Contents

Chapter 1

Sets and Number Systems

In this chapter, we will introduce concepts related to sets and give a review on properties of real numbers.

1.1 Sets

One of the most fundamental concepts in mathematics is that of a *set*. This concept can be used as a foundation of all known mathematics. Using set language, mathematical concepts can be described precisely and concisely.

In Section 1.1.1, we will introduce the idea of a a set and give definitions and notations for some concepts related to sets. In Section 1.1.2, we will introduce some set operations; these operations will be used in a later chapter on probability.

1.1.1 Basic Concepts

Idea of Definition A ***set*** is a collection of object(s).

Explanation In the above idea of definition, the use of the word '*object(s)*' means that there can be more than one objects, exactly one object or no object. However, writing in this way (adding an *s* in parenthesis) is cumbersome and will not be used in most cases.

Inquisitive readers may ask '*What is a collection? What is an object?*'. To explain what a collection is, one may use words that are "simpler" and have "clearer" meaning. However, this doesn't help. Readers can see why from the following:

Analogy Suppose a small child doesn't know the meaning of the word '*big*'. The following is what he or she obtains from a dictionary.

big, **bigger**, **biggest** Something that is **big** is large in size.

large, **lager**, **largest** Something that is **large** is greater in size than usual or average.

great, **greater**, **greatest** You use **great** to describe something that is very large in size, or unusually large. Great is more formal than big and is used instead of large when you are particularly impressed by the size.

Although you can't learn the meaning of the word 'big' from dictionaries, you "understand" the meaning of the word and you know how to "use" it. This is because you have seen the word many times, in many different situations.

In this section, we will give a brief discussion on concepts and notations related to sets. From the terminologies and notations introduced as well as from the examples given, readers will be able to "understand" what a set is and know how to "use" set concepts and notations.

When you continue reading this book, you will probably find something that you don't understand. Don't worry. Read that example or definition etc. again carefully. Relate it with what you have learnt. If you still find it difficult, you may leave it for some time. By reading more examples, you will gain experiences and you may suddenly understand something that you find difficult before. Remember, you know the meaning of 'big'.

Terminology 1.1.1 An object in a set is called an ***element*** or a ***member*** of the set.

To specify a set, we can state what elements are in the set. For example,

- The set whose elements are 1, 2 and 3.

  In mathematics, numbers are considered to be objects (although they don't have sizes and masses).

Besides using words, we will introduce two methods, called the *listing method* and *description method*, to specify (or to represent) a set. The listing method is to list the elements of the set and the description method is to describe a common property of the elements of the set.

Listing (I) To denote a set with finitely many elements, we list the elements of the set and enclose them by braces.

Example 1.1.1 The set whose elements are 1, 2 and 3 is denoted by

$$\{1, 2, 3\}$$

Remark We may represent the set $\{1, 2, 3\}$ by a diagram like that in Figure 1.1.1. The dots represent the objects and the ellipse can be considered as a bag. A set can be considered as a bag together with the objects inside. To represent a bag, we may use a circle, an ellipse, a rectangle or other geometric figures. In written style, the bag is denoted by a pair of braces.

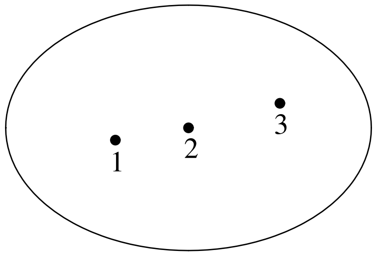

Figure 1.1.1

Some students may ask whether we can represent the set in Example 1.1.1 using notations like $[1, 2, 3]$ or $(1, 2, 3)$ or $\boxed{1, 2, 3}$. Although such notations are quite creative, other people may not understand. Suppose, instead of using Arabic numerals, I have my own way of writing numbers and I show you the following

$$\beta\iota\epsilon + \zeta\theta = \gamma\epsilon\beta$$

do you know the meaning of the calculation?

For a set with 100 elements (for example), it is impractical to write down all its elements. If there is a "pattern" for the elements, we use three dots '...' (read *and so on*) to represent the elements that are not written down.

Listing (II) To denote a set with finitely many elements having a pattern, we list the first few elements of the set, add three dots '...', list the last few elements of the set and enclose everything by braces.

Example 1.1.2 The set whose elements are the first one hundred positive integers is denoted by

$$\{1, 2, 3, \ldots, 99, 100\} \qquad \text{or simply} \qquad \{1, 2, 3, \ldots, 100\}.$$

Explanation The three dots '...' means that the pattern is repeated up to the number(s) listed at the end.

Remark It is difficult to tell what a pattern is. A pattern may be obvious to a student but not obvious to another student. In writing down the first few elements, use at least 3 elements.

For a set with infinitely many elements, it is impossible to write down all its elements. If there is a "pattern" for the elements, we also use three dots '$\ldots$' to represent the elements that are not written down.

Listing (III) To denote a set with infinitely many elements having a pattern, we list the first few elements of the set, add three dots '$\ldots$' and enclose everything by braces.

Example 1.1.3 The set whose elements are the even positive integers greater than 10 is denoted by

$$\{12, 14, 16, 18, 20, \ldots\} \qquad \text{or simply} \qquad \{12, 14, 16, \ldots\}$$

Explanation The three dots '$\ldots$' means that the pattern is repeated indefinitely.

Suppose in a problem, we consider a set, say $\{1, 2, 3, 4, 5\}$. We may have to refer to the set later many times. Instead of writing $\{1, 2, 3, 4, 5\}$ repeatedly, we can give it a name by using a symbol to represent the set. Usually, we use small letters (eg. a, b, c, d) to denote objects and capital letters (eg. A, B, C) to denote sets. For example, we may write

- "Denote A to be $\{1, 2, 3, 4, 5\}$."

which means that the set $\{1, 2, 3, 4, 5\}$ is given the "name" A. If we want to refer to the set later, we can just write A.

Example 1.1.4 Denote A to be $\{1, 2, 3, 4, 5\}$. Then 5 is an element of A but 6 is not an element of A.

Remark The situation in Example 1.1.4 can be represented by Figure 1.1.2. The object 6 is outside the ellipse; this indicates that 6 is not an element of the set A.

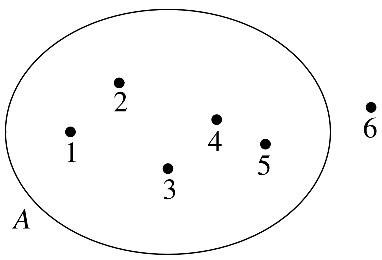

Figure 1.1.2

In junior forms, students have seen that letters like a, b, c or x, y can be used to denote numbers that are arbitrary. For example, the following is a property of squares of real numbers.

- For every real number a, we always have $a^2 \geq 0$.

Similarly, to denote objects that are arbitrary, small letters like a, b, c can be used; and to denote sets that are arbitrary, capital letters like A, B, C can be used. Thus a symbol like A may denote a specific set (see Example 1.1.4) or an arbitrary set (see Notation 1.1.2).

Suppose that x is an object and A is a set. Then either x is an element of A or x is not an element of A. There are special notations to denote these two cases.

Notation 1.1.2 For an object x and a set A,

(1) we write $x \in A$ (read x *belongs to* A) to mean that x is an element of A;

(2) we write $x \notin A$ (read x *does not belong to* A) to mean that x is not an element of A.

Example 1.1.5 Denote A to be $\{1, 2, 3, 4, 5\}$. Then $5 \in A$ but $6 \notin A$.

Remark The results given in Example 1.1.4 and Example 1.1.5 are the same. With the use of the symbols $\in$ and $\notin$, Example 1.1.5 is very concise.

Suppose in a problem, we have already denoted A to be the set $\{1, 2, 3, 4, 5\}$ and suppose we consider another set, say $\{1, 2, 3, \ldots, 100\}$, and we want to give it a name, we must not use the symbol A again; otherwise, the meaning of the symbol is ambiguous.

Example 1.1.6 Denote A to be $\{1, 2, 3, 4, 5\}$ and denote B to be $\{1, 2, 3, \ldots, 100\}$. Then every element of A is (also) an element of B.

In Example 1.1.6, omitting the word '*also*' doesn't change the meaning of the statement '*every element of A is also an element of B*'. Similarly, in Example 1.1.5, the word '*but*' can be replaced by '*and*'.
In written or spoken English, adding the word '*also*' and using '*but*' instead of '*and*' make the description more interesting and carry some judgment or emotion of the writer or speaker. We will not use such words when we write down definitions and theorems etc. (if mathematics depends on emotions, it will not be useful).

In primary schools, students have learnt the meaning of the equality sign '='. For example, in $1+2 = 5-2$, the numbers obtained from the left-side and right-side of the equality sign are the same. Similarly, we also use $A = B$ to mean that the sets on the left-side and right-side of the equality sign are the same (having same elements).

Definition 1.1.3 Let A and B be sets. We say that ***A and B are equal***, written $A = B$, to mean that every element of A is an element of B and vice versa.

Explanation In the definition, the first sentence describes the general setting. It tells that the definition for *equal* applies to sets.

The word *let* is used very often in mathematics. It means *suppose*. The first sentence can also be written as '*Suppose A and B are sets*'. In this book, we will write '*let* ... *be* ...' to describe general settings for definitions and theorems etc.

The use of different letters A and B and the plural '*sets*' does not rule out the possibility that A and B are the same set. We may imagine the following situation:

> There is a pool of sets. First we pick a set randomly and call it A. The set A is put back into the pool. Next we pick a set randomly and call it B. The set B may be different from A or it may be the same as A.

Example 1.1.7 Denote A to be $\{3, 5, 7\}$ and denote B to be the set whose elements are the first three odd prime numbers. Then $A = B$.

Explanation A *prime number* is a positive integer that is divisible by exactly two positive integers. Thus 1 is not a prime number, 2 is a prime number, 6 is not a prime number etc.

If we denote A to be the set $\{3, 5, 7\}$, then A and $\{3, 5, 7\}$ are the same set. Thus we may write $A = \{3, 5, 7\}$. In view of this, the sentence '*Denote A to be* $\{3, 5, 7\}$' is also written as '*Denote* $A = \{3, 5, 7\}$'. This kind of shorthand will be used whenever it is convenient.

Example 1.1.8 Denote $A = \{1, 2, 3, 4, 5\}$ and denote $B = \{3, 1, 2, 5, 4\}$. Then $A = B$.

Remark In listing elements of a set, order is not important. Moreover, there is no need to repeat the elements; we should avoid writing $\{1, 2, 3, 4, 5, 2, 1\}$, for example.

Definition 1.1.4 Let A and B be sets. We say that ***A is a <u>subset</u> of B***, written $A \subseteq B$, to mean that every element of A is an element of B.

Remark Instead of $A \subseteq B$, many authors write $A \subset B$ to denote A is a subset of B.

Example 1.1.9 Denote $A = \{1, 2, 3, 4, 5\}$ and denote $B = \{1, 2, 3, \ldots, 100\}$. Then $A \subseteq B$.

Remark The result in Example 1.1.9 is the same as that in Example 1.1.6.

To indicate that a set A is a subset of a set B, we may use a diagram like that in Figure 1.1.3.

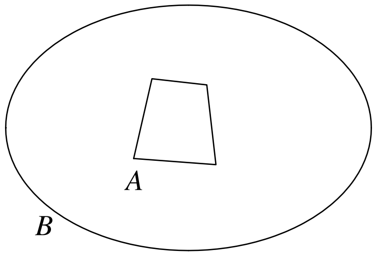

Figure 1.1.3

Example 1.1.10 Denote O to be the set consisting of all odd positive integers and denote P to be the set consisting of all prime numbers greater than 2. Then $P \subseteq O$.

