Existence and Uniqueness Theorem

Version 1 *Consider the following IVP*

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Suppose f *and* f_y *are continuous on a neighborhood of* (t_0, y_0). *Then in some neighborhood of* t_0*, the IVP has a unique solution.*

Existence and Uniqueness Theorem

Version 1 *Consider the following IVP* $\begin{cases} y'(t) = f(t,y) \\ y(t_0) = y_0 \end{cases}$

Suppose f *and* f_y *are continuous on a neighborhood of* (t_0, y_0)*. Then in some neighborhood of* t_0*, the IVP has a unique solution.*

Version 2 *Consider the IVP* $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

Suppose

(1) f *is continuous on* $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$*;*

(2) there exists $L \in \mathbb{R}$ *such that* $|f(t,y_1) - f(t,y_2)| \leq L|y_1 - y_2| \quad \forall\ (t,y_1),(t,y_2) \in R$

Then in some neighborhood of t_0*, the IVP has a unique solution.*

Metric Spaces

- In $\mathbb{R}$, *distance* between two points a and b is

$$\mathrm{dist}(a, b) = |a - b|$$

Metric Spaces

- In $\mathbb{R}$, *distance* between two points a and b is

$$\text{dist}(a, b) = |a - b|$$

- In $\mathbb{R}^n$, distance between $\mathbf{p} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{q} = (b_1, b_2, \ldots, b_n)$ is

$$\text{dist}(\mathbf{p}, \mathbf{q}) = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

Metric Spaces

- In $\mathbb{R}$, *distance* between two points a and b is

$$\text{dist}(a,b) = |a-b|$$

- In $\mathbb{R}^n$, distance between $\mathbf{p} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{q} = (b_1, b_2, \ldots, b_n)$ is

$$\text{dist}(\mathbf{p},\mathbf{q}) = \sqrt{\sum_{i=1}^{n}(a_i - b_i)^2}$$

Distance satisfies

- $\text{dist}(a,b) \geq 0$ and $\text{dist}(a,b) = 0$ iff $a = b$
- $\text{dist}(a,b) = \text{dist}(b,a)$
- $\text{dist}(a,b) + \text{dist}(b,c) \geq \text{dist}(a,c)$ (triangle inequality)

Definition Let X be a set. A *metric* on X is a function $d : X \times X \longrightarrow \mathbb{R}$ such that

(1) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

Definition Let X be a set. A *metric* on X is a function $d : X \times X \longrightarrow \mathbb{R}$ such that

(1) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

X together with d, denoted by (X, d), is called a *metric space*.

Definition Let X be a set. A *metric* on X is a function $d : X \times X \longrightarrow \mathbb{R}$ such that

(1) $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$

(3) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$

X together with d, denoted by (X, d), is called a *metric space*.

If the metric is clear, we may denote the metric space by X.

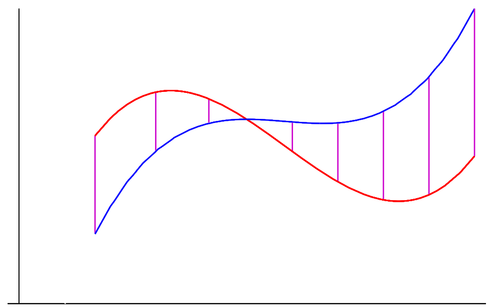
Example In $C[a,b]$, the set of all continuous functions on $[a,b]$, define

$$d(f,g) = \max_{a \le x \le b} |f(x) - g(x)|$$

called uniform metric.

Example In $C[a,b]$, the set of all continuous functions on $[a,b]$, define

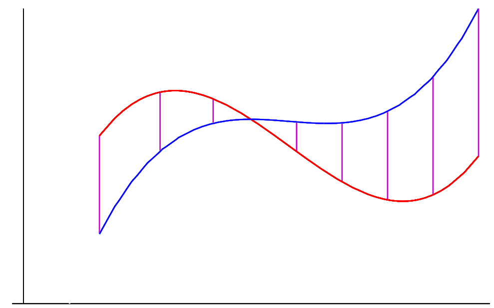
$$d(f,g) = \max_{a \le x \le b} |f(x) - g(x)|$$



called uniform metric.

Example In $C[a,b]$, the set of all continuous functions on $[a,b]$, define

$$d(f,g) = \max_{a \le x \le b} |f(x) - g(x)|$$



called uniform metric.

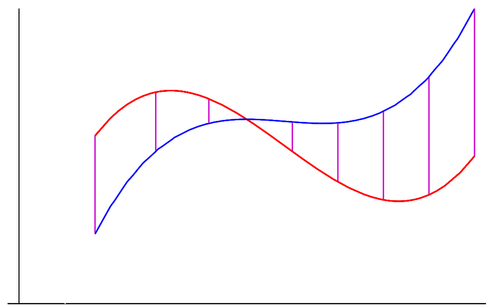
Example In $C[a,b]$, define

$$d(f,g) = \int_a^b |f(x) - g(x)|\,\mathrm{d}x$$

called L^1-metric.

Example In $C[a,b]$, the set of all continuous functions on $[a,b]$, define

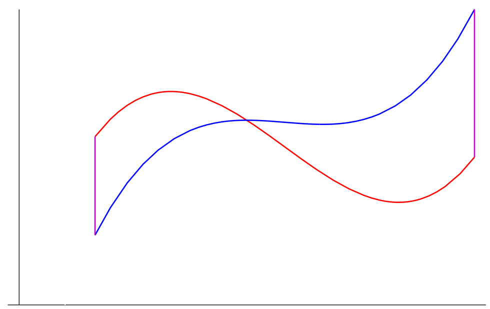
$$d(f,g) = \max_{a \le x \le b} |f(x) - g(x)|$$



called uniform metric.

Example In $C[a,b]$, define

$$d(f,g) = \int_a^b |f(x) - g(x)| \, \mathrm{d}x$$



called L^1-metric.

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

$$\forall\, \epsilon > 0,\ \ \exists\, N \in \mathbb{N} \ \text{ such that } \ d(x_n, a) < \epsilon \ \text{ whenever } \ n \geq N$$

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

$$\forall\, \epsilon > 0, \;\; \exists\, N \in \mathbb{N} \;\; \text{such that} \;\; d(x_n, a) < \epsilon \;\; \text{whenever} \;\; n \geq N$$

Note If a exists, it is unique.

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

$$\forall\ \epsilon > 0,\ \ \exists\ N \in \mathbb{N} \ \text{ such that } \ d(x_n, a) < \epsilon \ \text{ whenever } \ n \geq N$$

Note If a exists, it is unique.

a is called the *limit* of (x_n)

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

$$\forall\, \epsilon > 0, \;\; \exists\, N \in \mathbb{N} \;\; \text{such that } d(x_n, a) < \epsilon \;\text{ whenever }\; n \geq N$$

  Note If a exists, it is unique.

  a is called the *limit* of (x_n)

- *divergent* if it is not convergent.

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

$$\forall\, \epsilon > 0,\ \ \exists\, N \in \mathbb{N} \ \text{ such that } d(x_n, a) < \epsilon \ \text{ whenever } n \geq N$$

  Note If a exists, it is unique.

  a is called the *limit* of (x_n)

- *divergent* if it is not convergent.

Remark In $C[a, b]$ with uniform metric, a sequence (f_n) converges to f means

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

$$\forall\, \epsilon > 0,\ \ \exists\, N \in \mathbb{N} \ \text{ such that } \ d(x_n, a) < \epsilon \ \text{ whenever } \ n \geq N$$

  Note If a exists, it is unique.

  a is called the *limit* of (x_n)

- *divergent* if it is not convergent.

Remark In $C[a, b]$ with uniform metric, a sequence (f_n) converges to f means (f_n) *converges uniformly* to f on $[a, b]$.

Convergence

Definition A sequence (x_n) in a metric space X is said to be

- *convergent* if there exists $a \in X$ such that

$$d(x_n, a) \to 0 \qquad \text{as } n \to \infty$$

$$\forall\, \epsilon > 0, \;\; \exists\, N \in \mathbb{N} \;\text{ such that }\; d(x_n, a) < \epsilon \;\text{ whenever }\; n \geq N$$

  Note If a exists, it is unique.

  a is called the *limit* of (x_n)

- *divergent* if it is not convergent.

Remark In $C[a, b]$ with uniform metric, a sequence (f_n) converges to f means (f_n) *converges uniformly* to f on $[a, b]$.

$$d(f_n, f) \leq \epsilon \qquad \text{means} \qquad \max_{x \in [a,b]} |f_n(x) - f(x)| \leq \epsilon$$

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \quad \text{such that } d(x_n, x_m) < \epsilon \text{ whenever } m, n > N$$

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \quad \text{such that} \ d(x_n, x_m) < \epsilon \ \text{ whenever } \ m, n > N$$

Example Let (x_n) be a sequence in a metric space.

(x_n) convergent $\implies$ it is a Cauchy sequence.

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

$$\forall\ \epsilon > 0,\ \exists\, N \in \mathbb{N} \quad \text{such that } d(x_n, x_m) < \epsilon \text{ whenever } m, n > N$$

Example Let (x_n) be a sequence in a metric space.

(x_n) convergent $\Longrightarrow$ it is a Cauchy sequence.

Note In $\mathbb{R}$ (usual metric), if (x_n) is a Cauchy sequence, then it is convergent.

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

$$\forall\ \epsilon > 0,\ \exists\, N \in \mathbb{N} \quad \text{such that } d(x_n, x_m) < \epsilon \ \text{ whenever } \ m, n > N$$

Example Let (x_n) be a sequence in a metric space.

