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Consider cases where $r_1, r_2 \in \mathbb{R}$. Assume $r_1 \geq r_2$.

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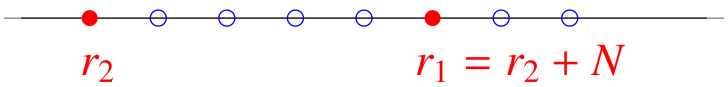

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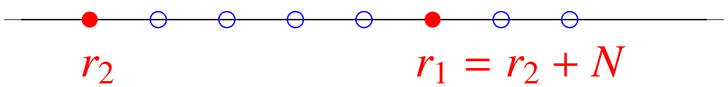

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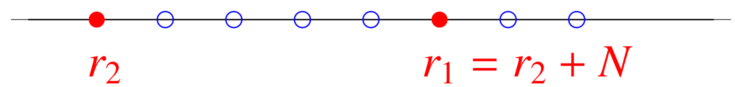

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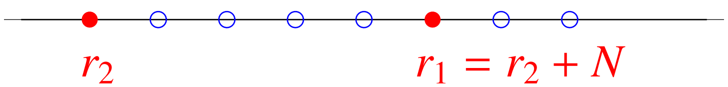

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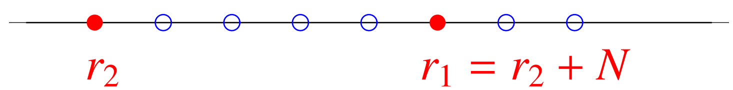

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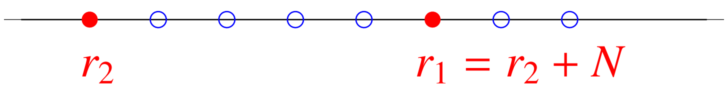

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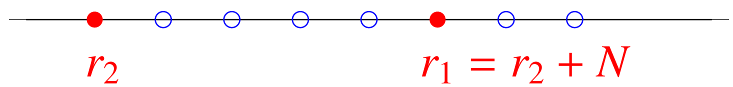

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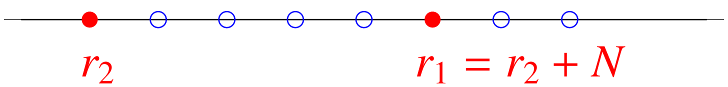

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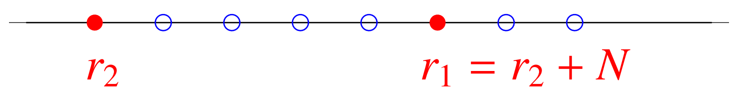

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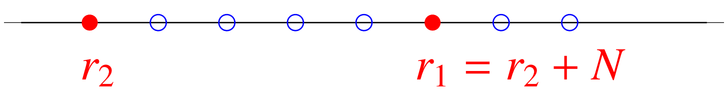

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  Since $F(r_2 + N + 1) \neq 0$, $F(r_2 + N + 2) \neq 0$, $\ldots$, $a_{N+1}, a_{N+2}, \ldots$ can be solved.

Case 2 and **Case 3**

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Note that $y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) > 0$ if $0 < t$ is small

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put $w = v'$, find positive w $\quad \frac{w'}{w} = \frac{-2y_1'}{y_1} - p$

$$\ln w = -2\ln y_1 - \int p(t)\,\mathrm{d}t$$

$$
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In some interval about 0, $g(t) = 1 + g_1 t + g_2 t^2 + \cdots$

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Case 3 $r_1 - r_2 \in \mathbb{Z}^+$

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&= g_{k-1} y_1(t) \ln t + t^{r_1 - k + 1} \left(\sum_{n=0}^{\infty} c_n t^n \right)
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$$= g_{k-1} y_1(t) \ln t + t^{r_1 - k + 1} \left(\sum_{n=0}^{\infty} c_n t^n \right)$$

where $c_0 = \frac{1}{-k+1} \neq 0$, note that $r_1 - k + 1 = r_1 - 2r_1 - p_0 + 1$

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$$\begin{aligned}
y_2(t) &= y_1(t)g_{k-1}\ln t + y_1(t)\cdot t^{-k+1}\left(\frac{1}{-k+1} + \sum_{n=1,n\neq k-1}^{\infty}\frac{g_n t^n}{-k+1+n}\right) \\
&= g_{k-1}y_1(t)\ln t + t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n t^n\right)\cdot t^{-k+1}\left(\frac{1}{-k+1} + \sum_{n=1,n\neq k-1}^{\infty}\frac{g_n t^n}{-k+1+n}\right) \\
&= g_{k-1}y_1(t)\ln t + t^{r_1-k+1}\left(\sum_{n=0}^{\infty} c_n t^n\right)
\end{aligned}$$

where $c_0 = \frac{1}{-k+1} \neq 0$, note that

$$\begin{aligned}
r_1 - k + 1 &= r_1 - 2r_1 - p_0 + 1 \\
&= -r_1 + (r_1 + r_2) = r_2
\end{aligned}$$

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Substitute form of y_1

$$2t\sum_{n=0}^{\infty} \frac{n+1}{(n!)^2}t^n - 2\sum_{n=0}^{\infty} \frac{1}{(n!)^2}t^{n+1} +$$

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Substitute into DE (use y_1 is a solution)

