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Methods to find $y_2(t)$ for cases 2 and 3

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t(2v' + tv'') + t(v + tv') - tv &= 0 \\
t^2 v'' + (2t + t^2)v' &= 0 \\
v'' + \left(\frac{2}{t} + 1\right)v' &= 0 \\
w' &= -\left(1 + \frac{2}{t}\right)w \qquad \text{put } w = v' \\
\frac{w'}{w} &= -\left(1 + \frac{2}{t}\right) \qquad \text{consider } w > 0
\end{aligned}$$

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Thus $$y_2(t) = -1 - t\ln t + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!(n-1)} t^n \qquad y_{2a}(t) = t\ln t + 1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n!(n-1)} t^n$$

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Substitution yields

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-1} + \sum_{n=0}^{\infty} 4(n+r)a_n t^{n+r-1} - \sum_{n=0}^{\infty} a_n t^{n+r+1} = 0$$

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$$\frac{\mathrm{d}}{\mathrm{d}r}(r+3)a_{2k}(r)\bigg|_{r=-3} \quad = \quad (r+3)a'_{2k}(r) + a_{2k}(r)\Big|_{r=-3}$$

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Thus $\quad y_2(t) = t^{-3} - \frac{1}{2}t^{-1} - \sum_{k=2}^{\infty} \frac{1}{2^k k!(2k-3)!!} t^{2k-3}$

Alternative way for regular singular point Case 3 $r_1 - r_2 \in \mathbb{Z}^+$

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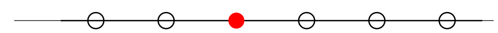

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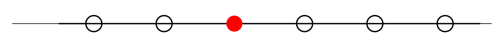

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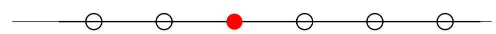

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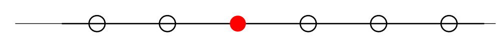

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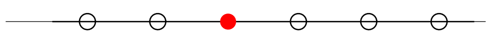

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For $r \in (r_2 - 1, r_2 + 1)$,

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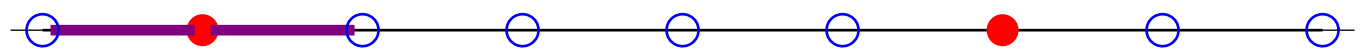

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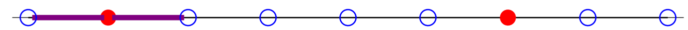

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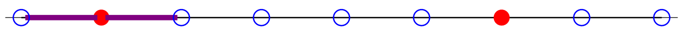

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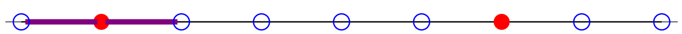
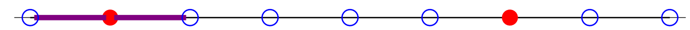
For $r \in (r_2 - 1, r_2 + 1) \setminus \{r_2\}$,

since $F(r+n) \neq 0$ for all $n \geq N$,



For $r \in (r_2 - 1, r_2 + 1)$,

since $F(r+n) \neq 0 \quad n = 1, \ldots, N-1$,

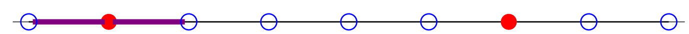


can find $a_1(r), a_2(r), \ldots, a_{N-1}(r)$ such that

$$a_n(r)F(r+n) + \sum_{k=0}^{n-1} a_k(r)\Big((r+k)p_{n-k} + q_{n-k}\Big) = 0 \quad 1 \leq n \leq N-1$$

For $r \in (r_2 - 1, r_2 + 1) \setminus \{r_2\}$,

since $F(r+n) \neq 0$ for all $n \geq N$,



can find $a_N(r)$, $a_{N+1}(r)$, $\ldots$ such that

$$a_n(r)F(r+n) + \sum_{k=0}^{n-1} a_k(r)\Big((r+k)p_{n-k} + q_{n-k}\Big) = 0 \quad \forall\, n \geq N$$