Definitions can be used in two ways. For example, we can use Definition 1.1.3 as follows:

(1) If it is given that sets A and B are equal, then we know that every element of A is an element of B and every element of B is an element of A.

(2) If we can show that every element of a set A is an element of a set B and every element of B is an element of A, then we know that the sets A and B are equal.

It is clear from Definition 1.1.3 and Definition 1.1.4 that if $A = B$, then $A \subseteq B$ and $B \subseteq A$. The following example means that the converse is true.

Example 1.1.11 Let A and B be sets. Suppose that $A \subseteq B$ and $B \subseteq A$. Show that $A = B$.

Proof Since $A \subseteq B$, it follows from Definition 1.1.4 that

(1) every element of A is an element of B.

Similarly, since $B \subseteq A$, it follows that

(2) every element of B is an element of A.

In view of (1) and (2), by Definition 1.1.3, we have $A = B$. □

Remark We use the notation □ to denote the *end of proof* (of theorem etc.) or *end of solution* (to examples).

Example 1.1.11 illustrates how we use definition to prove mathematical results. Some students may think that the result is obvious and thus does not need a proof. Sometimes, mathematicians also say "*obvious*" or "*clear*" in proofs of theorems (see the discussion preceding Example 1.1.11). To some people, a result may be obvious; but, it may not be obvious to other people. If you say obvious, make sure that it is really obvious— if your classmates ask you why, you should be able to explain to them.

There is another method (called *description*) to represent sets. The method is used to specify a subset of a set by describing a common property for the elements in that subset.

Denote $A = \{1, 2, 3, 4, 5\}$. Note that some elements of A are odd numbers and some are not odd numbers. Now consider the following

(∗) x is an odd number

If we substitute $x = 1, 3$ or 5 in (∗), then we get a true statement; if we substitute $x = 2$ or 4 in (∗), then we get a false statement. Thus, if we substitute x by any element of A, we get either a true statement or a false statement. We call (∗) a *property for elements of A* (with variable x). The set consisting of all the elements x of A such that x is an odd number is denoted by

$$\{x \in A : x \text{ is an odd number}\}.$$

Description For a set A and a property $P(x)$ for elements of A, we write

$$\{x \in A : P(x)\}$$

to denote the set consisting of all the elements x of A for which $P(x)$ is true.

Remark Instead of $\{x \in A : P(x)\}$, some authors write $\{x \in A | P(x)\}$. Both notations are read *the set of all x belonging to A such that P(x)*.

Example 1.1.12 Denote $A = \{1, 2, 3, 4, 5\}$. Denote $B = \{x \in A : x \text{ is an odd number}\}$. Then $B = \{1, 3, 5\}$.

Remark In the notation $\{x \in A : x \text{ is an odd number}\}$, the symbol x is called a *dummy variable*. We can use any symbol to denote a dummy variable. For example, the set B can be written as $\{a \in A : a \text{ is an odd number}\}$.

There are some special sets that we want to have special notations.

Definition 1.1.5 We call the ***the empty set***, and write $\{\,\}$, to mean the set that has no element.

Explanation The empty set has no element, if we list all the elements of the empty set and enclose "them" by braces, we get $\{\,\}$.

Remark To denote the empty set, many authors write $\emptyset$ (a Scandinavian letter—a zero 0 together with a slash /).

Notation 1.1.6 We denote

(1) $\mathbb{Z}$ to be the set of all integers;

(2) $\mathbb{Z}^+$ to be the set of all positive integers;

(3) $\mathbb{Q}$ to be the set of all rational numbers;

(4) $\mathbb{Q}^+$ to be the set of all positive rational numbers;

(5) $\mathbb{R}$ to be the set of all real numbers;

(6) $\mathbb{R}^+$ to be the set of all positive real numbers;

(7) $\mathbb{C}$ to be the set of all complex numbers.

The above sets will be discussed in more details in later sections. Note that

$$\mathbb{Z}^+ = \{x \in \mathbb{Z} : x > 0\}, \qquad \mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}, \qquad \mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$$

In mathematics, the set of all integers, the set of all real numbers etc. are important. We want to have specific symbols to denote the sets. If we use R to denote the set of all real numbers, then whenever we see the symbol R, we know its meaning. However, this will decrease the number of symbols that we can use. In many books printed before the computer age, the set of all real numbers is denoted by $\mathbf{R}$ (boldface). Using this convention, we can use R (usual capital letter) to denote other objects. Since it is inconvenient to write boldface letters on blackboard, mathematicians write the boldface $\mathbf{R}$ as $\mathbb{R}$. The font for such handwritten boldface is called *blackboard boldface*. Nowadays, using computers, many fancy fonts can be generated. The most commonly used blackboard boldface font for the letters Z,Q,R and C are the ones given in Notation 1.1.6. On paper, we can write $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$.

Example 1.1.13 In the following table, the column on the left gives some results written using notations introduced in this section and the column on the right gives the corresponding meaning in daily language.

(1)	$2 \in \mathbb{Z}^+$	(1′)	2 is a positive integer.
(2)	$1.5 \notin \mathbb{Z}$	(2′)	1.5 is not an integer.
(3)	$3 \in \mathbb{Q}$	(3′)	3 is a rational number.
(4)	$\pi \notin \mathbb{Q}$	(4′)	π is not a rational number.
(5)	$\mathbb{Z} \subseteq \mathbb{Q}$	(5′)	Every integer is a rational number.

Explanation A rational number is a number that can be written as $\frac{m}{n}$ where m and n are integers and $n \neq 0$. Note that $3 = \frac{3}{1}$.

Example 1.1.14 Use listing to rewrite $\{x \in \mathbb{Z}^+ : x^2 + 4 = 13\}$.

Explanation The given set consists of all positive integers x such that $x^2 + 4 = 13$. To find the set, we first solve the equation and then take the solution(s) that is/are positive integer(s).

Solution Solving

$$\begin{aligned} x^2 + 4 &= 13 \\ x^2 &= 9 \end{aligned}$$

we get $x = 3$ or $x = -3$. Note that $3 \in \mathbb{Z}^+$ but $-3 \notin \mathbb{Z}^+$.

The given set is $\{3\}$. □

Example 1.1.15 Use listing to rewrite $\{x \in \mathbb{Z} : 4x^2 - 5 = 20\}$.

Explanation The method is similar to that in Example 1.1.14. Since the solutions to the given equation are not integers, the given set has no element.

Solution Solving

$$\begin{aligned} 4x^2 - 5 &= 20 \\ 4x^2 &= 25 \\ (2x)^2 &= 25 \end{aligned}$$

we get $2x = 5$ or $2x = -5$ which gives $x = \frac{5}{2}$ or $x = -\frac{5}{2}$.

Note that $\frac{5}{2} \notin \mathbb{Z}$ and $-\frac{5}{2} \notin \mathbb{Z}$.

The given set is $\{\,\}$. □

Example 1.1.16 Use listing to rewrite $\{x \in \mathbb{Z}^+ : x \text{ is a factor of } 6\}$.

Explanation We say that a positive integer a is a *factor* of a positive integer n to mean that there is a positive integer b such that $n = a \cdot b$, that is, $\frac{n}{a}$ is a positive integer.

Solution The factors of 6 are $1, 2, 3$ and 6.

The given set is $\{1, 2, 3, 6\}$. □

Example 1.1.17 Use listing to rewrite the following set:

$$\{x \in \mathbb{Q} : x = \frac{m}{n} \text{ for some } m, n \in \mathbb{Z}^+ \text{ with } m, n < 3\}$$

Explanation "$m, n \in \mathbb{Z}^+$ with $m, n < 3$" means that m and n are positive integers less than 3. Thus m and n can be 1 or 2.

"$x = \frac{m}{n}$ for some $m, n \in \mathbb{Z}^+$ with $m, n < 3$" means that there exist positive integers m, n less that 3 such that $x = \frac{m}{n}$.

The elements in the given set are the rational numbers x such that x can be written as $\frac{m}{n}$ where $m, n = 1$ or 2.

Solution When $m = 1, n = 1$, we have $\frac{m}{n} = 1$; when $m = 1, n = 2$, we have $\frac{m}{n} = \frac{1}{2}$; when $m = 2, n = 1$, we have $\frac{m}{n} = 2$; when $m = 2, n = 2$, we have $\frac{m}{n} = 1$.

The given set is $\{\frac{1}{2}, 1, 2\}$. □

Example 1.1.18 Determine whether the following statement is true or false:

For every $m, n \in \mathbb{Z}^+$, we have $m - n \in \mathbb{Z}^+$.

If it is false, give a counterexample.

Explanation The statement means that for every positive integer m and for every positive integer n, the difference $m - n$ is (always) a positive integer.

To show that the given statement is false, it is enough to give positive integers a, b such that $a - b$ is not a positive integer. Such an example is called a *counterexample*.

Solution The given statement is false.

Counterexample: $1, 2 \in \mathbb{Z}^+$ but $1 - 2 = -1 \notin \mathbb{Z}^+$. □

To close this section, we consider the famous ancient Chinese paradox

"*White horse isn't horse*"

(1) When we say "*white horses are horses*", we mean "*every white horse is a horse*". Using set language, if we denote W to be the set of all white horses and denote H to be the set of all horses, then the statement "*every white horse is a horse*" means $W \subseteq H$.

(2) The word '*is*' also means "*is equal to*". For example, when we say "*one plus one is two*", we mean "*one plus one is equal to two*", that is, "$1 + 1 = 2$".

(3) Note that $W \neq H$ (there is at least one horse that is not white). Using the interpretation of the word '*is*' as in (2), we obtain the paradox "*W isn't H*".

Exercise 1.1.1

1. For each of the following sets, use listing to rewrite it.
   - (a) $\{x \in \mathbb{Z} : 2x + 3 = 15\}$
   - (b) $\{x \in \mathbb{Z} : x^2 = 4\}$
   - (c) $\{x \in \mathbb{Z}^+ : x^2 + 16 = 25\}$
   - (d) $\{x \in \mathbb{Z} : 10 \leq x \leq 20$ and x is an odd number$\}$
   - (e) $\{x \in \mathbb{Z} : x^2 < 6\}$
   - (f) $\{x \in \mathbb{Z}^+ : x$ is a factor of $18\}$
   - (g) $\{x \in \mathbb{Z}^+ : x < 30$ and x is a multiple of $7\}$
   - (h) $\{x \in \mathbb{Z} : x = n^2$ for some positive integer n less than $7\}$
2. For each of the following statements, rewrite it using set notations as in Example 1.1.13 (left column).
   - (a) -3 is not a positive integer.
   - (b) 1.25 is a rational number.
   - (c) -7 is a rational number.
   - (d) $\sqrt{2}$ is not a rational number.
3. For each of the following statements, determine whether it is true or false.
   - (a) $\dfrac{12}{3} \in \mathbb{Z}^+$
   - (b) $\pi + 1 \in \mathbb{Q}$
   - (c) $(\sqrt{2})^4 \in \mathbb{Z}$
4. For each of the following statements, determine whether it is true or false. If it is false, give a counterexample.
   - (a) For every $m, n \in \mathbb{Z}$, we have $m - n \in \mathbb{Z}$.
   - (b) For every $m, n \in \mathbb{Z}^+$, we have $m \div n \in \mathbb{Z}^+$.
   - (c) For every $a, b \in \mathbb{Q}$ with $b \neq 0$, we have $a \div b \in \mathbb{Q}$.
   - (d) For every $a \in \mathbb{Q}$ with $a > 0$, we have $\sqrt{a} \in \mathbb{Q}$.