(x_n) convergent $\Longrightarrow$ it is a Cauchy sequence.

Note In $\mathbb{R}$ (usual metric), if (x_n) is a Cauchy sequence, then it is convergent.

However, in some metric spaces, some Cauchy sequences may be divergent.

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

$$\forall\ \epsilon > 0,\ \exists\, N \in \mathbb{N} \quad \text{such that } d(x_n, x_m) < \epsilon \text{ whenever } m, n > N$$

Example Let (x_n) be a sequence in a metric space.

(x_n) convergent $\Longrightarrow$ it is a Cauchy sequence.

Note In $\mathbb{R}$ (usual metric), if (x_n) is a Cauchy sequence, then it is convergent.

However, in some metric spaces, some Cauchy sequences may be divergent.

Definition A metric space X is said to be *complete* if *all Cauchy sequences in* X *are convergent*.

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

$$\forall\ \epsilon > 0,\ \exists\, N \in \mathbb{N} \quad \text{such that } d(x_n, x_m) < \epsilon \ \text{ whenever } \ m, n > N$$

Example Let (x_n) be a sequence in a metric space.

$$(x_n) \text{ convergent} \implies \text{it is a Cauchy sequence.}$$

Note In $\mathbb{R}$ (usual metric), if (x_n) is a Cauchy sequence, then it is convergent.

However, in some metric spaces, some Cauchy sequences may be divergent.

Definition A metric space X is said to be *complete* if *all Cauchy sequences in X are convergent*.

Example It can be shown that $C[a, b]$ with uniform metric is complete

Definition A sequence (x_n) in a metric space X is called a *Cauchy sequence* if

$$d(x_n, x_m) \to 0 \quad \text{as } m, n \to \infty$$

$$\forall\, \epsilon > 0,\ \exists\, N \in \mathbb{N} \ \text{ such that } \ d(x_n, x_m) < \epsilon \ \text{ whenever } \ m, n > N$$

Example Let (x_n) be a sequence in a metric space.

(x_n) convergent $\Longrightarrow$ it is a Cauchy sequence.

Note In $\mathbb{R}$ (usual metric), if (x_n) is a Cauchy sequence, then it is convergent.

However, in some metric spaces, some Cauchy sequences may be divergent.

Definition A metric space X is said to be *complete* if *all Cauchy sequences in X are convergent*.

Example It can be shown that $C[a, b]$ with uniform metric is complete

$C[a, b]$ with L_1-metric is not complete

Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \ \text{such that}\ B(a;r) \subset V$$

where $B(a;r) = \{x \in X : d(x,a) < r\}$.

Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \text{ such that } B(a;r) \subset V$$

  where $B(a;r) = \{x \in X : d(x,a) < r\}$.

- A subset F of X is said to be *closed* if $X \setminus F$ is open.

Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \text{ such that } B(a;r) \subset V$$

  where $B(a;r) = \{x \in X : d(x,a) < r\}$.

- A subset F of X is said to be *closed* if $X \setminus F$ is open.

Example In $\mathbb{R}$ *(usual metric)*

- interval (α, β) is open

Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \text{ such that } B(a; r) \subset V$$

where $B(a; r) = \{x \in X : d(x, a) < r\}$.

- A subset F of X is said to be *closed* if $X \setminus F$ is open.

Example In $\mathbb{R}$ *(usual metric)*

- interval (α, β) is open

Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \text{ such that } B(a;r) \subset V$$

where $B(a;r) = \{x \in X : d(x,a) < r\}$.

- A subset F of X is said to be *closed* if $X \setminus F$ is open.

Example In $\mathbb{R}$ *(usual metric)*

- interval (α,β) is open

Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \text{ such that } B(a;r) \subset V$$

where $B(a;r) = \{x \in X : d(x,a) < r\}$.

- A subset F of X is said to be *closed* if $X \setminus F$ is open.

Example In $\mathbb{R}$ *(usual metric)*

- interval (α, β) is open

Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \text{ such that } B(a;r) \subset V$$

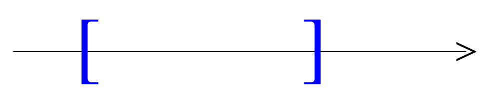
where $B(a;r) = \{x \in X : d(x,a) < r\}$.

- A subset F of X is said to be *closed* if $X \setminus F$ is open.

Example In $\mathbb{R}$ *(usual metric)*

- interval (α,β) is open

- interval $[\alpha,\beta]$ is closed



Definition Let X be a metric space.

- A subset V of X is said to be *open* if

$$\forall\, a \in V,\ \exists\, r > 0 \text{ such that } B(a;r) \subset V$$

  where $B(a;r) = \{x \in X : d(x,a) < r\}$.

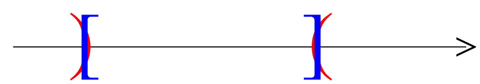
- A subset F of X is said to be *closed* if $X \setminus F$ is open.

Example In $\mathbb{R}$ *(usual metric)*

- interval (α, β) is open

- interval $[\alpha, \beta]$ is closed

  because its complement is open



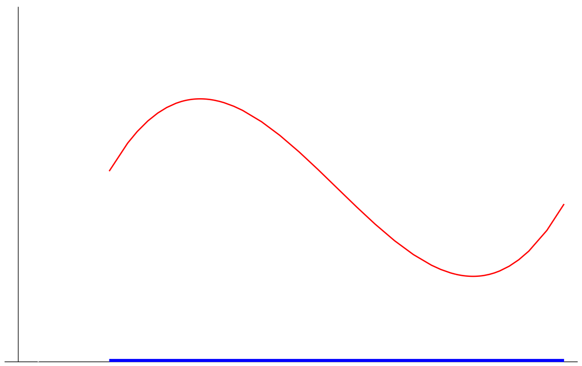
Example In $C[a,b]$ with *uniform metric*,

- The following subset is open

$$V = \left\{ f \in C[a,b] : f(x) > 0 \ \forall \ x \in [a,b] \right\}$$

Example In $C[a,b]$ with *uniform metric*,

- The following subset is open



$$V = \{f \in C[a,b] : f(x) > 0 \; \forall \; x \in [a,b]\}$$

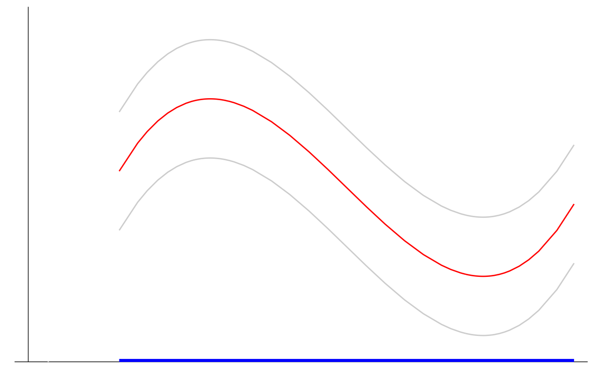
$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie above the } x\text{-axis}\}$$

Example In $C[a,b]$ with *uniform metric*,

- The following subset is open

$$V = \{f \in C[a,b] : f(x) > 0 \ \forall \ x \in [a,b]\}$$

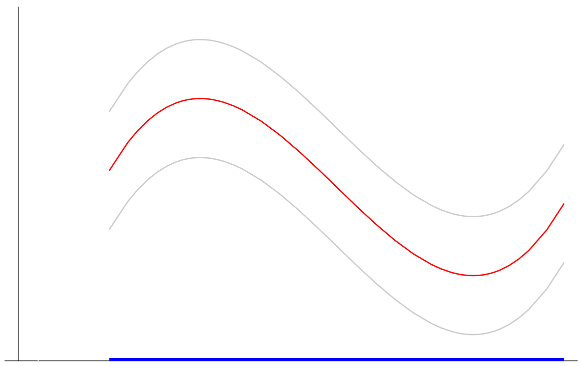
$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie above the } x\text{-axis}\}$$



Example In $C[a,b]$ with *uniform metric*,

- The following subset is open

$$V = \{f \in C[a,b] : f(x) > 0 \;\forall\; x \in [a,b]\}$$
$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie above the } x\text{-axis}\}$$

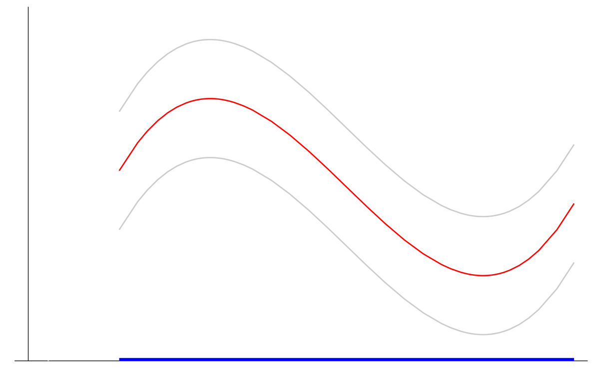


- The following subset is closed

$$F = \{f \in C[a,b] : 0 \le f(x) \le 1 \;\forall\; x \in [a,b]\}$$

Example In $C[a,b]$ with *uniform metric*,

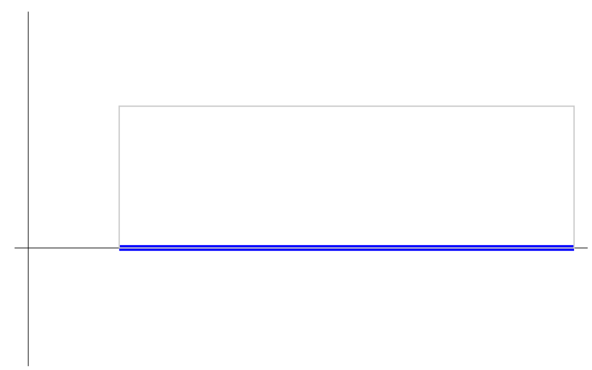
- The following subset is open



$$V = \{f \in C[a,b] : f(x) > 0 \ \forall \ x \in [a,b]\}$$

$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie above the } x\text{-axis}\}$$

