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$n \geq 1$ $\quad (n + r - 1)^2 a_n(r) - a_{n-1}(r) = 0$

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First solution

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y_1(t) = t\left(1 + \sum_{n=1}^{\infty} \frac{1}{(n!)^2} t^n\right)
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Use logarithmic differentiation to find $a_n'(r)$

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Second solution $$y_2(t) = y_1(t)\ln t + t\left(\sum_{n=1}^{\infty}\frac{-2}{(n!)^2}H_n t^n\right)$$

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Substitute into DE

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c_2 is *arbitrary* and $c_4 = -\frac{c_2}{8} - \frac{3}{64}$

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c_2 is *arbitrary* and $c_4 = -\dfrac{c_2}{8} - \dfrac{3}{64}$

With $c_2 = \dfrac{1}{4}$, $\quad c_{2n} = \dfrac{(-1)^{n+1}}{2^{2n} n!\,(n-1)!}(H_n + H_{n-1})$ where $H_0 := 0$

Proof by M.I.

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$$L[y] := y'' + p(t)y' + q(t)y = 0$$

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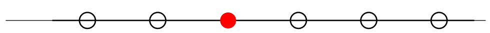

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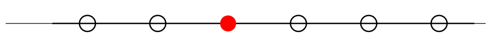

can find $a_1(r)$, $a_2(r)$, $\ldots$ such that

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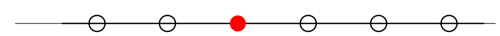

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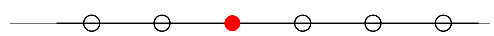

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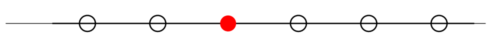

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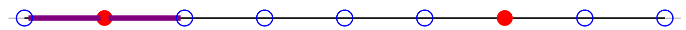
Put $r = r_1$, first solution $\quad y_1(t) \;=\; \phi(r_1, t)$

Idea to get a formula for the coefficients in $y_2(t)$

For $r \in (r_2 - 1, r_2 + 1) \setminus \{r_2\}$,

since $F(r+n) \neq 0$ for all $n \geq 1$,



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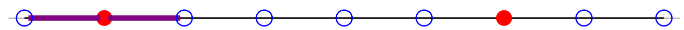

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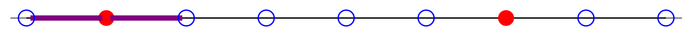

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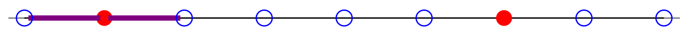

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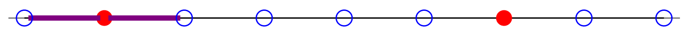

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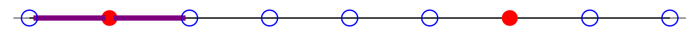

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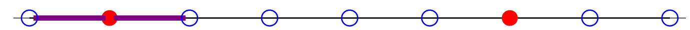

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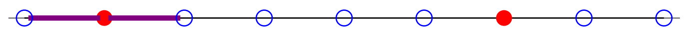

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L\left[\frac{\partial \widetilde{\phi}}{\partial r}\right](r,t) &= \frac{\partial}{\partial r}\Big((r - r_1)(r - r_2)^2 t^r\Big)
\end{aligned}$$

Take limit $r \to r_2$

$$L\left[\lim_{r\to r_2} \frac{\partial\widetilde{\phi}}{\partial r}\right](r,t) \quad =$$

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&= \frac{(-1)^n}{\prod_{k=0}^{n-1}(2n-2k)\cdot\prod_{k=0}^{n-2}(2n-2-2k)}\left(-\sum_{k=0}^{n-1}\frac{1}{2n-2k} - \sum_{k=0}^{n-2}\frac{1}{2n-2-2k}\right) \qquad \text{by } \ddagger \\
&= \frac{(-1)^{n+1}}{2^n n!\cdot 2^{n-1}(n-1)!}\cdot\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{1}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots+\frac{1}{1}\right)
\end{aligned}
$$

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&= \frac{(-1)^{n+1}}{2^n n!\cdot 2^{n-1}(n-1)!}\cdot\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{1}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots+\frac{1}{1}\right) \\
&= \frac{(-1)^{n+1}}{2^{2n}n!(n-1)!}(H_n+H_{n-1}) \qquad (n\geq 1, \quad \text{where } H_0 := 0)
\end{aligned}
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&= \frac{(-1)^{n+1}}{2^n n!\cdot 2^{n-1}(n-1)!}\cdot\frac{1}{2}\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+\frac{1}{1}+\frac{1}{n-1}+\frac{1}{n-2}+\cdots+\frac{1}{1}\right) \\
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$$

$$
\therefore\quad y_2(t) = -\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n}(n+1)!n!}t^{2n+1}\right)\ln t + t^{-1}\left(1+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2^{2n}n!(n-1)!}(H_n+H_{n-1})t^{2n}\right)
$$