Consider $$\widetilde{\phi}(r,t) \quad = \quad t^r\left((r-r_2)+\sum_{n=1}^{\infty}\widetilde{a}_n(r)t^n\right)$$

Consider

$$\widetilde{\phi}(r,t) = t^r\left((r-r_2)+\sum_{n=1}^{\infty}\widetilde{a}_n(r)t^n\right)$$

$$= \sum_{n=0}^{\infty}\widetilde{a}_n(r)t^{n+r}$$

where $\widetilde{a}_0(r) = r - r_2$

Consider

$$\widetilde{\phi}(r,t) = t^r\left((r-r_2) + \sum_{n=1}^{\infty} \widetilde{a}_n(r)t^n\right)$$

$$= \sum_{n=0}^{\infty} \widetilde{a}_n(r)t^{n+r} \qquad \text{where } \widetilde{a}_0(r) = r - r_2$$

Substitute $\widetilde{\phi}(r,t)$ into DE see (†)

$$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r + \sum_{n=1}^{\infty}\left[\widetilde{a}_n(r)F(r+n) + \sum_{k=0}^{n-1}\widetilde{a}_k(r)\Big((r+k)p_{n-k} + q_{n-k}\Big)\right]t^{r+n}$$

Consider $$\widetilde{\phi}(r,t) = t^r\left((r-r_2)+\sum_{n=1}^{\infty}\widetilde{a}_n(r)t^n\right)$$

$$= \sum_{n=0}^{\infty}\widetilde{a}_n(r)t^{n+r} \qquad \text{where } \widetilde{a}_0(r) = r-r_2$$

Substitute $\widetilde{\phi}(r,t)$ into DE see (†)

$$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r + \sum_{n=1}^{\infty}\left[\widetilde{a}_n(r)F(r+n)+\sum_{k=0}^{n-1}\widetilde{a}_k(r)\Big((r+k)p_{n-k}+q_{n-k}\Big)\right]t^{r+n}$$

Claim 1 For $r \in (r_2-1, r_2+1)$, $n \geq 1$

$\widetilde{a}_n(r)$ can be solved so that

$$\widetilde{a}_n(r)F(r+n)+\sum_{k=0}^{n-1}\widetilde{a}_k(r)\Big((r+k)p_{n-k}+q_{n-k}\Big) = 0 \qquad (*)$$

Compare with $(r - r_2)\phi(r,t) = \sum_{n=0}^{\infty}(r - r_2)a_n(r)t^{n+r}$ where $a_0(r) \equiv 1$

Compare with $$(r - r_2)\phi(r,t) = \sum_{n=0}^{\infty} (r - r_2)a_n(r)t^{n+r}$$ where $a_0(r) \equiv 1$

For $0 \le n \le N-1$, $r \in (r_2 - 1, r_2 + 1)$, $\widetilde{a}_n(r) = (r - r_2)a_n(r)$

Compare with $$(r-r_2)\phi(r,t) \;=\; \sum_{n=0}^{\infty}(r-r_2)a_n(r)t^{n+r}$$ where $a_0(r) \equiv 1$

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Compare with $(r-r_2)\phi(r,t) \;=\; \sum_{n=0}^{\infty}(r-r_2)a_n(r)t^{n+r}$ where $a_0(r)\equiv 1$

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Compare with $(r-r_2)\phi(r,t) \;=\; \sum_{n=0}^{\infty}(r-r_2)a_n(r)t^{n+r}$ where $a_0(r)\equiv 1$

For $0\le n\le N-1$, $r\in(r_2-1,r_2+1)$, $\widetilde{a}_n(r)=(r-r_2)a_n(r)$

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$$\widetilde{a}_N(r)F(r+N)+\sum_{k=0}^{N-1}(r-r_2)a_k(r)\Big((r+k)p_{N-k}+q_{N-k}\Big)=0$$