- The following subset is closed



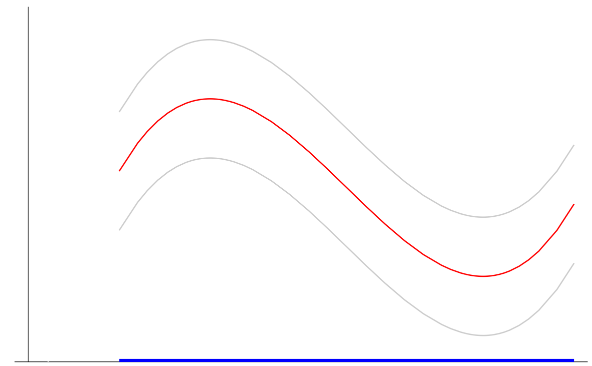
$$F = \{f \in C[a,b] : 0 \le f(x) \le 1 \ \forall \ x \in [a,b]\}$$

$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie inside the box}\}$$

Example In $C[a,b]$ with *uniform metric*,

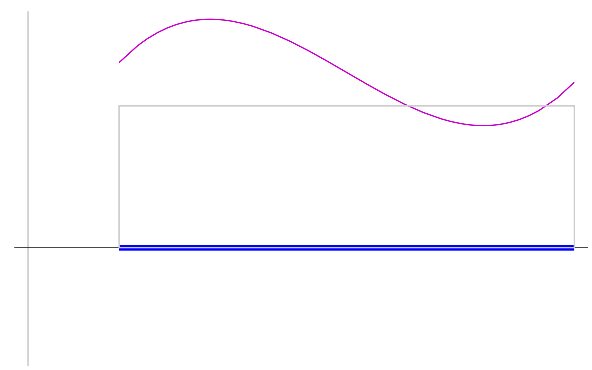
- The following subset is open

$$V = \{f \in C[a,b] : f(x) > 0 \ \forall \ x \in [a,b]\}$$
$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie above the } x\text{-axis}\}$$



- The following subset is closed

$$F = \{f \in C[a,b] : 0 \le f(x) \le 1 \ \forall \ x \in [a,b]\}$$
$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie inside the box}\}$$

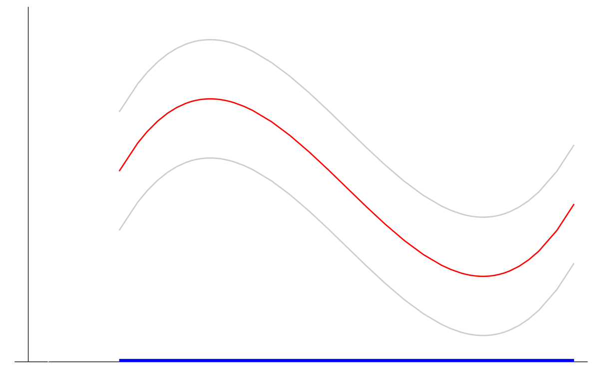


Complement of F

{continuous functions whose graphs are not entirely inside the box}

Example In $C[a,b]$ with *uniform metric*,

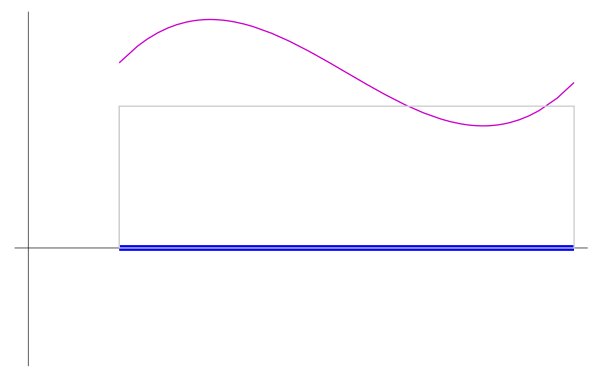
- The following subset is open



$$V = \{f \in C[a,b] : f(x) > 0 \ \forall \ x \in [a,b]\}$$

$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie above the } x\text{-axis}\}$$

- The following subset is closed



$$F = \{f \in C[a,b] : 0 \le f(x) \le 1 \ \forall \ x \in [a,b]\}$$

$$= \{\text{continuous functions on } [a,b] \text{ whose graphs lie inside the box}\}$$

Complement of F

{continuous functions whose graphs are not entirely inside the box}

at least partly outside the box

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

So we may consider the metric space (A, d),

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

So we may consider the metric space (A, d), called a *metric subspace* of (X, d)

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

So we may consider the metric space (A, d), called a *metric subspace* of (X, d)

Example Consider $\mathbb{R}$ *(usual metric)*.

Consider $[0, 1] \subset \mathbb{R}$.

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

So we may consider the metric space (A, d), called a *metric subspace* of (X, d)

Example Consider $\mathbb{R}$ *(usual metric)*.

Consider $[0, 1] \subset \mathbb{R}$.

May consider usual metric in $[0, 1]$.

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

So we may consider the metric space (A, d), called a *metric subspace* of (X, d)

Example Consider $\mathbb{R}$ *(usual metric)*.

Consider $[0, 1] \subset \mathbb{R}$.

May consider usual metric in $[0, 1]$.

So $[0, 1]$ becomes a *metric space*.

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

So we may consider the metric space (A, d), called a *metric subspace* of (X, d)

Example Consider $\mathbb{R}$ *(usual metric)*.

Consider $[0, 1] \subset \mathbb{R}$.

May consider usual metric in $[0, 1]$.

So $[0, 1]$ becomes a *metric space*.

Theorem *Let (X, d) be a complete metric space. Let A be a closed subset of X. Then*

Metric Subspaces

Let (X, d) be a metric space.

Let $A \subseteq X$.

Then d *"restricted" on* A (still denoted by d) is also a metric.

So we may consider the metric space (A, d), called a *metric subspace* of (X, d)

Example Consider $\mathbb{R}$ *(usual metric)*.

Consider $[0, 1] \subset \mathbb{R}$.

May consider usual metric in $[0, 1]$.

So $[0, 1]$ becomes a *metric space*.

Theorem *Let (X, d) be a complete metric space. Let A be a closed subset of X. Then (A, d) is complete.*

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\quad \forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \ \leq\ \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\quad \forall\, x, y \in X, \quad d(T^k x, T^k, y) \ \leq\ \alpha^k \cdot d(x, y)$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+,\qquad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\forall\, x, y \in X,\quad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

$P(1)$ is true

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\forall\, x, y \in X, \quad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

$P(1)$ is true $\because$ by definition, $\forall\, x, y \in X,\ d(Tx, Ty) \leq \alpha \cdot d(x, y)$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\quad \forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \ \leq\ \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\ \forall\, x, y \in X, \quad d(T^k x, T^k, y) \ \leq\ \alpha^k \cdot d(x, y)$

$P(1)$ is true $\ \because$ by definition, $\ \forall\, x, y \in X,\ d(Tx, Ty) \ \leq\ \alpha \cdot d(x, y)$

Suppose $P(k)$ is true.

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \ \leq\ \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\ \forall\, x, y \in X, \quad d(T^k x, T^k, y) \ \leq\ \alpha^k \cdot d(x, y)$

$P(1)$ is true $\because$ by definition, $\ \forall\, x, y \in X,\ \ d(Tx, Ty) \ \leq\ \alpha \cdot d(x, y)$

Suppose $P(k)$ is true.

Then $\forall\, x, y \in X,\ \ d(T^{k+1}x, T^{k+1}y)$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \le \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\quad \forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \le \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\quad \forall\, x, y \in X, \quad d(T^k x, T^k, y) \le \alpha^k \cdot d(x, y)$

$P(1)$ is true $\quad \because$ by definition, $\quad \forall\, x, y \in X,\ d(Tx, Ty) \le \alpha \cdot d(x, y)$

Suppose $P(k)$ is true.

Then $\forall\, x, y \in X,\ d(T^{k+1}x, T^{k+1}y) \;=\; d(T^k(Tx), T^k(Ty))$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\forall\, x, y \in X, \quad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

$P(1)$ is true $\because$ by definition, $\forall\, x, y \in X,\ d(Tx, Ty) \leq \alpha \cdot d(x, y)$

Suppose $P(k)$ is true.

Then $\forall\, x, y \in X,\ d(T^{k+1}x, T^{k+1}y) = d(T^k(Tx), T^k(Ty))$

$\leq \alpha^k \cdot d(Tx, Ty)$ induction assumption

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\forall\, x, y \in X$, $\forall\, k \in \mathbb{Z}^+$, $\quad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\forall\, x, y \in X, \quad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

$P(1)$ is true $\because$ by definition, $\forall\, x, y \in X,\ d(Tx, Ty) \leq \alpha \cdot d(x, y)$

Suppose $P(k)$ is true.