1.1.2 Set Operations

For numbers, we have the arithmetic operations: addition, subtraction, multiplication and division. For sets, we also have some operations. In this section, we will discuss three operations on sets, namely, *intersection*, *union* and *complement*.

Figure 1.1.4 shows two sets A and B represented by two circular regions. The shaded region is the common part of A and B; it consists of all everything that are in both A and B and is called the *intersection* of A and B.

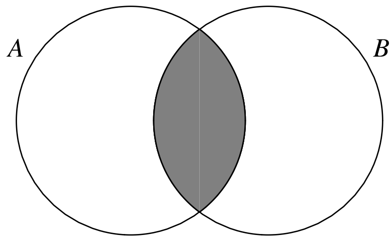

Figure 1.1.4

Definition 1.1.7 Let A and B be sets. We call ***the intersection of** A **and** B*, and write $A \cap B$, to mean the set whose elements are those that belong to both A and B.

Remark It is clear that $A \cap B = B \cap A$.

Example 1.1.19 Denote A and B to be the sets given as follows:

$$A = \{x \in \mathbb{Z}^+ : x \text{ is a factor of } 12\}, \qquad B = \{x \in \mathbb{Z}^+ : x \text{ is a factor of } 18\}$$

Then $A = \{1, 2, 3, 4, 6, 12\}$ and $B = \{1, 2, 3, 6, 9, 18\}$.

Thus $A \cap B = \{1, 2, 3, 6\}$.

Remark $A \cap B$ is the set consisting of all the common factors of 12 and 18.
The largest number in $A \cap B$ is the *H.C.F.* of 12 and 18.

Example 1.1.20 Denote C and D to be the sets given as follows:

$$C = \{x \in \mathbb{Z}^+ : x \text{ is a multiple of } 12\}, \qquad D = \{x \in \mathbb{Z}^+ : x \text{ is a multiple of } 18\}$$

Then $C = \{12, 24, 36, 48, \ldots\}$ and $D = \{18, 36, 54, 72, \ldots\}$.

Thus $C \cap D = \{36, 72, 108, \ldots\}$.

Remark $C \cap D$ is the set consisting of all the common multiples of 12 and 18.
The smallest number in $A \cap B$ is the *L.C.M.* of 12 and 18.

Example 1.1.21 In a certain class, there are five students. Table 1.1.5 shows the grades obtained by five students in two Math tests.

	Chan Tai Man	Li Siu Man	Cheung Wai Yi	Wong Lap Yan	Ho Kwok Yan
Test 1	A	B	A	C	B
Test 2	C	B	A	A	B

Table 1.1.5

If we denote X and Y to be the sets of all students in the class getting an A in Test 1 and Test 2 respectively. Then

$$X = \{\text{Chan Tai Man, Cheung Wai Yin}\}, \quad Y = \{\text{Cheung Wai Yin, Wong Lap Yan}\}$$

Thus $X \cap Y = \{\text{Cheung Wai Yin}\}$; it is the set of all students in the class getting two A's in the two tests.

Example 1.1.22 Denote A and B to be the sets given as follows:

$$A = \{1, 2, 3\}, \quad B = \{4, 5\}$$

Then $A \cap B = \{\,\}$.

Remark We say that *A and B are disjoint*, or *A and B have empty intersection*.

Example 1.1.23 Denote A, B and C to be the sets given as follows:

$$A = \{1, 3, 5, 7, 9\}, \qquad B = \{2, 3, 5, 7\}, \qquad C = \{1, 2, 3\}$$

Find the following sets.

(a) $A \cap B$ (b) $B \cap C$

(c) $(A \cap B) \cap C$ (d) $A \cap (B \cap C)$

Solution (a)
$$A \cap B = \{1, 3, 5, 7, 9\} \cap \{2, 3, 5, 7\} = \{3, 5, 7\}$$

(b)
$$B \cap C = \{2, 3, 5, 7\} \cap \{1, 2, 3\} = \{2, 3\}$$

(c)
$$(A \cap B) \cap C = \{3, 5, 7\} \cap \{1, 2, 3\} = \{3\} \qquad \text{By (1)}$$

(d)
$$A \cap (B \cap C) = \{1, 3, 5, 7, 9\} \cap \{2, 3\} = \{3\} \qquad \text{By (2)}$$

□

Remark In general, we have $(X \cap Y) \cap Z = X \cap (Y \cap Z)$. Thus there is no ambiguity in writing $X \cap Y \cap Z$. We say that set intersection is *associative*.

Figure 1.1.6 shows two sets A and B represented by two circular regions. The shaded region (colored light red or light blue or pink) consists of everything that are in A or B (or both) and is called the *union* of A and B.

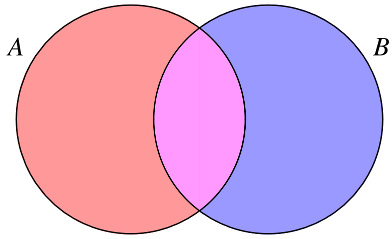

Figure 1.1.6

Remark In logic, usually we use a single letter, for example p, to represent a statement. For statements p and q, the statement '*p or q*' means '*p or q or both*'.

Definition 1.1.8 Let A and B be sets. We call ***the union of A and B***, and write $A \cup B$, to mean the set whose elements are those that belong to A or B.

Example 1.1.24 Consider the sets X and Y given in Example 1.1.21.

By Definition 1.1.8,

$$X \cup Y = \{\text{Chan Tai Man, Cheung Wai Yin, Wong Lap Yan}\};$$

it is the set of all students in the class getting at least one A in the two tests.

Example 1.1.25 Denote A, B and C to be the sets given as follows:

$$A = \{1, 2, 3, 4, 5\}, \qquad B = \{2, 3, 5, 7\}, \qquad C = \{3, 6, 9\}$$

Find the following sets.

(*a*) $A \cup B$ (*b*) $B \cup C$

(*c*) $(A \cup B) \cup C$ (*d*) $A \cup (B \cup C)$

Solution (*a*)
$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5\} \cup \{2, 3, 5, 7\} \\ &= \{1, 2, 3, 4, 5, 7\} \end{aligned}$$

(*b*)
$$\begin{aligned} B \cup C &= \{2, 3, 5, 7\} \cup \{3, 6, 9\} \\ &= \{2, 3, 5, 6, 7, 9\} \end{aligned}$$

(*c*)
$$\begin{aligned} (A \cup B) \cap C &= \{1, 2, 3, 4, 5, 7\} \cup \{3, 6, 9\} && \text{By (1)} \\ &= \{1, 2, 3, 4, 5, 6, 7, 9\} \end{aligned}$$

(*d*)
$$\begin{aligned} A \cup (B \cup C) &= \{1, 2, 3, 4, 5\} \cup \{2, 3, 5, 6, 7, 9\} && \text{By (2)} \\ &= \{1, 2, 3, 4, 5, 6, 7, 9\} \end{aligned}$$

□

Remark In general, we have $(X \cup Y) \cup Z = X \cup (Y \cup Z)$. Thus there is no ambiguity in writing $X \cup Y \cup Z$. We say that set union is *associative*.

Example 1.1.26 Denote A, B and C to be the sets given as follows:

$$A = \{1, 2, 3, 4, 5\}, \qquad B = \{4, 5, 6\}, \qquad C = \{1, 2\}$$

Find the following sets.

(a) $(A \cup B) \cap C$

(b) $A \cup (B \cap C)$

Explanation For (1), we have to find $A \cup B$ first. For (2), we have to find $B \cap C$ first.

Solution (a)

$$\begin{aligned} A \cup B &= \{1, 2, 3, 4, 5\} \cup \{4, 5, 6\} \\ &= \{1, 2, 3, 4, 5, 6\} \end{aligned}$$

Thus

$$\begin{aligned} (A \cup B) \cap C &= \{1, 2, 3, 4, 5, 6\} \cap \{1, 2\} \\ &= \{1, 2\} \end{aligned}$$

(b)

$$\begin{aligned} B \cap C &= \{4, 5, 6\} \cap \{1, 2\} \\ &= \{\,\} \end{aligned}$$

$$\begin{aligned} A \cup (B \cap C) &= \{1, 2, 3, 4, 5\} \cup \{\,\} \\ &= \{1, 2, 3, 4, 5\} \end{aligned}$$

□

Remark The example illustrates that $X \cup (Y \cap Z)$ and $(X \cup Y) \cap Z$ may not be equal. Thus, the expression $X \cup Y \cap Z$ is ambiguous (unless we have a convention for the order of set operations).

Figure 1.1.7 shows two sets A and B. The shaded region consists of all everything that are in A but not in B and is called the *relative complement* of B in A.

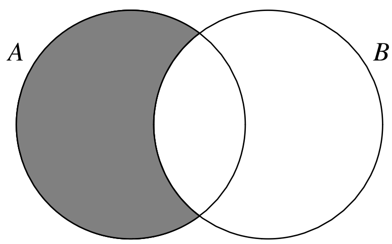

Figure 1.1.7

Definition 1.1.9 Let A and B be sets. We call ***the relative complement of B in A***, and write $A \setminus B$, to mean $\{x \in A : x \notin B\}$.

Example 1.1.27 Consider the sets X and Y given in Example 1.1.21. Then

$$X \setminus Y = \{\text{Chan Tai Man}\}$$

It is the set of all students in the class getting an A in Test 1 and not an A in Test 2.

Example 1.1.28 Denote A and B to be the sets given as follows:

$$A = \{1, 2, 3, 4, 5\}, \qquad B = \{4, 5, 6, 7\}$$

Find the following sets.

(a) $A \setminus B$

(b) $B \setminus A$

Solution (a)

$$A \setminus B = \{1, 2, 3, 4, 5\} \setminus \{4, 5, 6, 7\} = \{1, 2, 3\}$$

(b)

$$B \setminus A = \{4, 5, 6, 7\} \setminus \{1, 2, 3, 4, 5\} = \{6, 7\}$$

□

Figure 1.1.8 shows two sets A and B with A represented by the larger circular region and B by the smaller one. Note that the smaller circle is inside the larger circle—it means that B is a subset of A. The shaded region represents $A \setminus B$, that is, the relative complement of B in A.

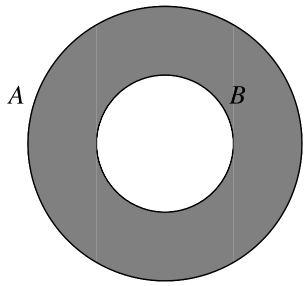

Figure 1.1.8

For the case where $B \subseteq A$, the set $A \setminus B$ is simply called the *complement* of B in A.

Terminology Let A and B be sets with $B \subseteq A$. We call *the complement of B in A* to mean the relative complement of B in A.

Example 1.1.29 Denote X, Y, Z and B to be the sets given as follows:

$$X = \{1, 2, 3, 4, 5\}, \qquad Y = \{4, 5, 6, 7\}, \qquad Z = \{3, 4, 5\}, \qquad B = \{3, 4\}$$

Find the following sets.

(a) The complement of B in X.

(b) The relative complement of B in Y.

(c) The complement of B in Z.

Solution (a)

$$\text{The complement of } B \text{ in } X = \{1, 2, 3, 4, 5\} \setminus \{3, 4\} = \{1, 2, 5\}$$

(b)

$$\text{The relative complement of } B \text{ in } Y = \{4, 5, 6, 7\} \setminus \{3, 4\} = \{5, 6, 7\}$$

(c)

$$\text{The complement of } B \text{ in } Z = \{3, 4, 5\} \setminus \{3, 4\} = \{5\}$$

□

Remark We can't call $Y \setminus B$ *the complement of B in Y*; this is because B is not a subset of Y.