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$$\widetilde{a}_N(r)F(r+N) + \sum_{k=0}^{N-1}(r-r_2)a_k(r)\Big((r+k)p_{N-k} + q_{N-k}\Big) = 0$$

Since $F(r) = (r-r_1)(r-r_2)$

Compare with $(r - r_2)\phi(r,t) \; = \; \sum_{n=0}^{\infty} (r - r_2) a_n(r) t^{n+r}$ where $a_0(r) \equiv 1$

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$$\widetilde{a}_N(r) F(r + N) + \sum_{k=0}^{N-1} (r - r_2) a_k(r) \Big((r + k) p_{N-k} + q_{N-k} \Big) = 0$$

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$$\widetilde{a}_N(r)(r + N - r_1)(r + N - r_2) +$$

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$$\widetilde{a}_N(r)F(r+N) + \sum_{k=0}^{N-1}(r - r_2)a_k(r)\Big((r+k)p_{N-k} + q_{N-k}\Big) = 0$$

Since $F(r) = (r - r_1)(r - r_2)$

$$\widetilde{a}_N(r)(r+N-r_1)(r+N-r_2) + \sum_{k=0}^{N-1}(r - r_2)a_k(r)\Big((r+k)p_{N-k} + q_{N-k}\Big) \;=\; 0$$

$$\widetilde{a}_N(r)(r - r_2)(r+N-r_2) +$$

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For $0\le n\le N-1$, $r\in(r_2-1,r_2+1)$, $\widetilde{a}_n(r)=(r-r_2)a_n(r)$

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$$\widetilde{a}_N(r)(r+N-r_1)(r+N-r_2)+\sum_{k=0}^{N-1}(r-r_2)a_k(r)\Big((r+k)p_{N-k}+q_{N-k}\Big) \;=\; 0$$

$$\widetilde{a}_N(r)(r-r_2)(r+N-r_2)+\sum_{k=0}^{N-1}(r-r_2)a_k(r)\Big((r+k)p_{N-k}+q_{N-k}\Big) \;=\; 0$$

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$$\widetilde{a}_N(r)F(r + N) + \sum_{k=0}^{N-1}(r - r_2)a_k(r)\Big((r + k)p_{N-k} + q_{N-k}\Big) = 0$$

Since $F(r) = (r - r_1)(r - r_2)$

$$\widetilde{a}_N(r)(r + N - r_1)(r + N - r_2) + \sum_{k=0}^{N-1}(r - r_2)a_k(r)\Big((r + k)p_{N-k} + q_{N-k}\Big) = 0$$

$$\widetilde{a}_N(r)(r - r_2)(r + N - r_2) + \sum_{k=0}^{N-1}(r - r_2)a_k(r)\Big((r + k)p_{N-k} + q_{N-k}\Big) = 0$$

Define $\widetilde{a}_N(r) = \dfrac{-1}{r + N - r_2}\sum_{k=0}^{N-1} a_k(r)\Big((r+k)p_{N-k}+q_{N-k}\Big)$ for $r \in (r_2-1, r_2+1)$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r + n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

□

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

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Claim 2 for $n \geq 0$, $\widetilde{a}_{N+n}(r_2) = \widetilde{a}_N(r_2) a_n(r_1)$

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$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) +$$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

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$$\widetilde{a}_{N+1}(r_2) F(r_2 + N + 1) + \sum_{k=0}^{N+1-1} \widetilde{a}_k(r_2)\Big((r_2 + k) p_{N+1-k} + q_{N+1-k}\Big) = 0$$

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$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + \sum_{k=0}^{N+1-1} \widetilde{a}_k(r_2)\Big((r_2 + k)p_{N+1-k} + q_{N+1-k}\Big) \quad = \quad 0$$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + 0 +$$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

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Claim 2 for $n \geq 0$, $\widetilde{a}_{N+n}(r_2) = \widetilde{a}_N(r_2) a_n(r_1)$