Then $\forall\, x, y \in X$, $\ d(T^{k+1}x, T^{k+1}y) = d(T^k(Tx), T^k(Ty))$

$$\leq \alpha^k \cdot d(Tx, Ty) \qquad \text{induction assumption}$$

$$\leq \alpha^k \cdot \alpha \cdot d(x, y) = \alpha^{k+1} \cdot d(x, y)$$

Definition Let (X, d) be a metric space. A mapping $T : X \longrightarrow X$ is called a *contraction map* if there exists a constant α with $0 < \alpha < 1$ such that

$$d(Tx, Ty) \leq \alpha \cdot d(x, y) \qquad \forall\, x, y \in X$$

Notation For $k \in \mathbb{Z}^+$, denote $\quad T^k = \overbrace{T \circ T \circ \cdots \circ T}^{k \text{ times}}$

Note $\forall\, x, y \in X,\ \forall\, k \in \mathbb{Z}^+, \qquad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

Proof Denote $P(k)$ to be the statement $\forall\, x, y \in X, \quad d(T^k x, T^k, y) \leq \alpha^k \cdot d(x, y)$

$P(1)$ is true $\because$ by definition, $\forall\, x, y \in X,\ d(Tx, Ty) \leq \alpha \cdot d(x, y)$

Suppose $P(k)$ is true.

Then $\forall\, x, y \in X,$

$$\begin{aligned} d(T^{k+1}x, T^{k+1}y) &= d(T^k(Tx), T^k(Ty)) \\ &\leq \alpha^k \cdot d(Tx, Ty) \qquad \text{induction assumption} \\ &\leq \alpha^k \cdot \alpha \cdot d(x, y) = \alpha^{k+1} \cdot d(x, y) \end{aligned}$$

That is, $P(k+1)$ is true.

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

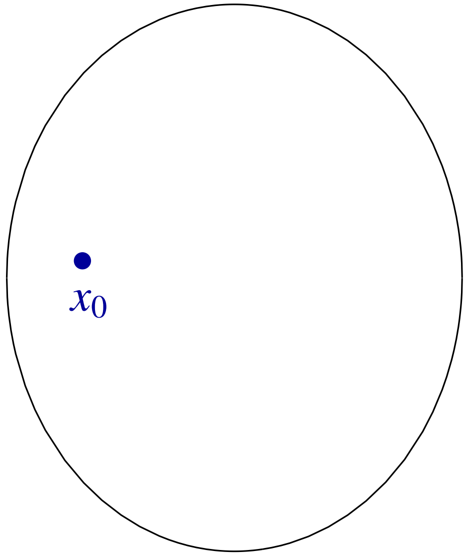
T *has a unique fixed point* x^**.*

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^**.*

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

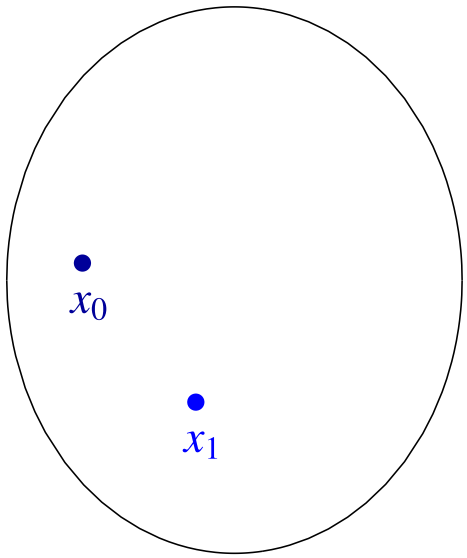
Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

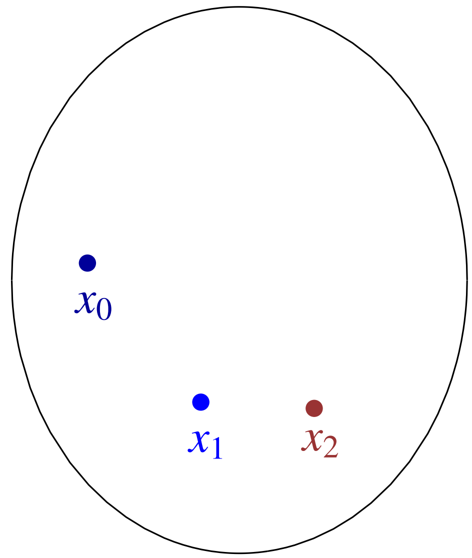
Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = T x_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



Fixed Point Theorem *Let (X, d) be a (non-empty) complete metric space. Suppose $T : X \longrightarrow X$ is a contraction map. Then*

T has a unique fixed point x^*.

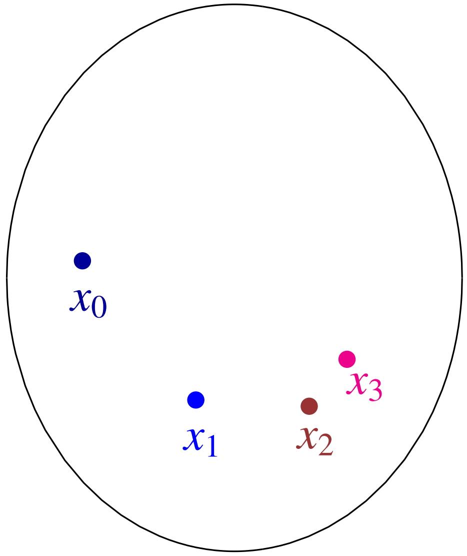
Moreover, *starting with any $x_0 \in X$, the sequence (x_k) given by $x_k = T x_{k-1}$ for $k = 1, 2, \ldots$, converges to the unique fixed point x^*.*



Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

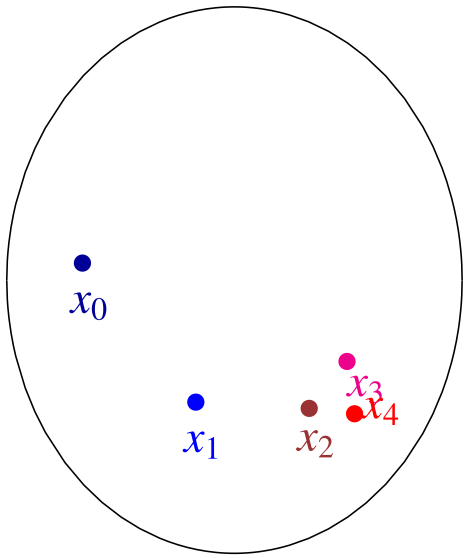
Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



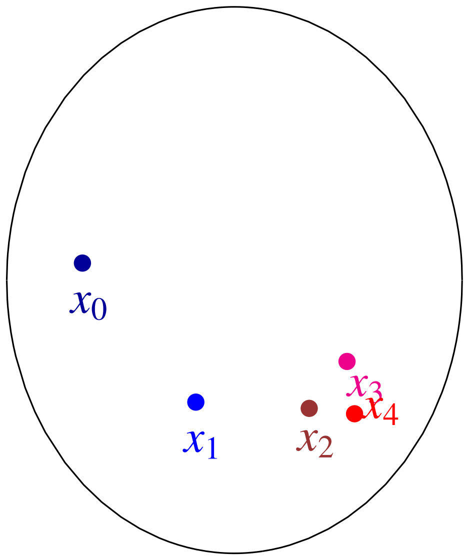
Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*



T *has a unique fixed point* x^*.

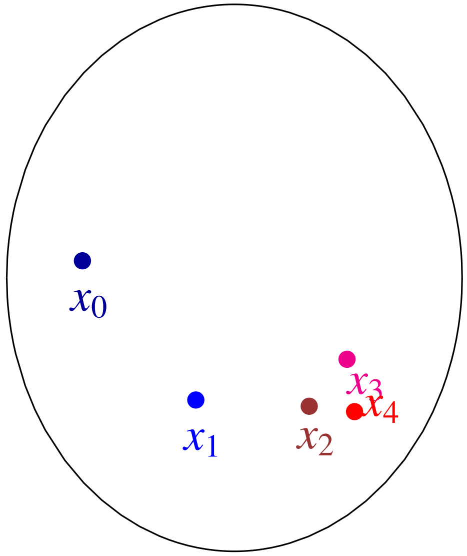
Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = T x_{k-1}$ *for* $k = 1, 2, \ldots,$ *converges to the unique fixed point* x^*.

Note $x_1 = T x_0$

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.

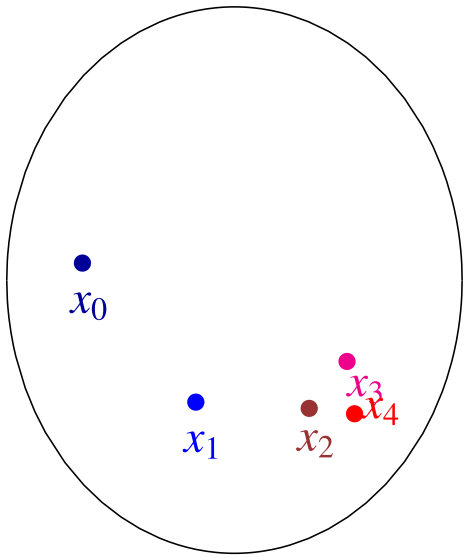


Note

$$x_1 = Tx_0$$

$$x_2 = Tx_1 =$$

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*



T *has a unique fixed point* x^*.

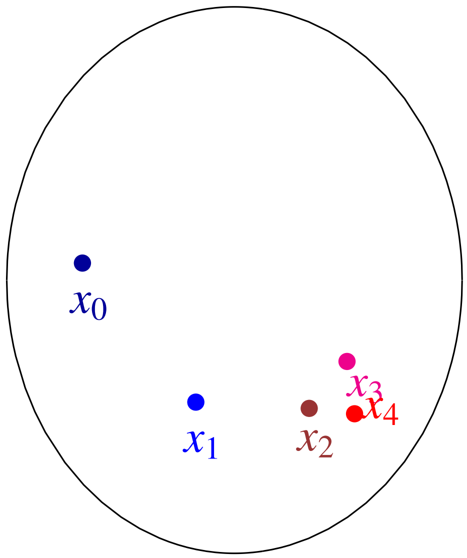
Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.