When we consider complement of a set, we have to tell *complement in which set* (see the first and third part in the last example). However, sometimes we consider a set that contains every objects that we are interested in. Such a set is called a *universal set*. For example, when primary school students learn how to count and how to do addition and multiplication, the set of all counting numbers $\mathbb{Z}^+$ can be taken as a universal set.

Convention If a set U is taken to be the universal set for a problem, then for every set S (note that S is a subset of U), *the complement of S*, denoted by S', means the complement of S in U, that is, $S' = U \setminus S$.

Remark Instead of S', some authors write $\overline{S}$ or S^c to denote the complement of S.

Example 1.1.30 For the tests results considered in Example 1.1.21, take the universal set to be

$$\{\text{Chan Tai Man, Li Siu Man, Cheung Wai Yi, Wong Lap Yan, Ho Kwok Yan}\}$$

Then for the set X given in the example, X' is the set of all students who didn't get an A in Test 1, that is,

$$X' = \{\text{Li Siu Man, Wong Lap Yan, Ho Kwok Yan}\}$$

Example 1.1.31 Take $\mathbb{Z}^+$ to be the universal set. Denote A to be the set of all odd positive integers. Then A' is the set of all even positive integers.

To consider relation between sets, we can use diagrams like that given in Figure 1.1.4, for example. Such a diagram is called a *Venn diagram*. For problems having a universal set, the universal set is usually represented by a rectangular region and subsets of the universal set are represented by geometric objects (for example, circular or elliptical regions) contained in the rectangular region.

- In the following two figures, U is the universal set and A (represented by the circular region) is a subset of U.

  (a) In Figure 1.1.9 (a), the shaded region is A.

  (b) In Figure 1.1.9 (b), the shaded region is A'.

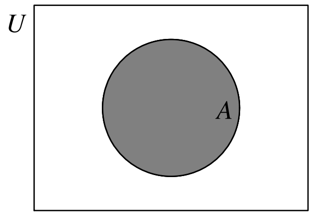

Figure 1.1.9 (a)

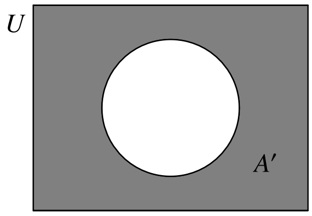

Figure 1.1.9 (b)

  Note that $A \cap A' = \{\}$ and $U = A \cup A'$. We say that U is the *disjoint union* of A and A'.

- In the following four figures, U is the universal set and A and B (represented by the circular regions on the left and right respectively) are subsets of U.

  (a) In Figure 1.1.10 (a), the shaded region is $A \setminus B$.

  (b) In Figure 1.1.10 (b), the shaded region is $A \cap B$.

  (c) In Figure 1.1.10 (c), the shaded region is $B \setminus A$.

  (d) In Figure 1.1.10 (d), the shaded region is $(A \cup B)'$.

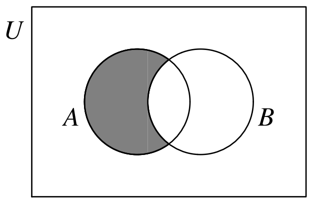

Figure 1.1.10 (a)

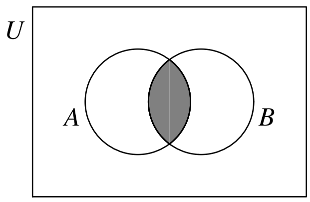

Figure 1.1.10 (b)

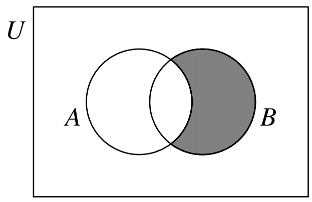

Figure 1.1.10 (c)

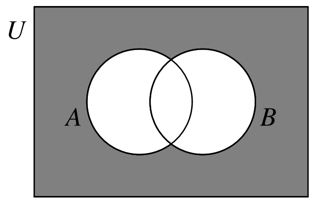

Figure 1.1.10 (d)

  Note that A is the disjoint union of $A \setminus B$ and $A \cap B$, B is the disjoint union of $B \setminus A$ and $A \cap B$ and that U is the disjoint union of $A \setminus B$, $A \cap B$, $B \setminus A$ and $(A \cap B)'$.

Example 1.1.32 Take the universal set U to be the set of all positive integers less than 10. Denote A and B to be the sets given as follows:

$$A = \{2, 3, 6, 8\}, \qquad B = \{6, 7, 8, 9\}$$

Find the following sets.

(a) $A' \cap B'$

(b) $(A \cup B)'$

Solution Note that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

(a)
$$\begin{aligned} A' &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{2, 3, 6, 8\} \\ &= \{1, 4, 5, 7, 9\} \end{aligned}$$

$$\begin{aligned} B' &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{6, 7, 8, 9\} \\ &= \{1, 2, 3, 4, 5\} \end{aligned}$$

Thus
$$\begin{aligned} A' \cap B' &= \{1, 4, 5, 7, 9\} \cap \{1, 2, 3, 4, 5\} \\ &= \{1, 4, 5\} \end{aligned}$$

(b)
$$\begin{aligned} A \cup B &= \{2, 3, 6, 8\} \cup \{6, 7, 8, 9\} \\ &= \{2, 3, 6, 7, 8, 9\} \end{aligned}$$

Thus
$$\begin{aligned} (A \cup B)' &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \setminus \{2, 3, 6, 7, 8, 9\} \\ &= \{1, 4, 5\} \end{aligned}$$

□

Remark Note that $(A \cup B)' = A \cap B'$.
In general, we have $(X \cup Y)' = X' \cap Y'$ and $(X \cap Y)' = X' \cup Y'$. These two results are known as the *De Morgan's Laws*.

To close this section, we discuss a result known as the *Inclusion-exclusion Principle*.

Notation 1.1.10 Let A be a set with finitely many elements. We write $n(A)$ to denote the number of elements of A.

Inclusion-exclusion Principle Let A and B be sets with finitely many elements. Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

Proof Denote $p = n(A \setminus B)$, $q = n(B \setminus A)$ and $r = n(A \cap B)$.

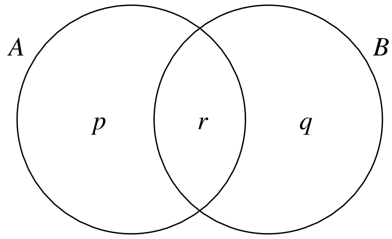

Figure 1.1.11

Since the sets $A \setminus B$, $B \setminus A$ and $A \cap B$ are pairwise disjoint, it follows that

$$n(A) = p + r, \qquad n(B) = q + r, \qquad n(A \cup B) = p + q + r$$

Hence
$$\begin{aligned} n(A) + n(B) - n(A \cap B) &= (p + r) + (q + r) - r \\ &= p + q + r \\ &= n(A \cup B) \end{aligned}$$

□

Example 1.1.33 A group of 200 students took exams on Chinese and English. It is known that 172 students passed Chinese, that 163 students passed English and that 149 students passed both Chinese and English. How many students

(1) passed at least one subjects;

(2) passed exactly one subject;

(3) failed both subjects.

Solution Denote U to be the set of all students in the group. Denote C and E to be the sets of all students who passed Chinese and English respectively.
It is given that

$$n(U) = 200, \quad n(C) = 172, \quad n(E) = 163, \quad n(C \cap E) = 149$$

(1) The set of all students who passed at least one subjects is $C \cup E$.

By the Inclusion-exclusion Principle,

$$\begin{aligned} n(C \cup E) &= n(C) + n(E) - n(C \cap E) \\ &= 172 + 163 - 149 \\ &= 186 \end{aligned}$$

186 students passed at least one subject.

(soln cont'd) (*b*) The set of all students who passed exactly one subject is the set $(C \cup E) \setminus (C \cap E)$

Note that $C \cap E \subseteq C \cup E$. Thus

$$\begin{aligned} n\big((C \cup E) \setminus (C \cap E)\big) &= n(C \cup E) - n(C \cap E) \\ &= 186 - 149 && \text{By (1)} \\ &= 37 \end{aligned}$$

37 students passed exactly one subject.

(*c*) The set of all students who failed both subjects is $(A \cup B)'$.

Note that $A \cup B \subseteq U$. Thus

$$\begin{aligned} n\big((A \cup B)'\big) &= n(U) - n(A \cup B) \\ &= 200 - 186 && \text{By (1)} \\ &= 14 \end{aligned}$$

14 students failed both subjects. □

Remark Alternatively, we can use "common sense" to obtain the following

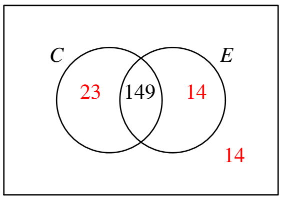

Figure 1.1.12

Exercise 1.1.2

1. Denote $A = \{2, 3, 5, 7\}$, denote $B = \{2, 4, 6, 8\}$ and denote $C = \{3, 4, 5\}$. Find the following sets.

(*a*) $A \cap B$ (*b*) $A \cap C$ (*c*) $B \cap C$

(*d*) $A \cup B$ (*e*) $A \cup C$ (*f*) $B \cup C$

(*g*) $A \cap B \cap C$ (*h*) $A \cup B \cup C$

(*i*) $(A \cup B) \cap C$ (*j*) $(A \cap C) \cup (B \cap C)$

(*k*) $(A \cap B) \cup C$ (*l*) $(A \cup C) \cap (B \cap C)$

(*m*) $A \setminus (B \cap C)$ (*n*) $(A \setminus B) \cup (A \setminus C)$

(*o*) $A \setminus (B \cup C)$ (*p*) $(A \setminus B) \cap (A \setminus C)$

2. Denote A and B to be the sets given as follows:

$$\begin{aligned} A &= \{x \in \mathbb{Z}^+ : x \text{ is an even number and } x \leq 12\} \\ B &= \{x \in \mathbb{Z}^+ : x \text{ is a multiple of 3 and } x < 12\} \end{aligned}$$

Find the following sets.

(a) A	(b) B	(c) $A \cap B$
(d) $A \cup B$	(e) $A \setminus B$	(f) $B \setminus A$

3. Take the universal set to be $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Denote $A = \{x \in U : x \text{ is a prime number}\}$ and denote $B = \{x \in U : x \text{ is divisible by } 60\}$. Find the following sets.

(a) A	(b) B	(c) A'	(d) B'
(e) $A \cap B$	(f) $A \cup B$	(g) $A \setminus B$	(h) $B \setminus A$
(i) $(A \cup B)'$	(j) $A' \cap B'$	(k) $(A \cap B)'$	(l) $A' \cup B'$

4. In a survey of 100 students, a marketing questionnaire included the following three questions:
   (I) Do you own a mobile phone?
   (II) Do you own a computer?
   (III) Do you own a mobile phone and a computer?

   95 students answered yes to (I), 72 students answered yes to (II) and 70 students answered yes to (III).

   (a) How many students own either a mobile phone or a computer or both?
   (b) How many students own a computer but not a mobile phone?
   (c) How many students own a mobile phone but not a computer?
   (d) How many students own neither a mobile phone nor a computer?

5. Find the number of integers in the set $\{x \in \mathbb{Z}^+ : x \leq 999\}$ that are divisible by both 21 and 35.

6. (a) Let A, B and C be finite sets. Show that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A\ B) - n(A\ C) - n(B\ C) + n(A\ B\ C)$$

   (b) Find the number of positive integers less than $10,000$ that are divisible by 7, 15 or 24.

1.2 The Real Number System

A number system is a set of numbers together with some arithmetic operations that fit together in a natural way. In this section, before considering real numbers in general, we give some reviews on counting numbers, fractions, integers and rational numbers.