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$$\widetilde{a}_{N+1}(r_2) F(r_2 + N + 1) + 0 + \widetilde{a}_N(r_2) \Big((r_2 + N) p_{N+1-N} + q_{N+1-N} \Big) = 0$$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

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$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + \sum_{k=0}^{N+1-1} \widetilde{a}_k(r_2)\Big((r_2 + k)p_{N+1-k} + q_{N+1-k}\Big) = 0$$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + 0 + \widetilde{a}_N(r_2)\Big((r_2 + N)p_{N+1-N} + q_{N+1-N}\Big) = 0$$

$$\widetilde{a}_{N+1}(r_2)F(r_1 + 1) + \widetilde{a}_N(r_2)(r_1 p_1 + q_1) = 0$$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

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Claim 2 for $n \geq 0$, $\widetilde{a}_{N+n}(r_2) = \widetilde{a}_N(r_2) a_n(r_1)$

$n = 0$ $\widetilde{a}_N(r_2) = \widetilde{a}_N(r_2) a_0(r_1)$

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$$\widetilde{a}_{N+1}(r_2) F(r_2 + N + 1) + 0 + \widetilde{a}_N(r_2)\Big((r_2 + N) p_{N+1-N} + q_{N+1-N}\Big) = 0$$

$$\widetilde{a}_{N+1}(r_2) F(r_1 + 1) + \widetilde{a}_N(r_2)(r_1 p_1 + q_1) = 0$$

In $(\ddagger)$, put $n = 1$ and $r = r_1$ $a_1(r_1) F(r_1 + 1) + a_0(r_1)(r_1 p_1 + q_1) = 0$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

□

Claim 2 for $n \geq 0$, $\widetilde{a}_{N+n}(r_2) = \widetilde{a}_N(r_2)a_n(r_1)$

$n = 0$ $\widetilde{a}_N(r_2) = \widetilde{a}_N(r_2)a_0(r_1)$

$n = 1$ by $(*)$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + \sum_{k=0}^{N+1-1} \widetilde{a}_k(r_2)\Big((r_2 + k)p_{N+1-k} + q_{N+1-k}\Big) = 0$$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + 0 + \widetilde{a}_N(r_2)\Big((r_2 + N)p_{N+1-N} + q_{N+1-N}\Big) = 0$$

$$\widetilde{a}_{N+1}(r_2)F(r_1 + 1) + \widetilde{a}_N(r_2)(r_1 p_1 + q_1) = 0$$

In (‡), put $n = 1$ and $r = r_1$ $a_1(r_1)F(r_1 + 1) + a_0(r_1)(r_1 p_1 + q_1) = 0$

Thus $\widetilde{a}_{N+1}(r_2)F(r_1 + 1) = \widetilde{a}_N(r_2)a_1(r_1)F(r_1 + 1)$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

□

Claim 2 for $n \geq 0$, $\widetilde{a}_{N+n}(r_2) = \widetilde{a}_N(r_2)a_n(r_1)$

$n = 0$ $\widetilde{a}_N(r_2) = \widetilde{a}_N(r_2)a_0(r_1)$

$n = 1$ by $(*)$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + \sum_{k=0}^{N+1-1} \widetilde{a}_k(r_2)\Big((r_2 + k)p_{N+1-k} + q_{N+1-k}\Big) = 0$$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + 0 + \widetilde{a}_N(r_2)\Big((r_2 + N)p_{N+1-N} + q_{N+1-N}\Big) = 0$$

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Thus

$$\widetilde{a}_{N+1}(r_2)F(r_1 + 1) = \widetilde{a}_N(r_2)a_1(r_1)F(r_1 + 1)$$

$$\widetilde{a}_{N+1}(r_2) = \widetilde{a}_N(r_2)a_1(r_1)$$

For $n > N$ and $r \in (r_2 - 1, r_2 + 1)$, since $F(r+n) \neq 0$, $\widetilde{a}_n(r)$ can be defined

□

Claim 2 for $n \geq 0$, $\widetilde{a}_{N+n}(r_2) = \widetilde{a}_N(r_2)a_n(r_1)$