Note

$$x_1 = Tx_0$$

$$x_2 = Tx_1 = T(Tx_0) =$$

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*



T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.

Note

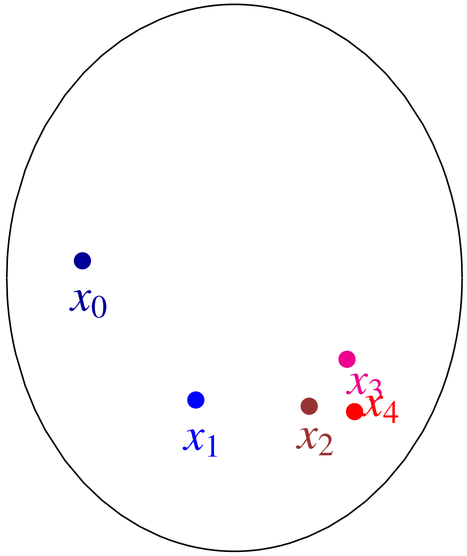
$$x_1 = Tx_0$$

$$x_2 = Tx_1 = T(Tx_0) = T^2x_0$$

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = T x_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



Note

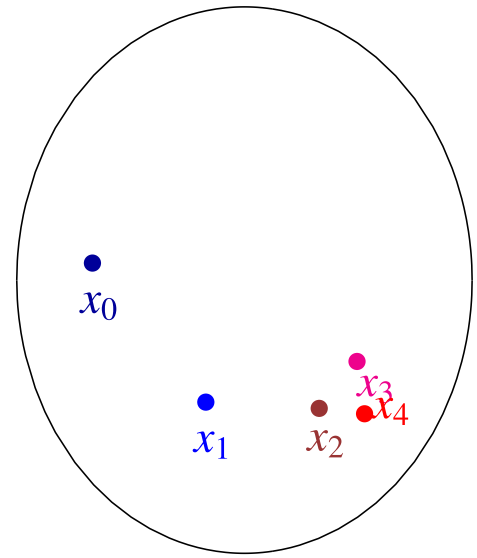
$$x_1 = T x_0$$

$$x_2 = T x_1 = T(T x_0) = T^2 x_0 \qquad \cdots \qquad x_k = T^k x_0$$

Fixed Point Theorem *Let (X, d) be a (non-empty) complete metric space. Suppose $T : X \longrightarrow X$ is a contraction map. Then*

T has a unique fixed point x^*.

Moreover, *starting with any $x_0 \in X$, the sequence (x_k) given by $x_k = Tx_{k-1}$ for $k = 1, 2, \ldots$, converges to the unique fixed point x^*.*



Note

$$x_1 = Tx_0$$

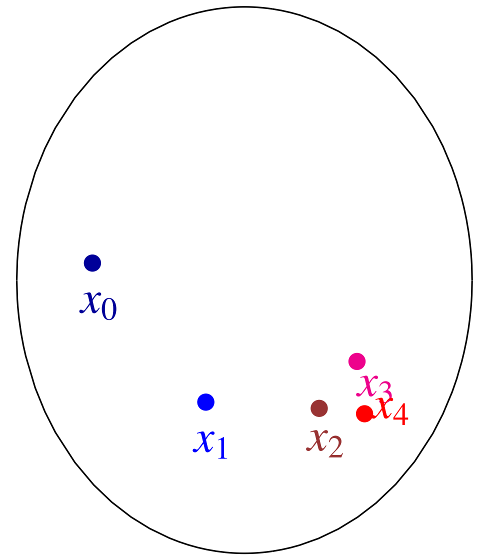
$$x_2 = Tx_1 = T(Tx_0) = T^2x_0 \qquad \cdots \qquad x_k = T^kx_0$$

Uniqueness Suppose x^* and x^{**} are fixed points of T.

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



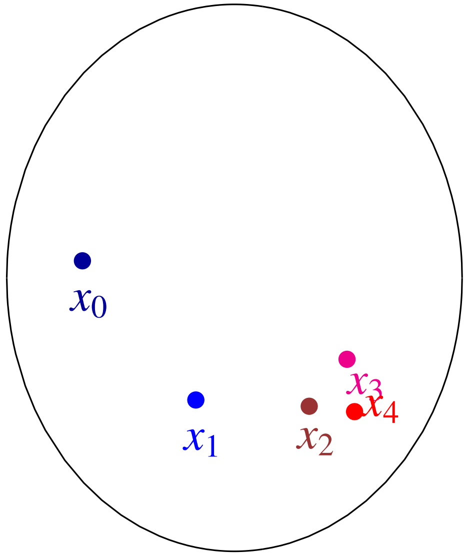
Note $x_1 = Tx_0$

$$x_2 = Tx_1 = T(Tx_0) = T^2x_0 \qquad \cdots \qquad x_k = T^k x_0$$

Uniqueness Suppose x^* and x^{**} are fixed points of T.

That is, $Tx^* = x^*$ and $Tx^{**} = x^{**}$

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*



T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = T x_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.

Note $x_1 = T x_0$

$x_2 = T x_1 = T(T x_0) = T^2 x_0 \quad \cdots \quad x_k = T^k x_0$

Uniqueness Suppose x^* and x^{**} are fixed points of T.

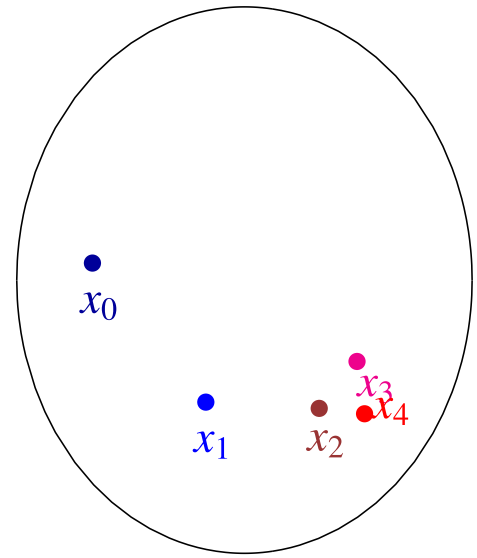
That is, $T x^* = x^*$ and $T x^{**} = x^{**}$

Consider $d(T x^*, T x^{**}) \leq$

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



Note $x_1 = Tx_0$

$$x_2 = Tx_1 = T(Tx_0) = T^2x_0 \qquad \cdots \qquad x_k = T^kx_0$$

Uniqueness Suppose x^* and x^{**} are fixed points of T.

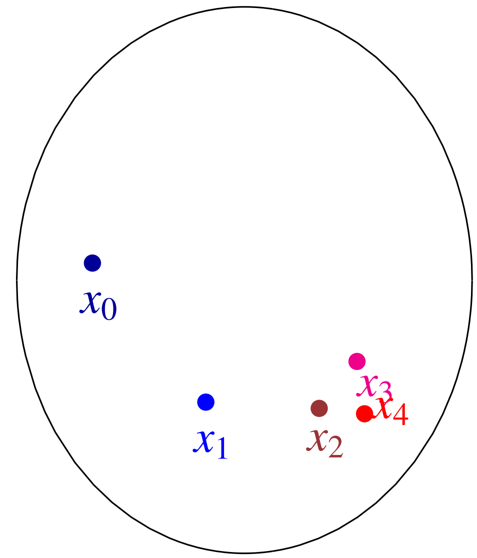
That is, $Tx^* = x^*$ and $Tx^{**} = x^{**}$

Consider $d(Tx^*, Tx^{**}) \leq \alpha \cdot d(x^*, x^{**})$

Fixed Point Theorem *Let (X, d) be a (non-empty) complete metric space. Suppose $T : X \longrightarrow X$ is a contraction map. Then*

T has a unique fixed point x^*.

Moreover, *starting with any $x_0 \in X$, the sequence (x_k) given by $x_k = Tx_{k-1}$ for $k = 1, 2, \ldots$, converges to the unique fixed point x^*.*



Note

$$x_1 = Tx_0$$

$$x_2 = Tx_1 = T(Tx_0) = T^2x_0 \qquad \cdots \qquad x_k = T^kx_0$$

Uniqueness Suppose x^* and x^{**} are fixed points of T.

That is, $Tx^* = x^*$ and $Tx^{**} = x^{**}$

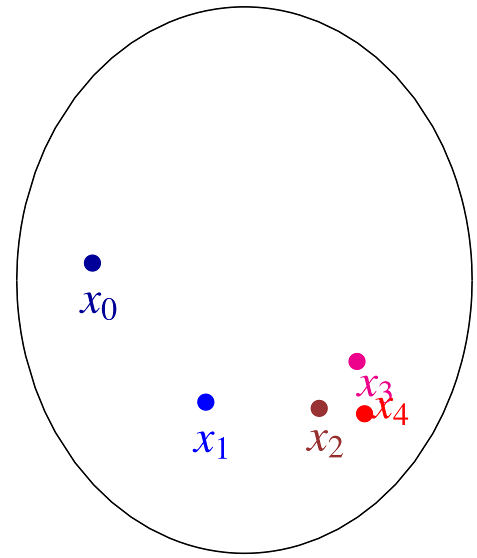
Consider $d(Tx^*, Tx^{**}) \leq \alpha \cdot d(x^*, x^{**})$

$$d(x^*, x^{**}) \leq \alpha \cdot d(x^*, x^{**})$$

Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a contraction map. Then*

T *has a unique fixed point* x^*.