(1) In the set of all counting numbers, we have the arithmetic operations addition + and multiplication $\times$.

(2) In the set of all fractions, besides addition and multiplication, we also have division $\div$.

(3) In the set of all integers, besides addition and multiplication, we also have subtraction −.

(4) In the set of all rational numbers, besides addition and multiplication, we also have subtraction and division (by non-zero numbers).

(5) In the set of all real numbers, besides addition, subtraction, multiplication and division (by non-zero numbers), we can also take square roots for positive numbers.

1.2.1 Counting Numbers

Ancient people used tools such as stones and knots on strings to record the number of objects. To count the number of objects, we use $1, 2, 3, 4$ and so on. These numbers are called ***counting numbers***. Since counting numbers are just integers that are positive (see later discussion), we use $\mathbb{Z}^+$ to denote the set of all counting numbers.

Remark Some authors call counting numbers as *natural numbers*. However, some authors also include 0 as a natural number. Because of this ambiguity, we will not use the term *natural number* in this course.

In $\mathbb{Z}^+$, we have arithmetic operations called *addition* (denoted by $+$) and *multiplication* (denoted by $\times$ or $\cdot$). Readers are assumed to be familiar with properties of these two operations.

For counting numbers a and b, the notation $a < b$ means that a is smaller than b (in quantity); it is equivalent to $b > a$ which means that b is greater than a. The notation $a \leq b$ means $a < b$ or $a = b$ and the notation $b \geq a$ means $b > a$ or $b = a$.

- Subtraction is the *reverse* process of addition. Suppose that a and b are counting numbers with $b > a$, then $b - a$ is defined to be the unique solution to the following equation:

$$a + x = b \tag{1.2.1}$$

  For example, if $a = 2$ and $b = 7$, the equation $2 + x = 7$ has one and only one solution, namely 5; we define $7 - 2$ to be 5. For the cases where $b < a$ or $b = a$, there is no counting number x satisfying (1.2.1). For example, if $a = 7$ and $b = 2$, the equation $7 + x = 2$ does not have any solution in $\mathbb{Z}^+$ (that is, there is no counting number x satisfying the equation). *How can we subtract* 7 *cattle from* 2 *cattle?*

- Division is the reverse process of multiplication. Suppose a and b are counting numbers, then there is at most one counting number x satisfying

$$a \cdot x = b$$

For example, if $a = 5$ and $b = 20$, the equation $5x = 20$ has one and only one solution, namely 4. We define $20 \div 5$ to be 4. On the other hand, if $a = 6$ and $b = 20$, the equation $6x = 20$ does not have any solution in $\mathbb{Z}^+$ (that is, there is no counting number x satisfying the equation). *How can we divide* 20 *cattle among* 6 *people evenly?*

1.2.2 Fractions

Suppose a cake is divided into 3 equal pieces and we take 1 piece. Then we get $\frac{1}{3}$ of a cake.

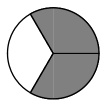

Figure 1.2.1

Suppose there are 7 cakes each of which is divided into 4 equal pieces and we take 1 piece from each cake. Then we get $\frac{7}{4}$ or $1\frac{3}{4}$ of a cake.

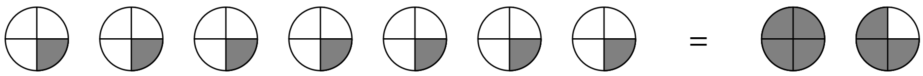

Figure 1.2.2

Numbers like $\frac{1}{3}$ and $\frac{7}{4}$ are called *fractions*. A ***fraction*** is a number that can be written in the form $\frac{m}{n}$ where m and n are counting numbers. For the fractions $\frac{2}{3}$ and $\frac{4}{6}$, although they have different forms, they are considered to be equal, written $\frac{2}{3} = \frac{4}{6}$.

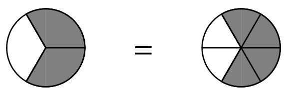

Figure 1.2.3

Since fractions are just rational numbers that are positive (see later discussion), we use $\mathbb{Q}^+$ to denote the set of all fractions. Note that every counting number is a fraction (that is, $\mathbb{Z}^+ \subseteq \mathbb{Q}^+$). For example, $2 = \frac{8}{4}$ is a fraction.

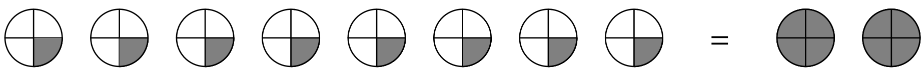

Figure 1.2.4

In $\mathbb{Q}^+$, besides addition and multiplication, we can also define division. For fractions $\frac{m}{n}$ and $\frac{p}{q}$ (where m, n, p, q are counting numbers), we define

$$\frac{p}{q} \div \frac{m}{n} = \frac{p}{q} \cdot \frac{n}{m}$$

which is the unique solution to the equation $\frac{m}{n} \cdot x = \frac{p}{q}$.

Remark If m and n are counting numbers, then $m \div n = \frac{m}{n}$ (for example, $3 \div 2 = \frac{3}{1} \div \frac{2}{1} = \frac{3}{1} \cdot \frac{1}{2} = \frac{3}{2}$). For this reason, fractions are also called *quotients*.

1.2.3 Integers

To denote the quantity "nothing", we use the number 0. For example, "*I have no cattle*" can be written as "*I have* 0 *cattle*". To denote "quantity n in the opposite direction", we use the number $-n$. For example, "*I owe* 5 *cattle*" can be written as "*I have* -5 *cattle*".

Counting numbers, 0 and numbers in the form $-n$ (where n is a counting number) are called ***integers***. The set of all integers is denoted by $\mathbb{Z}$ which stands for *Zhalen* (German word for numbers).

In $\mathbb{Z}$, besides addition and multiplication, we can also define *subtraction*. For integers a and b, we define

$$b - a = b + (-1)a$$

which is the unique solution to the equation $a + x = b$.

1.2.4 Rational Numbers

In Section 1.2.2 we have seen that $\mathbb{Z}^+$ can be enlarged to $\mathbb{Q}^+$ in which division can be defined. In Section 1.2.3, we have seen that $\mathbb{Z}^+$ can be enlarged to $\mathbb{Z}$ in which subtraction can be defined. It would be nice if we can obtain a set in which both subtraction and division (by non-zero numbers) can be defined.

A ***rational number*** is a number that can be written in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. The numbers a and b are called the *numerator* and *denominator* of the rational number $\frac{a}{b}$. The set of all rational numbers is denoted by $\mathbb{Q}$ (the letter Q stands for *quotient*).

Equality in $\mathbb{Q}$ Rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ (where $a, b, c, d \in \mathbb{Z}$ and $bd \neq 0$) are defined to be *equal* (written $\frac{a}{b} = \frac{c}{d}$) if and only if $ad = bc$.

Explanation $bd \neq 0$ is the same as $b \neq 0$ and $d \neq 0$.

The definition of equality of rational numbers tells that

$$\text{if } ad = bc, \text{ then } \frac{a}{b} = \frac{c}{d}; \quad \text{and} \quad \text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } ad = bc.$$

Convention Let a be an integer. The rational number $\frac{a}{1}$ and the integer a are considered to be equal. That is, $\frac{a}{1} = a$.

Remark Under the convention, every integer is a rational number, that is, $\mathbb{Z} \subseteq \mathbb{Q}$.

Example 1.2.1 (a) $\frac{0}{5} = \frac{0}{1} = 0$ *Check* $0 \cdot 1 = 5 \cdot 0$

(b) $\frac{-2}{3} = \frac{2}{-3} = \frac{4}{-6}$ *Check* $(-2) \cdot (-3) = 3 \cdot 2$ and $2 \cdot (-6) = (-3) \cdot 4$

(c) $\frac{-3}{-7} = \frac{3}{7}$ *Check* $(-3) \cdot 7 = (-7) \cdot 3$

Addition in $\mathbb{Q}$ Let a, b, c, d be integers with $bd \neq 0$. We define $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$.

Multiplication in $\mathbb{Q}$ Let a, b, c, d be integers with $bd \neq 0$. We define $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$.

Definition 1.2.1 Let s be a rational number. We call ***the negative of** s*, and write $-s$, to mean the rational number $(-1) \cdot s$.

Example 1.2.2 (a) Let $s = \frac{2}{3}$. Then $-s = \frac{-2}{3}$. That is, $-\left(\frac{2}{3}\right) = \frac{-2}{3}$.

(b) Let $t = \frac{5}{-4}$. Then $-t = \frac{5}{4}$. That is, $-\left(\frac{5}{-4}\right) = \frac{5}{4}$.

Note that every rational number can be expressed as either a fraction or 0 or the negative of a fraction. For example, $\frac{-3}{-7} = \frac{3}{7}$ is a fraction and $\frac{-2}{3} = -\frac{2}{3}$ is the negative of a fraction.

Subtraction in $\mathbb{Q}$ Let s and t be rational numbers. We define $t - s = t + (-s)$.

Remark $t - s$ is the unique solution to the equation $s + x = t$.

Definition 1.2.2 Let s be a rational number with $s \neq 0$. We call ***the reciprocal of** s*, and write s^{-1}, to mean the rational number obtained by interchanging the numerator and denominator of s.

Explanation Since $s \neq 0$, it follows that the numerator of s is not equal to 0.

Example 1.2.3 Let $s = \frac{-3}{4}$. Then $s^{-1} = \frac{4}{-3}$.

Division in $\mathbb{Q}$ Let s and t be rational numbers with $s \neq 0$. We define $t \div s = t \cdot s^{-1}$

Remark $t \div s$ is the unique solution to the equation $s \cdot x = t$.

Rational Numbers as Repeating Decimals Using long division, we can express rational numbers as decimal numbers. For example,

(1) $\frac{7}{4} = 1\frac{3}{4} = 1.75$

(2) $\frac{2}{11} = 0.18181818\ldots = 0.\dot{1}\dot{8}$

(3) $-\frac{49}{198} = -0.2474747\ldots = -0.2\dot{4}\dot{7}$

In (1), the decimal is *terminating*. In (2) and (3), the decimals are not terminating but *repeating*. Given a terminating decimal, it is easy to convert it to a rational number. For example,

$$3.45 = 3 + \frac{45}{100} = 3 + \frac{9}{20} = \frac{69}{20}$$

Given a repeating decimal, it can be converted to a rational number by writing it as a sum of infinitely many terms. For example, using a formula for the sum of *geometric series*, we get

$$0.333\ldots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = \frac{1}{3}$$

Details will be discussed in Chapter **??**. Using the above idea (sum of infinitely many terms), we can also write

$$\frac{7}{4} = 1.75 = 1 + \frac{7}{10} + \frac{5}{100} + \frac{0}{1000} + \frac{0}{10000} + \cdots = 1.75\dot{0}$$

Thus every rational number can be expressed as a repeating decimal.

Insufficiency of $\mathbb{Q}$ It can be shown that the equation $x^2 = 2$ does not have any solution in $\mathbb{Q}$, that is, there does not exist any rational number whose square is 2. However, from the Pythagoras Theorem, we see that there is a number whose square is 2. That number, denoted by $\sqrt{2}$, represents the length of the hypotenuse of the right-angled isosceles triangle with two sides having lengths equal to 1 (see Figure 1.2.5).

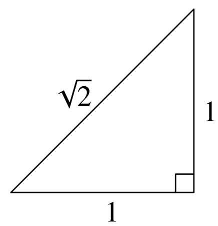

Figure 1.2.5

It would be nice if we can enlarge $\mathbb{Q}$ to a number system that includes $\sqrt{2}$ (and many more).