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Compare coefficients

$$2 + b_1 = 0$$

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$$\begin{aligned} 2 + b_1 &= 0 \\ \frac{2(n-1)}{\big((n-1)!\big)^2} + (n-1)^2 b_{n-1} - b_{n-2} &= 0 \qquad n \geq 3 \end{aligned}$$

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Solving $\quad n^2 b_n = b_{n-1} - \frac{2n}{(n!)^2}$

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$$\begin{aligned} n^2 b_n &= b_{n-1} - \frac{2n}{(n!)^2} \\ b_n &= \frac{1}{n^2} b_{n-1} - \frac{2}{n(n!)^2} \end{aligned}$$

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$$
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$$
\begin{aligned}
n^2 b_n &= b_{n-1} - \frac{2n}{(n!)^2} \\
b_n &= \frac{1}{n^2} b_{n-1} - \frac{2}{n(n!)^2} \\
&= \frac{1}{n^2}\left(\frac{1}{(n-1)^2} b_{n-2} - \frac{2}{(n-1)\left((n-1)!\right)^2}\right) - \frac{2}{n(n!)^2} \\
&= \frac{1}{n^2(n-1)^2} b_{n-2} - \frac{2}{(n-1)(n!)^2} - \frac{2}{n(n!)^2}
\end{aligned}
$$

Continue $$b_n = \frac{1}{n^2(n-1)^2} b_{n-2} - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right)$$

Continue

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$$= \frac{1}{n^2(n-1)^2}\left(\frac{1}{(n-2)^2} b_{n-3} - \frac{2}{(n-2)\big((n-2)!\big)^2}\right) - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right)$$

Continue

$$
\begin{aligned}
b_n &= \frac{1}{n^2(n-1)^2} b_{n-2} - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right) \\
&= \frac{1}{n^2(n-1)^2}\left(\frac{1}{(n-2)^2} b_{n-3} - \frac{2}{(n-2)\big((n-2)!\big)^2}\right) - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right) \\
&= \frac{1}{n^2(n-1)^2(n-2)^2} b_{n-3} - \frac{2}{(n!)^2}\left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\right)
\end{aligned}
$$

Continue

$$b_n = \frac{1}{n^2(n-1)^2} b_{n-2} - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right)$$

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$$\vdots$$

$$= \frac{1}{n^2(n-1)^2 \cdots 2^2} b_1 -$$

Continue

$$
\begin{aligned}
b_n &= \frac{1}{n^2(n-1)^2} b_{n-2} - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right) \\
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&= \frac{1}{n^2(n-1)^2(n-2)^2} b_{n-3} - \frac{2}{(n!)^2}\left(\frac{1}{n-2} + \frac{1}{n-1} + \frac{1}{n}\right) \\
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&= \frac{1}{n^2(n-1)^2 \cdots 2^2} b_1 - \frac{2}{(n!)^2}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right)
\end{aligned}
$$

Continue

$$
\begin{aligned}
b_n &= \frac{1}{n^2(n-1)^2} b_{n-2} - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right) \\
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&= \frac{-2}{(n!)^2} - \frac{2}{(n!)^2}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right)
\end{aligned}
$$

Continue

$$b_n = \frac{1}{n^2(n-1)^2} b_{n-2} - \frac{2}{(n!)^2}\left(\frac{1}{n-1} + \frac{1}{n}\right)$$

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$$= \frac{-2}{(n!)^2} - \frac{2}{(n!)^2}\left(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}\right)$$

$$= -\frac{2}{(n!)^2} H_n$$

where $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

Alternative method for Euler Equation, Case 2

$$t^2y'' + \alpha t y' + \beta y = 0, \qquad t > 0$$

where α and β are constants.

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$$L[t^r \ln t]$$

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$$\frac{\mathrm{d}}{\mathrm{d}x}b^x = b^x \cdot \ln b$$

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$$\frac{\partial}{\partial r} L[t^r] = \frac{\partial}{\partial r}\left(t^r (r - r_1)^2\right)$$

$$L\left[\frac{\partial}{\partial r} t^r\right] = (r - r_1)^2 \frac{\partial}{\partial r} t^r + t^r \frac{\partial}{\partial r}(r - r_1)^2$$

$$L[t^r \ln t] = (r - r_1)^2 t^r \ln t + 2(r - r_1)\, t^r$$

$$\frac{\mathrm{d}}{\mathrm{d}x} b^x = b^x \cdot \ln b$$

$$L[t^{r_1} \ln t] = 0$$

Alternative way for regular singular point Case 2 $r_1 = r_2$

$$L[y] := y'' + p(t)y' + q(t)y = 0$$

Denote $\phi(r,t) = t^r \left(1 + \sum_{n=1}^{\infty} a_n(r)\, t^n\right)$

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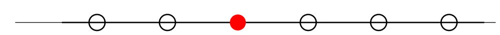

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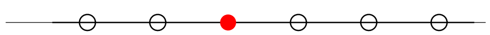

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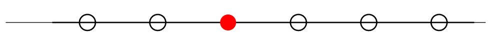

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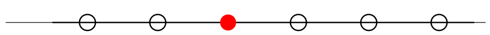

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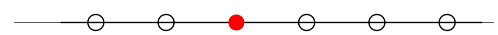

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Put $r = r_1$, first solution $y_1(t) \;=\; \phi(r_1, t)$

Differentiate (5) with respect to r

$$\frac{\partial}{\partial r} L[\phi] \quad = \quad \frac{\partial}{\partial r}\Big((r - r_1)^2 t^r\Big)$$

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