Moreover, *starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.



Note $x_1 = Tx_0$

$x_2 = Tx_1 = T(Tx_0) = T^2x_0 \qquad \cdots \qquad x_k = T^kx_0$

Uniqueness Suppose x^* and x^{**} are fixed points of T.

That is, $Tx^* = x^*$ and $Tx^{**} = x^{**}$

Consider $d(Tx^*, Tx^{**}) \leq \alpha \cdot d(x^*, x^{**})$

$d(x^*, x^{**}) \leq \alpha \cdot d(x^*, x^{**})$

Therefore, $d(x^*, x^{**}) = 0$

Existence Take any $x_0 \in X$.

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = T x_{k-1}$

Claim (x_k) is a Cauchy sequence.

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof $d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = T x_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof
$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$

$$= d(T^m x_0, T^m x_1) +$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = T x_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof
$$d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n)$$

$$= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) +$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \end{aligned}$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\leq \alpha^m d(x_0, x_1) + \end{aligned}$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = T x_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof
$$
\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\
&= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\
&\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) +
\end{aligned}
$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \end{aligned}$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = T x_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned}
d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\
&= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\
&\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\
&= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1)
\end{aligned}$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\ &= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1) \longrightarrow 0 \quad \text{as } m, n \to \infty \end{aligned}$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\ &= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1) \longrightarrow 0 \quad \text{as } m, n \to \infty \end{aligned}$$

Denote $x^* = \lim_{k \to \infty} x_k$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\ &= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1) \longrightarrow 0 \quad \text{as } m, n \to \infty \end{aligned}$$

Denote $x^* = \lim_{k \to \infty} x_k$

By continuity of T, $Tx^* = \lim_{k \to \infty} Tx_k$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\ &= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1) \longrightarrow 0 \quad \text{as } m, n \to \infty \end{aligned}$$

Denote $x^* = \lim_{k\to\infty} x_k$

By continuity of T,

$$\begin{aligned} Tx^* &= \lim_{k\to\infty} Tx_k \\ &= \lim_{k\to\infty} x_{k+1} \end{aligned}$$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\le \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\ &= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1) \longrightarrow 0 \quad \text{as } m, n \to \infty \end{aligned}$$

Denote $x^* = \lim_{k \to \infty} x_k$

By continuity of T,

$$\begin{aligned} Tx^* &= \lim_{k \to \infty} Tx_k \\ &= \lim_{k \to \infty} x_{k+1} \end{aligned}$$

But $\lim_{k \to \infty} x_{k+1} = x^*$

Existence Take any $x_0 \in X$.

Consider the sequence (x_k), where $x_k = Tx_{k-1}$

Claim (x_k) is a Cauchy sequence.

Proof

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \\ &= d(T^m x_0, T^m x_1) + d(T^{m+1} x_0, T^{m+1} x_1) + \cdots + d(T^{n-1} x_0, T^n x_1) \\ &\leq \alpha^m d(x_0, x_1) + \alpha^{m+1} d(x_0, x_1) + \cdots + \alpha^{n-1} d(x_0, x_1) \\ &= \frac{\alpha^m - \alpha^n}{1 - \alpha} d(x_0, x_1) \longrightarrow 0 \quad \text{as } m, n \to \infty \end{aligned}$$

Denote $x^* = \lim_{k \to \infty} x_k$

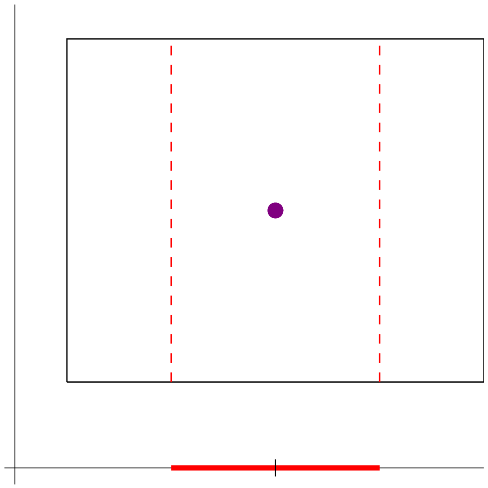
By continuity of T,

$$\begin{aligned} Tx^* &= \lim_{k \to \infty} Tx_k \\ &= \lim_{k \to \infty} x_{k+1} \end{aligned}$$

But $\lim_{k \to \infty} x_{k+1} = x^*$ Therefore, $Tx^* = x^*$

Proof of Existence and Uniqueness Theorem

Consider the IVP $\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$



Suppose

(1) f *is continuous on* R*, where*

$R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$;

(2) there exists $L \in \mathbb{R}$ *such that*

$$|f(t, y_1) - f(t, y_2)| \leq L \cdot |y_1 - y_2| \;\; \forall \; (t, y_1), (t, y_2) \in R$$

Then in some neighborhood of t_0*, the IVP has a unique solution.*

Idea of Proof

$y = \varphi(t)$ is solution IVP in neighborhood of t_0 $\Longleftrightarrow$ $\exists\, h > 0$ such that

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0, \quad \forall\, t \in [t_0 - h, t_0 + h]$$

Idea of Proof

$y = \varphi(t)$ is solution IVP in neighborhood of $t_0 \iff \exists\, h > 0$ such that

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0, \quad \forall\, t \in [t_0 - h, t_0 + h]$$

To solve the *integral equation*,

Introduce a continuous mapping, denoted by T, from a metric space into itself

$$\varphi(t) \mapsto \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

Idea of Proof

$y = \varphi(t)$ is solution IVP in neighborhood of t_0 $\iff$ $\exists\, h > 0$ such that

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\,\mathrm{d}s + y_0, \quad \forall\, t \in [t_0 - h, t_0 + h]$$

To solve the *integral equation*,

Introduce a continuous mapping, denoted by T, from a metric space into itself

$$\varphi(t) \mapsto \int_{t_0}^{t} f(s, \varphi(s))\,\mathrm{d}s + y_0$$

φ is solution to integral $\iff$ $T\varphi = \varphi$ (fixed point of T)

Idea of Proof

$y = \varphi(t)$ is solution IVP in neighborhood of $t_0 \iff \exists\, h > 0$ such that

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\,\mathrm{d}s + y_0, \quad \forall\, t \in [t_0 - h, t_0 + h]$$

To solve the *integral equation*,

Introduce a continuous mapping, denoted by T, from a metric space into itself

$$\varphi(t) \mapsto \int_{t_0}^{t} f(s, \varphi(s))\,\mathrm{d}s + y_0$$

φ is solution to integral $\iff T\varphi = \varphi$ (fixed point of T)

To find fixed point *which is unique*,

Idea of Proof

$y = \varphi(t)$ is solution IVP in neighborhood of $t_0 \iff \exists\, h > 0$ such that

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0, \quad \forall\, t \in [t_0 - h, t_0 + h]$$

To solve the *integral equation*,

Introduce a continuous mapping, denoted by T, from a metric space into itself

$$\varphi(t) \mapsto \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

φ is solution to integral $\iff T\varphi = \varphi$ (fixed point of T)

To find fixed point *which is unique*,

- Start with any (feasible) φ_0, eg. $\varphi_0 \equiv y_0$.

Idea of Proof

$y = \varphi(t)$ is solution IVP in neighborhood of $t_0 \iff \exists\, h > 0$ such that

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0, \quad \forall\, t \in [t_0 - h, t_0 + h]$$

To solve the *integral equation*,

Introduce a continuous mapping, denoted by T, from a metric space into itself

$$\varphi(t) \mapsto \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

φ is solution to integral $\iff T\varphi = \varphi$ (fixed point of T)

To find fixed point *which is unique*,

- Start with any (feasible) φ_0, eg. $\varphi_0 \equiv y_0$.
- Consider $\varphi_n = T\varphi_{n-1}$ $(n = 1, 2, \ldots)$

Idea of Proof

$y = \varphi(t)$ is solution IVP in neighborhood of $t_0 \iff \exists\, h > 0$ such that

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0, \quad \forall\, t \in [t_0 - h, t_0 + h]$$

To solve the *integral equation*,

Introduce a continuous mapping, denoted by T, from a metric space into itself

$$\varphi(t) \mapsto \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

φ is solution to integral $\iff T\varphi = \varphi$ (fixed point of T)

To find fixed point *which is unique*,

- Start with any (feasible) φ_0, eg. $\varphi_0 \equiv y_0$.
- Consider $\varphi_n = T\varphi_{n-1}$ $(n = 1, 2, \ldots)$
- The sequence of functions (φ_n) converges uniformly to a function φ^*

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{ |f(t, y)| \; : (t, y) \in R \right\}$.

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{ |f(t,y)| \ : (t,y) \in R \right\}$.

Let $0 < \alpha < 1$.

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{ |f(t,y)| \ : (t,y) \in R \right\}$.

Let $0 < \alpha < 1$.

Let $h = \min\left\{ a, \dfrac{b}{M}, \dfrac{\alpha}{L} \right\}$.

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{ |f(t,y)| \ : (t,y) \in R \right\}$.

Let $0 < \alpha < 1$.

Let $h = \min\left\{a, \frac{b}{M}, \frac{\alpha}{L}\right\}$.

Let $\mathcal{A} = \left\{\varphi \in C[t_0 - h, t_0 + h] : \ |\varphi(t) - y_0| \le b \ \ \forall\, t \in [t_0 - h, t_0 + h]\right\}$

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{|f(t,y)| \ : (t,y) \in R\right\}$.