Exercise 1.2.4

1. Rewrite each of the following rational numbers as (i) a terminating decimal; (ii) a repeating decimal.

   (a) $\frac{3}{8}$ (b) $\frac{7}{20}$ (c) $\frac{1}{16}$

2. Rewrite each of the following rational numbers as a non-terminating but repeating decimal.

   (a) $\frac{1}{6}$ (b) $\frac{2}{7}$ (c) $\frac{3}{11}$

   Can you find a rule to determine whether the decimal expression for a rational number is terminating or non-terminating?

1.2.5 Real Numbers

The construction and properties of real numbers are important in mathematics. However, a rigorous definition of real numbers is beyond the scope of this course. Below we will consider real numbers as points on the number line. Before that, we explain how to identify rational numbers as points.

(1) First we take any line (usually a horizontal one).

(2) Take a point on the line and identify that point as the number 0.

(3) Take any line segment.

(4) Locate the points on both sides of 0 such that the distance between any pairs of consecutive points equal to the length of the line segment taken in (3).

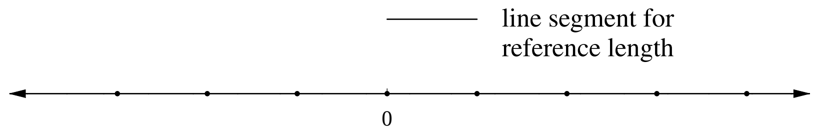

Figure 1.2.6

(5) Take one side of 0 (usually the right side). For the points obtained in (4) lying on this side, starting from the one nearest to 0, identify the points as the numbers 1, 2, 3 and so on; and for the points obtained in (4) lying on the other side, starting from the one nearest to 0, identify the points as the numbers −1, −2, −3 and so on.

Figure 1.2.7

(6) For the point that bisects the line segment from 0 to 1, it is identified as the fraction $\frac{1}{2}$. For the two points that trisect the line segment from 0 to 1, they are identified as the fractions $\frac{1}{3}$ and $\frac{2}{3}$, with $\frac{1}{3}$ nearer to 0.

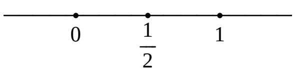

Figure 1.2.8 (a)

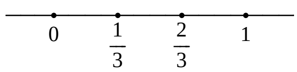

Figure 1.2.8 (b)

In this way, every fraction smaller than 1 can be identified as a point on the line. For fractions greater than 1, they can also be identified as points on the line. For example, the fractions $\frac{7}{3}$ and $\frac{8}{3}$ are identified as the points that trisect the line segment from 2 to 3 with $\frac{7}{3}$ nearer to 2. *Note that the points* $\frac{7}{3}$ *and* $\frac{8}{3}$ *can be obtained by moving the points* $\frac{1}{3}$ *and* $\frac{2}{3}$ *respectively two* *units to the right.*

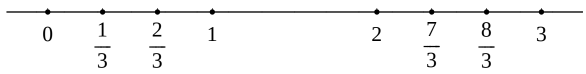

Figure 1.2.9

For every fraction $\frac{m}{n}$ (where m and n are counting numbers), the fraction $-\frac{m}{n}$ is identified as the point on the other side of 0 such that its distance to 0 is the same as that from $\frac{m}{n}$ to 0.

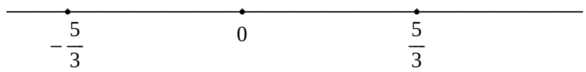

Figure 1.2.10

Although many points on the line are identified as rational numbers, there are many "holes". There are points on the line not identified as rational numbers. For example, the marked point X in Figure 1.2.11 is not identified as any rational number; it is identified as the number $\sqrt{2}$ which is the distance from the point X to 0.

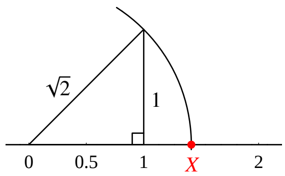

Figure 1.2.11

To enlarge the rational number system, we identify each point on the line as a number, called a ***real number***. With such identification, the line described above is called the *real number line*, or simply the *number line*. We denote the set of all real numbers by $\mathbb{R}$. Geometrically, the set $\mathbb{R}$ consists of all the points on the number line.

Note that every rational number is a real number, that is, $\mathbb{Q} \subseteq \mathbb{R}$. Real numbers that are not rational numbers are called ***irrational numbers***. For example, $\sqrt{2}$ is an irrational number. It can be shown that π is also an irrational numbers.

The number 0 divide the real number line into two sides. Real numbers on the the same side as the number 1 (usually the right-side) are said to be ***positive***.

Addition and Multiplication In $\mathbb{R}$, we can define addition and multiplication that are extensions of that in $\mathbb{Q}$. Figure 1.2.12 (a) and Figure 1.2.12 (b) show how to define $a+b$ and ab respectively for positive real numbers a and b using geometric construction.

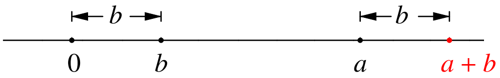

Figure 1.2.12 (a)

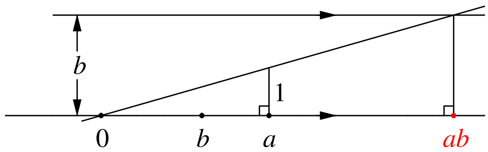

Figure 1.2.12 (b)

Below are some important properties of addition and multiplication.

Commutative Property of Addition	In $\mathbb{R}$, $a+b \equiv b+a$
Associative Property of Addition	In $\mathbb{R}$, $(a+b)+c \equiv a+(b+c)$
Zero Element of Addition	In $\mathbb{R}$, $0+a \equiv a$
Cancellation Principle for Addition	Let $a, b, c \in \mathbb{R}$. Then $a+b=a+c$ if and only if $b=c$.
Commutative Property of Multiplication	In $\mathbb{R}$, $a \cdot b \equiv b \cdot a$
Associative Property of Multiplication	In $\mathbb{R}$, $(a \cdot b) \cdot c \equiv a \cdot (b \cdot c)$
Distributive Property of Multiplication over Addition	In $\mathbb{R}$, $a \cdot (b+c) \equiv a \cdot b + a \cdot c$ $(a+b) \cdot c \equiv a \cdot c + b \cdot c$
Identity Element of Multiplication	In $\mathbb{R}$, $1 \cdot a \equiv a$
Product Zero Principle	Let $a, b \in \mathbb{R}$. Then $a \cdot b = 0$ if and only if $a=0$ or $b=0$.
Cancellation Principle for Multiplication	Let $a, b, c \in \mathbb{R}$ with $a \neq 0$. Then $a \cdot b = a \cdot c$ if and only if $b=c$.
Perfect Square Identities	In $\mathbb{R}$, $(a+b)^2 \equiv a^2 + 2ab + b^2$ $(a-b)^2 \equiv a^2 - 2ab + b^2$
Identity for Difference of Squares	In $\mathbb{R}$, $a^2 - b^2 \equiv (a+b)(a-b)$

Explain The statement "In $\mathbb{R}$, $a+b \equiv b+a$" means that for every real number a and every real number b, we always have $a+b=b+a$. It is called an *identity* in $\mathbb{R}$. If the underlying set is understood, we imply write $a+b \equiv b+a$.

Explain The sentence "$a+b=a+c$ if and only if $b=c$" means the following

$$\text{if } b=c, \text{ then } a+b=a+c; \quad \text{and} \quad \text{if } a+b=a+c, \text{ then } b=c.$$

The phrase '*if and only if*' is used very often in mathematics. The sentence "Condition (1) *if and only if* Condition (2)" means the following:

if Condition (2), *then* Condition (1); and *if* Condition (1), *then* Condition (2).

In other words, Condition (1) and Condition (2) are equivalent.

Definition 1.2.3 Let a be a real number. We call ***the negative of*** a, and write $-a$, to mean the real number $(-1) \cdot a$.

Remark $-a + a \equiv 0$

For $a \neq 0$, the real numbers a and $-a$ are on different sides of 0 and their distances to 0 are equal.

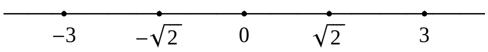

Figure 1.2.13

Terminology Real numbers that are on the same side as the number -1 (usually the left-side) are said to be *negative*.

Caution The negative of a real number may not be negative. For example, if $a = -2$, then $-a = 2$ is not negative.

Note Every real number is either positive, 0 or negative.

Subtraction Let $a, b \in \mathbb{R}$. We define $b - a = b + (-a)$.

Remark $b - a$ is the unique solution to the equation $a + x = b$.

Notation Let $a, b \in \mathbb{R}$. We write $a < b$ (or equivalently $b > a$) to mean that the real number $b - a$ is positive and we write $a \leq b$ (or equivalently $b \geq a$) to mean that $a < b$ or $a = b$.

Note A real number r is positive means that $r > 0$. The sets of all positive real numbers, positive rational numbers and positive integers are denoted by $\mathbb{R}^+$, $\mathbb{Q}^+$ and $\mathbb{Z}^+$ respectively, that is,

- $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$
- $\mathbb{Q}^+ = \{x \in \mathbb{Q} : x > 0\}$
- $\mathbb{Z}^+ = \{x \in \mathbb{Z} : x > 0\}$

Note that $\mathbb{Q}^+$ is the set of all fractions and $\mathbb{Z}^+$ is the set of all counting numbers.

Theorem 1.2.1 For every real number r with $r \neq 0$, there exists a unique real number, called the ***reciprocal*** of r and denoted by r^{-1}, such that $r \cdot r^{-1} = 1$.

Explanation Figure 1.2.14 (a) and Figure 1.2.14 (b) show how to construct a line segment with length r^{-1} for the cases where $r > 1$ and $0 < r < 1$ respectively (hence we can construct the real number r^{-1} on the number line).

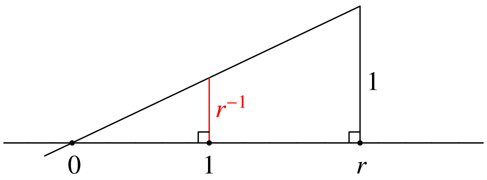

Figure 1.2.14 (a)

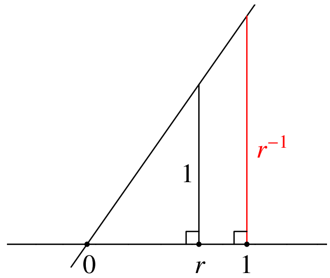

Figure 1.2.14 (b)

Division Let $a, b \in \mathbb{R}$ with $a \neq 0$. We define $b \div a = b \cdot a^{-1}$.

Remark The number $b \div a$ is the unique solution to the equation $a \cdot x = b$.
$b \div a$ is also written as $\frac{b}{a}$. Putting $b = 1$, we get $\frac{1}{a} = 1 \div a = 1 \cdot a^{-1} = a^{-1}$.

In Section 1.2.4, we have seen that rational numbers can be expressed as infinite repeating decimals. For irrational numbers, they can be expressed as infinite non-repeating decimals. Below we illustrate how to do this for the irrational number $\sqrt{2}$.

(1) First note that $1 < \sqrt{2} < 2$. This is because $1^2 < 2 < 2^2$. Thus

$$\sqrt{2} = 1.a_1a_2a_3a_4a_5\ldots$$

where a_1 denotes the first decimal digit, a_2 the second digit and so on.