$n = 0$ $\quad \widetilde{a}_N(r_2) = \widetilde{a}_N(r_2)a_0(r_1)$

$n = 1$ by $(*)$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + \sum_{k=0}^{N+1-1} \widetilde{a}_k(r_2)\Big((r_2 + k)p_{N+1-k} + q_{N+1-k}\Big) = 0$$

$$\widetilde{a}_{N+1}(r_2)F(r_2 + N + 1) + 0 + \widetilde{a}_N(r_2)\Big((r_2 + N)p_{N+1-N} + q_{N+1-N}\Big) = 0$$

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In (‡), put $n = 1$ and $r = r_1$ $\quad a_1(r_1)F(r_1 + 1) + a_0(r_1)(r_1 p_1 + q_1) = 0$

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$$\widetilde{a}_{N+1}(r_2)F(r_1 + 1) = \widetilde{a}_N(r_2)a_1(r_1)F(r_1 + 1)$$

$$\widetilde{a}_{N+1}(r_2) = \widetilde{a}_N(r_2)a_1(r_1)$$

□

Using these $\widetilde{a}_n(r)$'s, $$L[\widetilde{\phi}](r,t) \quad = \quad (r - r_2)F(r)t^r$$

Using these $\widetilde{a}_n(r)$'s,

$$
\begin{aligned}
L[\widetilde{\phi}](r,t) &= (r-r_2)F(r)t^r \\
&= (r-r_2)^2(r-r_1)t^r
\end{aligned}
$$

Using these $\widetilde{a}_n(r)$'s,

$$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r$$

$$= (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r

$$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$

Using these $\widetilde{a}_n(r)$'s, $$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r = (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r $$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) = 2(r-r_2)\cdot(r-r_1)t^r + (r-r_2)^2\cdot\frac{\partial}{\partial r}(r-r_1)t^r$$

Using these $\widetilde{a}_n(r)$'s,

$$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r = (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r

$$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) = 2(r-r_2)\cdot(r-r_1)t^r + (r-r_2)^2\cdot\frac{\partial}{\partial r}(r-r_1)t^r$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r_2,t) = 0$$

Using these $\widetilde{a}_n(r)$'s,

$$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r = (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r

$$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) = 2(r-r_2)\cdot(r-r_1)t^r + (r-r_2)^2\cdot\frac{\partial}{\partial r}(r-r_1)t^r$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r_2,t) = 0$$

Second solution

$$y_2(t) = \left.\frac{\partial}{\partial r}\widetilde{\phi}(r,t)\right|_{r=r_2}$$

Using these $\widetilde{a}_n(r)$'s, $$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r = (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r $$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) = 2(r-r_2)\cdot(r-r_1)t^r + (r-r_2)^2\cdot\frac{\partial}{\partial r}(r-r_1)t^r$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r_2,t) = 0$$

Second solution $$y_2(t) = \frac{\partial}{\partial r}\widetilde{\phi}(r,t)\Big|_{r=r_2}$$

$$= \left(t^r\ln t\right)\left((r-r_2) + \sum_{n=1}^{\infty}\widetilde{a}_n(r)t^n\right) + t^r\left(1 + \sum_{n=1}^{\infty}\widetilde{a}'_n(r)t^n\right)$$

Using these $\widetilde{a}_n(r)$'s,

$$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r$$
$$= (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r

$$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) = 2(r-r_2)\cdot(r-r_1)t^r + (r-r_2)^2\cdot\frac{\partial}{\partial r}(r-r_1)t^r$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r_2,t) = 0$$

Second solution

$$y_2(t) = \frac{\partial}{\partial r}\widetilde{\phi}(r,t)\Big|_{r=r_2}$$
$$= \left(t^r\ln t\right)\left((r-r_2)+\sum_{n=1}^{\infty}\widetilde{a}_n(r)t^n\right) + t^r\left(1+\sum_{n=1}^{\infty}\widetilde{a}_n'(r)t^n\right)\Bigg|_{r=r_2}$$