Let $0 < \alpha < 1$.

Let $h = \min\left\{a, \frac{b}{M}, \frac{\alpha}{L}\right\}$.

Let $\mathcal{A} = \left\{\varphi \in C[t_0 - h, t_0 + h] : \ |\varphi(t) - y_0| \le b \ \ \forall \ t \in [t_0 - h, t_0 + h]\right\}$

A continuous function $\varphi : [t_0 - h, t_0 + h] \longrightarrow \mathbb{R}$ *belongs to* $\mathcal{A}$ *means that its graph lies entirely inside the rectangular region* $[t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b]$

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{ |f(t,y)| \ : (t,y) \in R \right\}$.

Let $0 < \alpha < 1$.

Let $h = \min\left\{ a, \frac{b}{M}, \frac{\alpha}{L} \right\}$.

Let $\mathcal{A} = \left\{ \varphi \in C[t_0 - h, t_0 + h] : \ |\varphi(t) - y_0| \le b \ \ \forall\, t \in [t_0 - h, t_0 + h] \right\}$

A continuous function $\varphi : [t_0 - h, t_0 + h] \longrightarrow \mathbb{R}$ belongs to $\mathcal{A}$ means that its graph lies entirely inside the rectangular region $[t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b]$

$\mathcal{A}$ is a *closed* subset of $C[t_0 - h, t_0 + h]$ *(uniform metric)*

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{ |f(t,y)| \ : (t,y) \in R \right\}$.

Let $0 < \alpha < 1$.

Let $h = \min\left\{ a, \dfrac{b}{M}, \dfrac{\alpha}{L} \right\}$.

Let $\mathcal{A} = \left\{ \varphi \in C[t_0 - h, t_0 + h] : \ |\varphi(t) - y_0| \le b \ \ \forall\, t \in [t_0 - h, t_0 + h] \right\}$

A continuous function $\varphi : [t_0 - h, t_0 + h] \longrightarrow \mathbb{R}$ *belongs to* $\mathcal{A}$ *means that its graph lies entirely inside the rectangular region* $[t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b]$

$\mathcal{A}$ is a *closed* subset of $C[t_0 - h, t_0 + h]$ *(uniform metric)*

Hence $\mathcal{A}$ is a *complete* metric space.

Proof of the Existence and Uniqueness Theorem

Denote $M = \max\left\{ |f(t,y)| \ : (t,y) \in R \right\}$.

Let $0 < \alpha < 1$.

Let $h = \min\left\{ a, \frac{b}{M}, \frac{\alpha}{L} \right\}$.

Let $\mathcal{A} = \left\{ \varphi \in C[t_0 - h, t_0 + h] : \ |\varphi(t) - y_0| \le b \ \ \forall\, t \in [t_0 - h, t_0 + h] \right\}$

A continuous function $\varphi : [t_0 - h, t_0 + h] \longrightarrow \mathbb{R}$ belongs to $\mathcal{A}$ means that its graph lies entirely inside the rectangular region $[t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b]$

$\mathcal{A}$ is a *closed* subset of $C[t_0 - h, t_0 + h]$ *(uniform metric)*

Hence $\mathcal{A}$ is a *complete* metric space.

Consider the mapping $T : \mathcal{A} \longrightarrow \mathcal{A}$ given by

$$T\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

and so $|f(s, \varphi(s))| \leq M$.

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

and so $|f(s, \varphi(s))| \leq M$.

$\therefore$ for $|t - t_0| \leq h$, $\quad |T\varphi(t) - y_0| = \left| \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s \right|$

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

and so $|f(s, \varphi(s))| \leq M$.

$\therefore$ for $|t - t_0| \leq h$,
$$\begin{aligned} |T\varphi(t) - y_0| &= \left| \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s \right| \\ &\leq \int_{t_0}^{t} |f(s, \varphi(s))|\, |\mathrm{d}s| \end{aligned}$$

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

and so $|f(s, \varphi(s))| \leq M$.

$\therefore$ for $|t - t_0| \leq h$,

$$\begin{aligned} |T\varphi(t) - y_0| &= \left| \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s \right| \\ &\leq \int_{t_0}^{t} |f(s, \varphi(s))|\, |\mathrm{d}s| \\ &\leq M|t - t_0| \end{aligned}$$

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

and so $|f(s, \varphi(s))| \leq M$.

$\therefore$ for $|t - t_0| \leq h$,

$$\begin{aligned} |T\varphi(t) - y_0| &= \left| \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s \right| \\ &\leq \int_{t_0}^{t} |f(s, \varphi(s))|\, |\mathrm{d}s| \\ &\leq M|t - t_0| \\ &\leq Mh \end{aligned}$$

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

and so $|f(s, \varphi(s))| \leq M$.

$\therefore$ for $|t - t_0| \leq h$,

$$\begin{aligned} |T\varphi(t) - y_0| &= \left| \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s \right| \\ &\leq \int_{t_0}^{t} |f(s, \varphi(s))|\, |\mathrm{d}s| \\ &\leq M|t - t_0| \\ &\leq Mh \\ &\leq b \end{aligned}$$

To show that T maps $\mathcal{A}$ into $\mathcal{A}$

Let $\varphi \in \mathcal{A}$.

By construction, for all $s \in [t_0 - h, t_0 + h]$,

$$(s, \varphi(s)) \in [t_0 - h, t_0 + h] \times [y_0 - b, y_0 + b] \subseteq R$$

and so $|f(s, \varphi(s))| \le M$.

$\therefore$ for $|t - t_0| \le h$,

$$\begin{aligned} |T\varphi(t) - y_0| &= \left| \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s \right| \\ &\le \int_{t_0}^{t} |f(s, \varphi(s))|\, |\mathrm{d}s| \\ &\le M|t - t_0| \\ &\le Mh \\ &\le b \end{aligned}$$

Hence $T\varphi \in \mathcal{A}$.

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$$\forall\ t \in [t_0 - h, t_0 + h], \quad |T\varphi_1(t) - T\varphi_2(t)| = \left| \int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big) \, \mathrm{d}s \right|$$

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$$\forall\ t \in [t_0 - h, t_0 + h], \qquad |T\varphi_1(t) - T\varphi_2(t)| \;=\; \left|\int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s))\Big)\, \mathrm{d}s\right|$$

$$\leq \int_{t_0}^{t} \Big|f(s, \varphi_1(s)) - f(s, \varphi_2(s))\Big|\, |\mathrm{d}s|$$

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$$\forall\ t \in [t_0 - h, t_0 + h], \quad |T\varphi_1(t) - T\varphi_2(t)| = \left| \int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big)\, \mathrm{d}s \right|$$

$$\leq \int_{t_0}^{t} \Big| f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big|\, |\mathrm{d}s|$$

$$\leq \int_{t_0}^{t} L \cdot \Big| \varphi_1(s) - \varphi_2(s) \Big|\, |\mathrm{d}s|$$

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$$\forall\ t \in [t_0 - h, t_0 + h], \qquad |T\varphi_1(t) - T\varphi_2(t)| = \left| \int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big)\, \mathrm{d}s \right|$$

$$\leq \int_{t_0}^{t} \Big| f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big|\, |\mathrm{d}s|$$

$$\leq \int_{t_0}^{t} L \cdot \Big| \varphi_1(s) - \varphi_2(s) \Big|\, |\mathrm{d}s|$$

$$\leq \int_{t_0}^{t} L \cdot\ d(\varphi_1, \varphi_2)\, |\mathrm{d}s|$$

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$\forall\ t \in [t_0 - h, t_0 + h]$,

$$
\begin{aligned}
|T\varphi_1(t) - T\varphi_2(t)| &= \left| \int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big) \, \mathrm{d}s \right| \\
&\leq \int_{t_0}^{t} \Big| f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big| \, |\mathrm{d}s| \\
&\leq \int_{t_0}^{t} L \cdot \Big| \varphi_1(s) - \varphi_2(s) \Big| \, |\mathrm{d}s| \\
&\leq \int_{t_0}^{t} L \cdot \ d(\varphi_1, \varphi_2) \, |\mathrm{d}s| \\
&= L \cdot d(\varphi_1, \varphi_2) \cdot |t - t_0|
\end{aligned}
$$

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$\forall\ t \in [t_0 - h, t_0 + h]$,

$$
\begin{aligned}
|T\varphi_1(t) - T\varphi_2(t)| &= \left| \int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big) \, \mathrm{d}s \right| \\
&\le \int_{t_0}^{t} \Big| f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big| \, |\mathrm{d}s| \\
&\le \int_{t_0}^{t} L \cdot \Big| \varphi_1(s) - \varphi_2(s) \Big| \, |\mathrm{d}s| \\
&\le \int_{t_0}^{t} L \cdot \ d(\varphi_1, \varphi_2) \, |\mathrm{d}s| \\
&= L \cdot d(\varphi_1, \varphi_2) \cdot |t - t_0| \\
&\le L \cdot d(\varphi_1, \varphi_2) \cdot h
\end{aligned}
$$

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$\forall\ t \in [t_0 - h, t_0 + h]$,

$$
\begin{aligned}
|T\varphi_1(t) - T\varphi_2(t)| &= \left| \int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big)\, \mathrm{d}s \right| \\
&\le \int_{t_0}^{t} \Big| f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big|\, |\mathrm{d}s| \\
&\le \int_{t_0}^{t} L \cdot \Big| \varphi_1(s) - \varphi_2(s) \Big|\, |\mathrm{d}s| \\
&\le \int_{t_0}^{t} L \cdot\ d(\varphi_1, \varphi_2)\, |\mathrm{d}s| \\
&= L \cdot d(\varphi_1, \varphi_2) \cdot |t - t_0| \\
&\le L \cdot d(\varphi_1, \varphi_2) \cdot h \\
&\le \alpha \cdot d(\varphi_1, \varphi_2)
\end{aligned}
$$