(2) To determine the first decimal digit a_1, we consider the squares of the numbers

$$1,\ 1.1,\ 1.2,\ \ldots,\ 1.9,\ 2$$

Note that $1.4^2 = 1.96 < 2 < 2.25 = 1.5^2$. Thus $1.4 < \sqrt{2} < 1.5$ and so

$$\sqrt{2} = 1.4a_2a_3a_4a_5\ldots$$

(3) To determine the second decimal digit a_2, we consider the squares of the numbers

$$1.4,\ 1.41,\ 1.42,\ \ldots,\ 1.49,\ 1.5$$

Note that $1.41^2 = 1.9881 < 2 < 2.0164 = 1.42^2$. Thus $1.41 < \sqrt{2} < 1.42$ and so

$$\sqrt{2} = 1.41a_3a_4a_5\ldots$$

($\vdots$) Repeating the above steps, we can determine the digits a_3, a_4 and so on.

Remark The above method is not efficient. There are much quicker methods to find the decimal digits of $\sqrt{2}$ (using more advanced mathematics).

Exercise 1.2.5

1. For each of the following real numbers, determine whether it is (i) an integer; (ii) a rational number.

   (a) $\frac{6}{2}$ (b) $0.1\dot{2}\dot{3}$ (c) -31 (d) $-\sqrt{2}$

   (e) $\sqrt{25}$ (f) $1-\sqrt{3}$ (g) 13^{-1} (h) $0.1010010001000010000001\ldots$

2. For each of the following statements, determine whether it is true or false.

   (a) 0 is a rational number.

   (b) $\sqrt{2}$ is not an integer.

   (c) π can be expressed as a repeating decimal.

   (d) All integer are real numbers.

   (e) All real numbers are rational numbers.

   (f) Some integers are irrational numbers.

   (g) Some rational numbers are integers.

   (h) Some irrational numbers are not real numbers.

   (i) All irrational numbers can be expressed as non-repeating decimals.

   (j) Some irrational numbers are not rational numbers.

   (k) All rational numbers are not irrational numbers.

1.2.6 Square Roots of Real Numbers

Notation 1.2.4 Let $a \in \mathbb{R}$ and let n be a positive integer. We denote

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}$$

Remark The number a^n (read 'a to the n-th power') is called the *n-th power* of a. For the case where $n = 2$, the number a^2 (read 'a square') is called the *square* of a. For the case where $n = 3$, the number a^3 (read 'a cube') is called the *cube* of a.

Note that $\sqrt{2}$ is a real number whose square is 2. We may ask whether there is any real number whose square is π, for example. The answer is 'yes'. Before giving a general result, we introduce the following terminology.

Terminology 1.2.5 A ***non-negative real number*** means a real number that is not negative.

Theorem 1.2.2 For every non-negative real number b, there is a unique non-negative real number whose square is b.

Notation We denote $\sqrt{b}$ to be the unique non-negative real number whose square is b.

Explanation If $b = 0$, then 0 is the unique non-negative real number whose square is 0.

If $b > 0$, the real number $\sqrt{b}$ can be obtained using geometric construction. In Figure 1.2.15, the curve ABC is a semi-circle. By a property of circle (see Chapter **??**), the measure of $\angle ABC$ is 90°. Using similar triangles, we can show that the length of the line segment BD is $\sqrt{b}$.

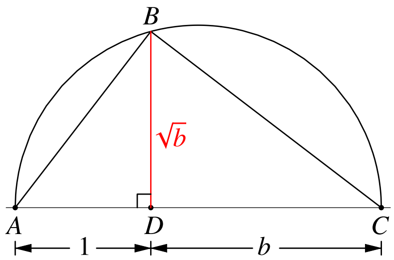

Figure 1.2.15

Example 1.2.4 $\sqrt{9} = 3$

$\sqrt{0} = 0$

Remark It is incorrect to write $\sqrt{9} = \pm 3$. By definition, $\sqrt{9}$ is the unique non-negative real number whose square is 9.

Let b be a positive real number. By Theorem 1.2.2, we know that $\sqrt{b}$ is a solution to the equation $x^2 = b$. By simple arithmetic, we see that the number $-\sqrt{b}$ is also a solution to the equation. We may ask whether there is any other solution. The answer is given in a result called the *Square-root Theorem*. First, we prove a simple result which will be used in the proof of the theorem.

Lemma 1.2.3 Let a be a real number. Suppose that $a \neq 0$. Then the equation $x^2 = a^2$ has exactly two solutions, namely, a and $-a$.

Proof

$$\begin{aligned}
x^2 &= a^2 \\
x^2 - a^2 &= 0 \\
(x+a)(x-a) &= 0 && \text{By Identity for Difference of Squares} \\
x + a = 0 \quad &\text{or} \quad x - a = 0 && \text{By Product Zero Principle} \\
x = -a \quad &\text{or} \quad x = a
\end{aligned}$$

□

Square-root Theorem Let b be a real number.

(1) If $b > 0$, then the equation $x^2 = b$ has exactly two solutions, namely $\sqrt{b}$ and $-\sqrt{b}$.

(2) If $b = 0$, then the equation $x^2 = b$ has exactly one solution, namely 0.

(3) If $b < 0$, then the equation $x^2 = b$ has no solution.

Proof (*a*) If $b > 0$, then $\sqrt{b} \neq 0$. The equation $x^2 = b$ can be written as

$$x^2 = \left(\sqrt{b}\right)^2$$

By Lemma 1.2.3, the equation has exactly two solutions, namely $\sqrt{b}$ and $-\sqrt{b}$.

(*b*) The equation $x^2 = 0$ can be written as $x \cdot x = 0$. The result follows from the Product Zero Principle.

(*c*) The result follows from the fact that the square of every real number is greater than or equal to 0. □

Remark In (3), the sentence "the equation $x^2 = b$ has no solution" means that there is no real number x satisfying the equality. In Chapter **??**, we will consider complex numbers—there are complex numbers u satisfying $u^2 = b$, even for $b < 0$. Thus, we should take into consideration the *universal set* when we talk about whether an equation has solutions or not.

Definition 1.2.6 Let b be a non-negative real number.

- We call ***a square root of*** b to mean a real number whose square is equal to b.
- We call ***the principle square root of*** b to mean the non-negative real number whose square is equal to b.

Example 1.2.5 The square roots of 4 are 2 and −2.
The principle square root of 4 is 2, that is, $\sqrt{4} = 2$.

Caution It is incorrect to write $\sqrt{4} = \pm 2$. See the remark for Example 1.2.4.

The symbol '$\sqrt{\ }$' is called the *radical* sign. The expression under the radical sign is called the *radicand*. For example, in the expression $\sqrt{5}$, the radicand is 5; in the expression $\sqrt{x^2 + y^2}$, the radicand is $x^2 + y^2$.

It can be shown that if n is a positive integer and $\sqrt{n}$ is not an integer, then $\sqrt{n}$ is an irrational number. Such numbers are called *surds*. For example, $\sqrt{2}$, $\sqrt{6}$ and $\sqrt{12}$ are surds but $\sqrt{9}$ is not a surd (since $\sqrt{9} = 3$ is an integer). In the rest of this section, we will discuss how to manipulate expressions involving surds.

Example 1.2.6

$$\begin{aligned} 3\sqrt{2} - 5\sqrt{2} + 6\sqrt{2} &= (3 - 5 + 6) \cdot \sqrt{2} && \text{Extract common factor } \sqrt{2} \\ &= 4\sqrt{2} \end{aligned}$$

Remark $4\sqrt{2}$ means $4 \cdot \sqrt{2}$ which is the same as $\sqrt{2} \cdot 4$ or $\sqrt{2}\,4$. Although both notations $4\sqrt{2}$ and $\sqrt{2}\,4$ represent the same real number, the first notation is preferred. *Can you find a reason for this?*

Example 1.2.7

$$\begin{aligned} \left(-\sqrt{3} + 3\sqrt{3}\right) \cdot 5\sqrt{3} &= (-1 + 3) \cdot \sqrt{3} \cdot 5\sqrt{3} && \text{Extract common factor} \\ &= 2 \cdot 5 \cdot \left(\sqrt{3}\right)^2 \\ &= 10 \cdot 3 && \sqrt{3} \text{ is a square root of } 3 \\ &= 30 \end{aligned}$$

Theorem 1.2.4 Let a and b be positive real numbers. Then

(1) $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$

(2) $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$

Proof To prove (1), we have to show that $\sqrt{a} \cdot \sqrt{b}$ is the principle square root of the positive real number ab.

Note that $\sqrt{a} \cdot \sqrt{b}$ is positive. It remains to show that the square of $\sqrt{a} \cdot \sqrt{b}$ is ab which can be checked directly.

$$\begin{aligned} \left(\sqrt{a} \cdot \sqrt{b}\right)^2 &= \sqrt{a} \cdot \sqrt{b} \cdot \sqrt{a} \cdot \sqrt{b} \\ &= \left(\sqrt{a} \cdot \sqrt{a}\right) \cdot \left(\sqrt{b} \cdot \sqrt{b}\right) && \text{Associative and Commutative Properties of } \times \\ &= a \cdot b \end{aligned}$$

The result in (2) can be proved similarly. □

Example 1.2.8

$$\begin{aligned} \sqrt{12} \cdot \sqrt{3} &= \sqrt{12 \cdot 3} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$

Example 1.2.9

$$\frac{\sqrt{54}}{\sqrt{6}} = \sqrt{\frac{54}{6}} = \sqrt{9} = 3$$

Remark There are alternative ways to obtain the result. See Example 1.2.13 and Example 1.2.19.

Example 1.2.10 Without using calculator, determine which of the following numbers is larger.

$$5\sqrt{6}, \qquad 7\sqrt{3}$$

Explanation We can use Theorem 1.2.4 to rewrite the given numbers in the form $\sqrt{n}$ and apply the following result: *If a and b are positive real numbers and $a > b$, then $\sqrt{a} > \sqrt{b}$.*

Solution

$$5\sqrt{6} = \sqrt{25}\cdot\sqrt{6} = \sqrt{25\cdot 6} = \sqrt{150} \quad \text{and} \quad 7\sqrt{3} = \sqrt{49}\cdot\sqrt{3} = \sqrt{49\cdot 3} = \sqrt{147}$$

Since $150 > 147$, it follows that $5\sqrt{6} > 7\sqrt{3}$.

$5\sqrt{6}$ is the larger number. □

The notations $5\sqrt{6}$ and $\sqrt{150}$ represent the same real number. Both forms are useful and it is difficult to tell which one is simpler *(it seems that the first one is more useful)*.

- To convert from $5\sqrt{6}$ to $\sqrt{150}$, we can rewrite 5 as $\sqrt{25}$ and apply Theorem 1.2.4 (see the calculation in Example 1.2.10).
- To convert from $\sqrt{150}$ to $5\sqrt{6}$, we can use the prime factorization of the integer 150.

In the expressions $5\sqrt{6}$ and $\sqrt{150}$, the numbers under the radical sign are 6 and 150 respectively. The prime factorizations of 6 and 150 are

$$6 = 2\cdot 3, \qquad 150 = 2\times 3\times 5^2$$

In the factorization of 150, there is a *square factor* 5^2, whereas there is no square factor in that of 6. We say that the positive integer 6 is *square free*.

Definition We say that a positive integer p is *square free* to mean that there does not exist any integer $n > 1$ such that n^2 is a factor of p.

Example 1.2.11 24 is not square free.

Reason $24 = 2^3 \cdot 3$; the factor 2 is repeated, hence 2^2 is a factor of 24.

30 is square free.

Reason $30 = 2 \cdot 3 \cdot 5$; there is no repeated factor.

Example 1.2.12 Rewrite $\sqrt{12}$ in the form '$a\sqrt{b}$' where a and b are integers and b is square free.

Solution
$$\begin{aligned}\sqrt{12} &= \sqrt{2^2 \cdot 3} && \text{Prime factorization}\\ &= \sqrt{2^2} \cdot \sqrt{3} && \text{By Theorem 1.2.4}\\ &= 2\sqrt{3}\end{aligned}$$
□

Example 1.2.13 Rewrite $\sqrt{54}$ in the form '$a\sqrt{b}$', where a and b are integers and b is square free and hence find $\dfrac{\sqrt{54}}{\sqrt{6}}$.