Using these $\widetilde{a}_n(r)$'s,
$$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r = (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r
$$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$
$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) = 2(r-r_2)\cdot(r-r_1)t^r + (r-r_2)^2\cdot\frac{\partial}{\partial r}(r-r_1)t^r$$
$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r_2,t) = 0$$

Second solution
$$y_2(t) = \left.\frac{\partial}{\partial r}\widetilde{\phi}(r,t)\right|_{r=r_2}$$
$$= \left.\Big(t^r\ln t\Big)\left((r-r_2)+\sum_{n=1}^{\infty}\widetilde{a}_n(r)t^n\right) + t^r\left(1+\sum_{n=1}^{\infty}\widetilde{a}_n'(r)t^n\right)\right|_{r=r_2}$$
$$= \Big(t^{r_2}\ln t\Big)\sum_{n=1}^{\infty}\widetilde{a}_n(r_2)t^n +$$

Using these $\widetilde{a}_n(r)$'s, $$L[\widetilde{\phi}](r,t) = (r-r_2)F(r)t^r = (r-r_2)^2(r-r_1)t^r$$

Differentiate w.r.t. r $$\frac{\partial}{\partial r}L[\widetilde{\phi}](r,t) = \frac{\partial}{\partial r}\Big((r-r_2)^2\cdot(r-r_1)t^r\Big)$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) = 2(r-r_2)\cdot(r-r_1)t^r + (r-r_2)^2\cdot\frac{\partial}{\partial r}(r-r_1)t^r$$

$$L\left[\frac{\partial\widetilde{\phi}}{\partial r}\right](r_2,t) = 0$$

Second solution
$$\begin{aligned} y_2(t) &= \left.\frac{\partial}{\partial r}\widetilde{\phi}(r,t)\right|_{r=r_2} \\ &= \left.\Big(t^r\ln t\Big)\left((r-r_2)+\sum_{n=1}^{\infty}\widetilde{a}_n(r)t^n\right)+t^r\left(1+\sum_{n=1}^{\infty}\widetilde{a}_n'(r)t^n\right)\right|_{r=r_2} \\ &= \Big(t^{r_2}\ln t\Big)\sum_{n=1}^{\infty}\widetilde{a}_n(r_2)t^n + t^{r_2}\left(1+\sum_{n=1}^{\infty}\widetilde{a}_n'(r_2)t^n\right) \end{aligned}$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n \quad =$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n \quad = \quad \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$\begin{aligned}(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\ &= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N}\end{aligned}$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$\begin{aligned}
(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\
&= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N} \\
&= \left(t^{r_1} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_N(r_2) a_n(r_1) t^n && \text{by Claim 2}
\end{aligned}$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$\begin{aligned}
(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\
&= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N} \\
&= \left(t^{r_1} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_N(r_2) a_n(r_1) t^n \qquad \text{by Claim 2} \\
&= \widetilde{a}_N(r_2) \cdot \left(t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n\right) \ln t
\end{aligned}$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$\begin{aligned}
(t^{r_2}\ln t)\sum_{n=1}^{\infty}\widetilde{a}_n(r_2)t^n &= \left(t^{r_2}\ln t\right)\sum_{n=N}^{\infty}\widetilde{a}_n(r_2)t^n \\
&= \left(t^{r_2}\ln t\right)\sum_{n=0}^{\infty}\widetilde{a}_{n+N}(r_2)t^{n+N} \\
&= \left(t^{r_1}\ln t\right)\sum_{n=0}^{\infty}\widetilde{a}_N(r_2)a_n(r_1)t^n \qquad \text{by Claim 2} \\
&= \widetilde{a}_N(r_2)\cdot\left(t^{r_1}\sum_{n=0}^{\infty}a_n(r_1)t^n\right)\ln t \\
&= \widetilde{a}_N(r_2)y_1(t)\ln t
\end{aligned}$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$
\begin{aligned}
(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\
&= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N} \\
&= \left(t^{r_1} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_N(r_2) a_n(r_1) t^n \qquad \text{by Claim 2} \\
&= \widetilde{a}_N(r_2) \cdot \left(t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n\right) \ln t \\
&= \widetilde{a}_N(r_2) y_1(t) \ln t
\end{aligned}
$$

