓ_2 to be the lines represented by the following equations respectively:

$$\begin{aligned} 5x + 6y - 10 &= 0 \\ 2x + 3y - 5 &= 0 \end{aligned}$$

Determine the number of points of intersection of ℓ_1 and ℓ_2.

Solution The equations in slope-intercept form of ℓ_1 and ℓ_2 are

$$\begin{aligned} y &= -\frac{5}{6}x + \frac{10}{6} \\ y &= -\frac{2}{3}x + \frac{5}{3} \end{aligned}$$

respectively. Note that

$$-\frac{5}{6} \neq -\frac{2}{3}$$

By Theorem 2.5.1, the lines ℓ_1 and ℓ_2 do not coincide and are not parallel.
The lines ℓ_1 and ℓ_2 have exactly one point of intersection. □

Remark Solving the system of equations, we get exactly one solution.

Example 2.5.7 Denote ℓ_1 to be the line given by the equation $x + 3y - 2 = 0$ and denote ℓ_2 to be the line given by the equation $ax - 6y + c = 0$, where a and c are real numbers. Suppose that the lines ℓ_1 and ℓ_2 are parallel. What can you tell about the numbers a and c?

Solution The equations in slope-intercept form of ℓ_1 and ℓ_2 are

$$y = -\frac{1}{3}x + \frac{2}{3}, \qquad y = \frac{a}{6}x + \frac{c}{6}$$

respectively. Since the lines ℓ_1 and ℓ_2 are parallel, it follows that

$$-\frac{1}{3} = \frac{a}{6} \qquad \text{and} \qquad \frac{2}{3} \neq \frac{c}{6}$$

Hence $a = -2$ and $c \neq 4$. □

Example 2.5.8 Denote ℓ_1 to be the line given by the equation $2x - 5y + 7 = 0$ and denote ℓ_2 to be the line given by the equation $ax + by + 1 = 0$, where a and b are real numbers with $b \neq 0$. Suppose that the lines ℓ_1 and ℓ_2 are perpendicular. Find the value of $a : b$.

Explanation $a : b$ means $\frac{a}{b}$.

Solution The equations in slope-intercept form of ℓ_1 and ℓ_2 are

$$y = \frac{2}{5}x + \frac{7}{5}, \qquad y = -\frac{a}{b}x - \frac{1}{b}$$

respectively. Since the lines ℓ_1 and ℓ_2 are perpendicular, it follows that

$$\frac{2}{5} \times \frac{-a}{b} = -1$$

Hence $a : b = 5 : 2$. □

Exercise 2.5

1. For each of the following, determine the number of points of intersection of the lines represented by the two given linear equations in two unknowns.

   (*a*) $x = 2, \quad y = 3$
   (*b*) $x + 2 = 0, \quad 5x + 10 = 0$
   (*c*) $3x + 1 = 0, \quad 2x + 3 = 0$
   (*d*) $y = -3x, \quad y = -3x + 1$
   (*e*) $y = 2x - 3, \quad x = 2y - 4$
   (*f*) $3x + 7y - 1 = 0, \quad 6x + 14y - 1 = 0$
   (*g*) $4x - y - 2 = 0, \quad 12x - 3y = 6$
   (*h*) $y = x, \quad 3x + 4y = 5$

2. Denote ℓ_1 and ℓ_2 to be the lines given by the equations $2x + y - 5 = 0$ and $4x + ky = 6$ respectively, where k is a real number. Suppose that the lines ℓ_1 and ℓ_2 have no point of intersection. Find the value of k.

3. Denote ℓ_1 and ℓ_2 to be the lines given by the equations $6x + py + 3 = 0$ and $2x - 3y + q = 0$ respectively, where p and q are real numbers. Suppose that the lines ℓ_1 and ℓ_2 have infinitely many points of intersection. Find the values of p and q.

4. Denote ℓ_1 and ℓ_2 to be the lines given by the equations $ax + y - 12 = 0$ and $x + by - 6 = 0$ respectively, where a and b are real numbers. Suppose that the lines ℓ_1 and ℓ_2 have no point of intersection. What can you tell about the numbers a and b?

5. Denote ℓ_1 and ℓ_2 to be the lines given by the equations $ax + 2y + 3 = 0$ and $3x + by + 5 = 0$ respectively, where a and b are real numbers. Suppose that the lines ℓ_1 and ℓ_2 intersect at a point on the x-axis. Find the values of a and b.

6. Denote ℓ_1, ℓ_2 and ℓ_3 to be the lines given by the equations $2x + 3y - 6 = 0$, $x + 2y - k = 0$ and $y = x$ respectively, where k is a real number.
   - (a) Suppose that ℓ_1 and ℓ_2 intersect at a point on the x-axis. Find the value of k.
   - (b) Suppose that ℓ_1 and ℓ_2 intersect at a point on the y-axis. Find the value of k.
   - (c) Suppose that ℓ_1 and ℓ_2 intersect at a point belonging to ℓ_3. Find the value of k.
   - (d) Suppose that ℓ_1 and ℓ_2 intersect at a point whose x-coordinate and y-coordinate are positive. Find the possible values of k.