To show that T is a contraction map

Let $\varphi_1, \varphi_2 \in \mathcal{A}$.

$$
\begin{aligned}
\forall\ t \in [t_0 - h, t_0 + h], \qquad |T\varphi_1(t) - T\varphi_2(t)| &= \left| \int_{t_0}^{t} \Big(f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big)\, \mathrm{d}s \right| \\
&\le \int_{t_0}^{t} \Big| f(s, \varphi_1(s)) - f(s, \varphi_2(s)) \Big|\, |\mathrm{d}s| \\
&\le \int_{t_0}^{t} L \cdot \Big| \varphi_1(s) - \varphi_2(s) \Big|\, |\mathrm{d}s| \\
&\le \int_{t_0}^{t} L \cdot\ d(\varphi_1, \varphi_2)\, |\mathrm{d}s| \\
&= L \cdot d(\varphi_1, \varphi_2) \cdot |t - t_0| \\
&\le L \cdot d(\varphi_1, \varphi_2) \cdot h \\
&\le \alpha \cdot d(\varphi_1, \varphi_2)
\end{aligned}
$$

Thus $d(T\varphi_1, T\varphi_2) \ \le\ \alpha \cdot d(\varphi_1, \varphi_2)$

Proof of Existence and Uniqueness Theorem (cont...)

From above $T : \mathcal{A} \longrightarrow \mathcal{A}$ is a contraction map.

Proof of Existence and Uniqueness Theorem (cont...)

From above $T : \mathcal{A} \longrightarrow \mathcal{A}$ is a contraction map.

By Fixed Point Theorem, T has a unique fixed point

Proof of Existence and Uniqueness Theorem (cont...)

From above $T : \mathcal{A} \longrightarrow \mathcal{A}$ is a contraction map.

By Fixed Point Theorem, T has a unique fixed point

(unique solution to the IVP)

Proof of Existence and Uniqueness Theorem (cont...)

From above $T : \mathcal{A} \longrightarrow \mathcal{A}$ is a contraction map.

By Fixed Point Theorem, T has a unique fixed point

(unique solution to the IVP)

Moreover, starting from any $\varphi_0 \in \mathcal{A}$,

the sequence of Picard iterates $\varphi_{n+1} = T\varphi_n$

converges to the unique solution uniformly on $[t_0 - h, t_0 + h]$

Remark

Can take $h = \min\left\{a, \dfrac{b}{M}\right\}$

Remark

Can take $h = \min\left\{a, \dfrac{b}{M}\right\}$

In this case,

T *may not be* a contraction map.

Remark

Can take $h = \min\left\{a, \dfrac{b}{M}\right\}$

In this case,

T *may not be* a contraction map.

However T^N is a contraction map if N *is large enough*

Remark

Can take $h = \min\left\{a, \dfrac{b}{M}\right\}$

In this case,

T *may not be* a contraction map.

However T^N is a contraction map if N *is large enough*

Use induction to show

$$|T^n\varphi_1(t) - T^n\varphi_2(t)| \leq \frac{L^n|t-t_0|^n}{n!}d(\varphi_1,\varphi_2) \quad \text{for} \quad \begin{aligned} &\varphi_1,\varphi_2 \in \mathcal{A}, \\ &|t-t_0| \leq h, \\ &n \in \mathbb{Z}^+ \end{aligned}$$

Remark

Can take $h = \min\left\{a, \dfrac{b}{M}\right\}$

In this case,

T *may not be* a contraction map.

However T^N is a contraction map if N *is large enough*

Use induction to show

$$|T^n\varphi_1(t) - T^n\varphi_2(t)| \leq \frac{L^n|t-t_0|^n}{n!} d(\varphi_1, \varphi_2) \quad \text{for} \quad \varphi_1, \varphi_2 \in \mathcal{A},$$

$$|t - t_0| \leq h,$$

$$n \in \mathbb{Z}^+$$

Note $\dfrac{L^n|t-t_0|^n}{n!} \leq \dfrac{L^n a^n}{n!} \to 0$ as $n \to \infty$

Remark

Can take $h = \min\left\{a, \dfrac{b}{M}\right\}$

In this case,

T *may not be* a contraction map.

However T^N is a contraction map if N *is large enough*

Use induction to show

$$|T^n\varphi_1(t) - T^n\varphi_2(t)| \le \frac{L^n|t-t_0|^n}{n!}d(\varphi_1,\varphi_2) \quad \text{for} \quad \varphi_1,\varphi_2 \in \mathcal{A},$$

$$|t-t_0| \le h,$$

$$n \in \mathbb{Z}^+$$

Note $\dfrac{L^n|t-t_0|^n}{n!} \le \dfrac{L^n a^n}{n!} \to 0$ as $n \to \infty$

Required result follows from the Generalized Fixed Point Theorem.

Generalized Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a mapping such that* $T^N : X \longrightarrow X$ *is a contraction map for some natural number* N*. Then*

Generalized Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a mapping such that* $T^N : X \longrightarrow X$ *is a contraction map for some natural number* N*. Then*

- T *has a unique fixed point* x^**.*

Generalized Fixed Point Theorem *Let* (X, d) *be a (non-empty) complete metric space. Suppose* $T : X \longrightarrow X$ *is a mapping such that* $T^N : X \longrightarrow X$ *is a contraction map for some natural number* N. *Then*

- T *has a unique fixed point* x^*.
- *Moreover, starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = Tx_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.

Generalized Fixed Point Theorem *Let (X, d) be a (non-empty) complete metric space. Suppose $T : X \longrightarrow X$ is a mapping such that $T^N : X \longrightarrow X$ is a contraction map for some natural number N. Then*

- T *has a unique fixed point* x^*.
- *Moreover, starting with any* $x_0 \in X$, *the sequence* (x_k) *given by* $x_k = T x_{k-1}$ *for* $k = 1, 2, \ldots$, *converges to the unique fixed point* x^*.

Remark Distance between the iterates and the unique fixed point

$$d(x^*, x_k) \le \frac{\alpha^{[\frac{k}{N}]}}{1 - \alpha} \cdot \max_{0 \le i < N} d(T^i x_0, T^{N+i} x_0)$$

where $0 < \alpha < 1$, $d(T^N x, T^N y) \le \alpha \cdot d(x, y)$ for all $x, y \in X$.

Existence and Uniqueness Theorem for First Order Linear IVP

Consider the following IVP:
$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

Suppose $p, g \in C(a,b)$, $t_0 \in (a,b)$ *and* $y_0 \in \mathbb{R}$. *Then the IVP has a unique solution on* (a,b).

Existence and Uniqueness Theorem for First Order Linear IVP

Consider the following IVP:
$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

Suppose $p, g \in C(a, b)$*,* $t_0 \in (a, b)$ *and* $y_0 \in \mathbb{R}$*. Then the IVP has a unique solution on* (a, b)*.*

Remark Result also holds for 1st order linear system IVP.

Existence and Uniqueness Theorem for First Order Linear IVP

Consider the following IVP: $\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$

Suppose $p, g \in C(a, b)$, $t_0 \in (a, b)$ *and* $y_0 \in \mathbb{R}$. *Then the IVP has a unique solution on* (a, b).

Remark Result also holds for 1st order linear system IVP.

Proof Rewrite DE $y' = f(t, y)$ where $f(t, y) = g(t) - p(t)y$.

Suffices to show *for all* α, β *with* $a < \alpha < t_0 < \beta < b$, *the integral equation*

$$\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\,\mathrm{d}s + y_0$$

has unique solution on $[\alpha, \beta]$.

- Let $\mathcal{A} = C[\alpha, \beta]$ (a complete metric space).

  Consider $T : \mathcal{A} \longrightarrow \mathcal{A}$

$$T\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

If $\varphi \in \mathcal{A}$, *then* $T\varphi \in \mathcal{A}$

- Let $\mathcal{A} = C[\alpha, \beta]$ (a complete metric space).

  Consider $T : \mathcal{A} \longrightarrow \mathcal{A}$

$$T\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

  If $\varphi \in \mathcal{A}$, *then* $T\varphi \in \mathcal{A}$

- Consider $R = [\alpha, \beta] \times (-\infty, \infty)$.

  f satisfies *Lipschitz condition w.r.t. 2nd argument*

$$|f_y(t, y)| = |p(t)| \leq L \qquad \forall\ (t, y) \in R$$

  where $L = \max_{t \in [\alpha, \beta]} |p(t)|$.

- Let $\mathcal{A} = C[\alpha, \beta]$ (a complete metric space).

  Consider $T : \mathcal{A} \longrightarrow \mathcal{A}$

$$T\varphi(t) = \int_{t_0}^{t} f(s, \varphi(s))\, \mathrm{d}s + y_0$$

  If $\varphi \in \mathcal{A}$, *then* $T\varphi \in \mathcal{A}$

- Consider $R = [\alpha, \beta] \times (-\infty, \infty)$.

  f satisfies *Lipschitz condition w.r.t. 2nd argument*

$$|f_y(t, y)| = |p(t)| \le L \qquad \forall\ (t, y) \in R$$

  where $L = \max_{t \in [\alpha, \beta]} |p(t)|$.

- T^N is a contraction map if N is large enough.