Therefore $$y_2(t) = A y_1(t) \ln t + t^{r_2} \left(1 + \sum_{n=1}^{\infty} c_n t^n\right)$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$\begin{aligned}
(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\
&= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N} \\
&= \left(t^{r_1} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_N(r_2) a_n(r_1) t^n \qquad \text{by Claim 2} \\
&= \widetilde{a}_N(r_2) \cdot \left(t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n\right) \ln t \\
&= \widetilde{a}_N(r_2) y_1(t) \ln t
\end{aligned}$$

Therefore $$y_2(t) = A y_1(t) \ln t + t^{r_2} \left(1 + \sum_{n=1}^{\infty} c_n t^n\right)$$

where $$A = \widetilde{a}_N(r_2)$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$\begin{aligned}
(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\
&= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N} \\
&= \left(t^{r_1} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_N(r_2) a_n(r_1) t^n \qquad \text{by Claim 2} \\
&= \widetilde{a}_N(r_2) \cdot \left(t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n\right) \ln t \\
&= \widetilde{a}_N(r_2) y_1(t) \ln t
\end{aligned}$$

Therefore $$y_2(t) = A y_1(t) \ln t + t^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n t^n\right)$$

where $$\begin{aligned} A &= \widetilde{a}_N(r_2) \\ &= \lim_{r \to r_2} (r - r_2) a_N(r) \end{aligned}$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$
\begin{aligned}
(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\
&= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N} \\
&= \left(t^{r_1} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_N(r_2) a_n(r_1) t^n \qquad \text{by Claim 2} \\
&= \widetilde{a}_N(r_2) \cdot \left(t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n\right) \ln t \\
&= \widetilde{a}_N(r_2) y_1(t) \ln t
\end{aligned}
$$

Therefore $$y_2(t) = A y_1(t) \ln t + t^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n t^n\right)$$

where $A = \widetilde{a}_N(r_2)$ $\qquad c_n = \widetilde{a}'_n(r_2)$

$$= \lim_{r \to r_2} (r - r_2) a_N(r)$$

Since $\widetilde{a}_0(r_2) = \widetilde{a}_1(r_2) = \cdots = \widetilde{a}_{N-1}(r_2) = 0$,

$$
\begin{aligned}
(t^{r_2} \ln t) \sum_{n=1}^{\infty} \widetilde{a}_n(r_2) t^n &= \left(t^{r_2} \ln t\right) \sum_{n=N}^{\infty} \widetilde{a}_n(r_2) t^n \\
&= \left(t^{r_2} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_{n+N}(r_2) t^{n+N} \\
&= \left(t^{r_1} \ln t\right) \sum_{n=0}^{\infty} \widetilde{a}_N(r_2) a_n(r_1) t^n \qquad \text{by Claim 2} \\
&= \widetilde{a}_N(r_2) \cdot \left(t^{r_1} \sum_{n=0}^{\infty} a_n(r_1) t^n\right) \ln t \\
&= \widetilde{a}_N(r_2) y_1(t) \ln t
\end{aligned}
$$

Therefore $$y_2(t) = A y_1(t) \ln t + t^{r_2} \left(1 + \sum_{n=1}^{\infty} c_n t^n\right)$$

where
$$
\begin{aligned}
A &= \widetilde{a}_N(r_2) & c_n &= \widetilde{a}_n'(r_2) \\
&= \lim_{r \to r_2} (r - r_2) a_N(r) & &= \lim_{r \to r_2} \frac{\mathrm{d}}{\mathrm{d}r} (r - r_2) a_n(r)
\end{aligned}
$$