Solution
$$\begin{aligned}\sqrt{54} &= \sqrt{2 \cdot 3^3}\\ &= \sqrt{2 \cdot 3 \cdot 3^2} && \text{To get a square}\\ &= \sqrt{2 \cdot 3} \cdot \sqrt{3^2}\\ &= \sqrt{6} \cdot 3\\ &= 3\sqrt{6}\end{aligned}$$

Hence
$$\begin{aligned}\frac{\sqrt{54}}{\sqrt{6}} &= \frac{3\sqrt{6}}{\sqrt{6}}\\ &= 3\end{aligned}$$
□

Example 1.2.14 Rewrite $7\sqrt{6} \cdot 2\sqrt{3}$ in the form '$a\sqrt{b}$' where a and b are integers and b is square free.

Solution
$$\begin{aligned}7\sqrt{6} \cdot 2\sqrt{3} &= 7 \cdot 2 \cdot \sqrt{6} \cdot \sqrt{3}\\ &= 14 \cdot \sqrt{6 \cdot 3}\\ &= 14 \cdot \sqrt{2 \cdot 3^2} && \text{Prime Factorization of } 6 \cdot 3\\ &= 14 \cdot \sqrt{2} \cdot \sqrt{3^2}\\ &= 14 \cdot \sqrt{2} \cdot 3\\ &= 42\sqrt{2}\end{aligned}$$
□

Example 1.2.15 Rewrite $3\sqrt{20} - 8\sqrt{5}$ in the form '$a\sqrt{b}$' where a and b are integers and b is square free.

Solution
$$\begin{aligned} 3\sqrt{20} - 8\sqrt{5} &= 3 \cdot \sqrt{2^2 \cdot 5} - 8\sqrt{5} && \text{Prime Factorization of 20} \\ &= 3 \cdot \sqrt{2^2} \cdot \sqrt{5} - 8\sqrt{4} \\ &= 3 \cdot 2\sqrt{5} - 8\sqrt{5} \\ &= (6-8) \cdot \sqrt{5} \\ &= -2\sqrt{5} \end{aligned}$$
□

Example 1.2.16 Rewrite $\sqrt{12}\left(\sqrt{3} + \sqrt{2}\right)$ in the form '$a + b\sqrt{c}$' where a, b, c are integers and c is square free.

Solution
$$\begin{aligned} \sqrt{12}\left(\sqrt{3} + \sqrt{2}\right) &= \sqrt{2^2 \cdot 3} \cdot \left(\sqrt{3} + \sqrt{2}\right) \\ &= \sqrt{2^2} \cdot \sqrt{3} \cdot \left(\sqrt{3} + \sqrt{2}\right) \\ &= 2 \cdot \sqrt{3} \cdot \left(\sqrt{3} + \sqrt{2}\right) \\ &= 2 \cdot \sqrt{3} \cdot \sqrt{3} + 2 \cdot \sqrt{3} \cdot \sqrt{2} \\ &= 2 \cdot 3 + 2 \cdot \sqrt{3 \cdot 2} \\ &= 6 + 2\sqrt{6} \end{aligned}$$
□

Example 1.2.17 Rewrite the following expression
$$3\sqrt{75} + 4\sqrt{50} - 5\sqrt{18} - 6\sqrt{12}$$
in the form '$a\sqrt{b} + c\sqrt{d}$' where a, b, c, d are integers and b and d are square free.

Solution $3\sqrt{75} + 4\sqrt{50} - 5\sqrt{18} - 6\sqrt{12}$
$$\begin{aligned} &= 3 \cdot \sqrt{3 \cdot 5^2} + 4 \cdot \sqrt{2 \cdot 5^2} - 5 \cdot \sqrt{2 \cdot 3^2} - 6 \cdot \sqrt{2^2 \cdot 3} \\ &= 3 \cdot \sqrt{3} \cdot 5 + 4 \cdot \sqrt{2} \cdot 5 - 5 \cdot \sqrt{2} \cdot 3 - 6 \cdot 2 \cdot \sqrt{3} \\ &= 15\sqrt{3} + 20\sqrt{2} - 15\sqrt{2} - 12\sqrt{3} \\ &= \left(15\sqrt{3} - 12\sqrt{3}\right) + \left(20\sqrt{2} - 15\sqrt{2}\right) \\ &= (15-12) \cdot \sqrt{3} + (20-15) \cdot \sqrt{2} \\ &= 3\sqrt{3} + 5\sqrt{2} \end{aligned}$$
□

Given an expression in the form $\frac{a}{\sqrt{b}}$, it may be useful to rewrite it in a form such that the denominator does not involve the radical sign. The following example is an illustration.

Example 1.2.18 Given that $\sqrt{2} \approx 1.41421$ correct to 5 decimal places. Find the value of $\frac{1}{\sqrt{2}}$ correct to 3 decimal places.

Explanation Division by decimals is difficult whereas division by integers is easier. We try to rewrite $\frac{1}{\sqrt{2}}$ in the form $\frac{\text{a number}}{n}$ where n is an integer. This can be done by multiplying the numerator and denominator by $\sqrt{2}$.

Solution
$$\begin{aligned}\frac{1}{\sqrt{2}} &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \qquad \text{the value is unchanged since } \frac{\sqrt{2}}{\sqrt{2}} = 1\\ &= \frac{\sqrt{2}}{2}\\ &\approx \frac{1.41421}{2}\\ &\approx 0.707 \quad \text{(correct to 3 decimal places)}\end{aligned}$$

Expressions like $\frac{1}{\sqrt{2}}$ or $\frac{3}{4+\sqrt{5}}$ can be written in a form such that the denominators do not involve the radical sign. The process of removing the radical sign in the denominator is called *rationalization of the denominator*.

(1) To rationalize the denominator for $\frac{\text{an expression}}{\sqrt{b}}$, we can multiply the numerator and denominator by $\sqrt{b}$ and use the fact that $\left(\sqrt{b}\right)^2 = b$.

(2) To rationalize the denominator for $\frac{\text{an expression}}{b \pm \sqrt{c}}$, we can multiply the numerator and denominator by $b \mp \sqrt{c}$, and use

$$\left(b+\sqrt{c}\right)\left(b-\sqrt{c}\right) = b^2 - (\sqrt{c})^2 = b^2 - c$$

(3) To rationalize the denominator for $\frac{\text{an expression}}{a\sqrt{b} \pm c\sqrt{d}}$, we can use a technique similar to that in (2).

Example 1.2.19 Rationalize the denominator for $\frac{\sqrt{54}}{\sqrt{6}}$.

Solution
$$\begin{aligned}\frac{\sqrt{54}}{\sqrt{6}} &= \frac{\sqrt{54}}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}\\ &= \frac{\sqrt{54 \times 6}}{6} = \frac{\sqrt{324}}{6}\\ &= \frac{18}{6} = 3\end{aligned}$$

□

Example 1.2.20 Rewrite $\dfrac{3}{\sqrt{8}}$ in the form $\dfrac{a\sqrt{b}}{c}$ where a, b, c are integers and b is square free.

Solution
$$\begin{aligned}\frac{3}{\sqrt{8}} &= \frac{3}{\sqrt{8}} \times \frac{\sqrt{8}}{\sqrt{8}} \\ &= \frac{3 \cdot \sqrt{2^2 \cdot 2}}{8} \\ &= \frac{3 \cdot 2 \cdot \sqrt{2}}{8} \\ &= \frac{3\sqrt{2}}{4}\end{aligned}$$
□

Remark Alternatively, we may express the denominator $\sqrt{8}$ as $2\sqrt{2}$ first and then multiply the numerator and denominator by $\sqrt{2}$.

Example 1.2.21 Rewrite $\dfrac{2}{3+\sqrt{5}}$ in the form $\dfrac{a \pm \sqrt{b}}{c}$ where a, b, c are integers and b is square free.

Solution
$$\begin{aligned}\frac{2}{3+\sqrt{5}} &= \frac{2}{3+\sqrt{5}} \times \frac{3-\sqrt{5}}{3-\sqrt{5}} \\ &= \frac{2 \cdot (3-\sqrt{5})}{3^2 - 5} \\ &= \frac{2 \cdot (3-\sqrt{5})}{4} \\ &= \frac{3-\sqrt{5}}{2}\end{aligned}$$
□

Remark In view of the Square-root Theorem, we can only define square root for non-negative real numbers. It will be nice if we can enlarge the real number system to one in which every number has a square root. In Chapter **??**, we will consider the complex number system and we will show that every non-zero complex number has two square roots.

Exercise 1.2.6

1. For each of the following expressions, rewrite it in the form $a\sqrt{b}$, where a and b are integers and b is square free.

   (a) $\sqrt{8}$ (b) $\sqrt{27}$ (c) $\sqrt{45}$ (d) $\sqrt{32}$

   (e) $3\sqrt{12}$ (f) $-7\sqrt{20}$ (g) $2\sqrt{125}$

2. For each of the following expressions, rewrite it in the form $a\sqrt{b}$, where a and b are integers and b is square free.

   (a) $5\sqrt{3} \times 4\sqrt{2}$ (b) $-2\sqrt{3} \times 5\sqrt{6}$ (c) $3\sqrt{5} - 7\sqrt{5}$

   (d) $4\sqrt{2} + 2\sqrt{8}$ (e) $\sqrt{54} - \sqrt{24}$ (f) $\sqrt{12}\,(\sqrt{45} - \sqrt{80})$

3. For each of the following expressions, rewrite it in the form $a\sqrt{b}$ or $\frac{a\sqrt{b}}{c}$, where a, b and c are integers and b is square free.

   (a) $\sqrt{\frac{8}{9}}$ (b) $\sqrt{\frac{27}{4}}$ (c) $\frac{3}{\sqrt{5}}$ (d) $\frac{4\sqrt{5}}{\sqrt{2}}$

   (e) $\frac{2\sqrt{5}}{\sqrt{18}}$ (f) $\sqrt{\frac{125}{32}}$ (g) $5\sqrt{\frac{8}{27}}$ (h) $-2\sqrt{\frac{54}{7}}$

4. Expand and simplify the following

   (a) $(2 + \sqrt{5})^2$ (b) $(3 - \sqrt{5})^2$

   (c) $(2 + \sqrt{3})(2 - \sqrt{3})$ (d) $(3 + 4\sqrt{5})^2$

   (e) $(3 - 4\sqrt{5})(3 + 4\sqrt{5})$ (f) $(-3 + 2\sqrt{5})(3 + 2\sqrt{5})$

5. For each of the following, rationalize the denominator.

   (a) $\frac{12}{5\sqrt{3}}$ (b) $\frac{3}{2 - \sqrt{3}}$ (c) $\frac{5\sqrt{2}}{\sqrt{2} + 1}$ (d) $\frac{1}{\sqrt{2} + \sqrt{3}}$

6. For each of the following expressions, rewrite it in the form $\frac{\text{numerator}}{\text{denominator}}$ such that the denominator does not involve the radical sign.

   (a) $\frac{27}{\sqrt{7}} - \frac{12}{\sqrt{7}}$ (b) $\frac{4}{\sqrt{2}} + \frac{4}{\sqrt{8}}$ (c) $\frac{\sqrt{5}}{11} - \frac{2}{\sqrt{11}}$

   (d) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{6}}$ (e) $\frac{1 + \sqrt{2}}{\sqrt{8}}$ (f) $\frac{\sqrt{5} - \sqrt{2}}{2} + \frac{5}{\sqrt{5}}$