Example Solve the IVP

$$\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

Example Solve the IVP
$$\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

Example Solve the IVP
$$\begin{cases} (1+t^2)^2 y'' + (1+t^2) y' + 2(1+t-3t^2) y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ

Example Solve the IVP $\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

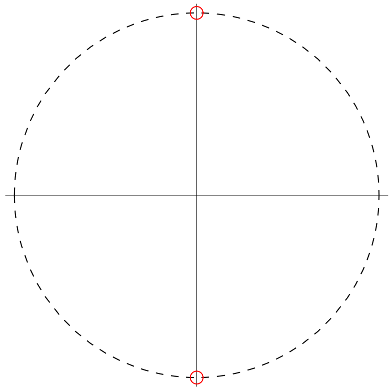
Example Solve the IVP
$$\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$$

Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

- p and q are analytic at $t = 0$ $\quad \rho(p) = \rho(q) = 1$.



Example Solve the IVP $\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

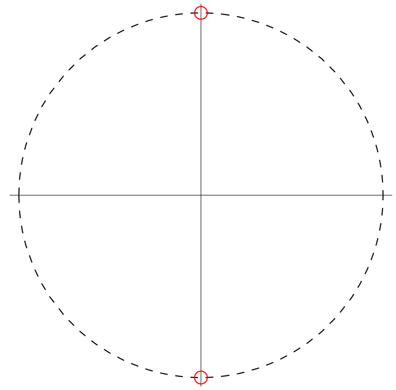
Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

- p and q are analytic at $t = 0$ $\quad \rho(p) = \rho(q) = 1$.

  By Thm, IVP has power series solution, $\rho \geq 1$.



Example Solve the IVP $\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

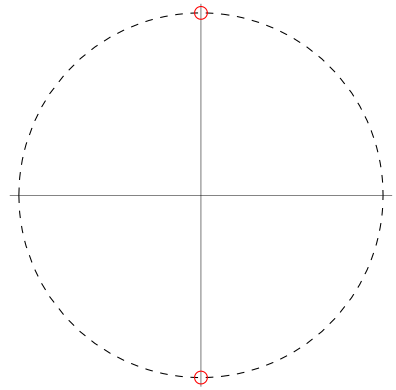
Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

- p and q are analytic at $t = 0$ $\quad \rho(p) = \rho(q) = 1$.

  By Thm, IVP has power series solution, $\quad \rho \geq 1$.



Question φ = power series solution *on* $\mathbb{R}$?

Example Solve the IVP $\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

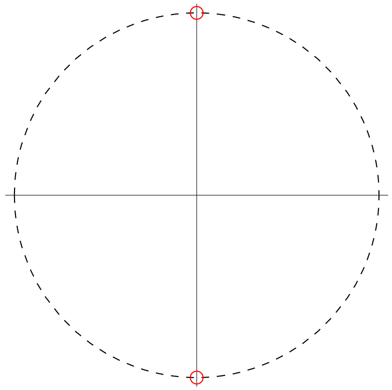
Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

- p and q are analytic at $t = 0$ $\quad \rho(p) = \rho(q) = 1$.

  By Thm, IVP has power series solution, $\quad \rho \geq 1$.



Question φ = power series solution *on* $\mathbb{R}$?

Put $y(t) = \sum_{n=0}^{\infty} a_n t^n$ into the DE

Example Solve the IVP $\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

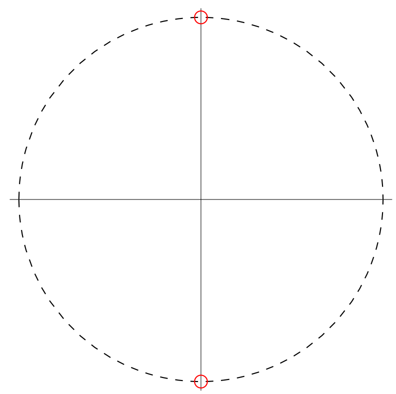
Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

- p and q are analytic at $t = 0$ $\quad \rho(p) = \rho(q) = 1$.

  By Thm, IVP has power series solution, $\quad \rho \geq 1$.



Question φ = power series solution *on* $\mathbb{R}$?

Put $y(t) = \sum_{n=0}^{\infty} a_n t^n$ into the DE

$$(1 + 2t^2 + t^4) \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} +$$

Example Solve the IVP $\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

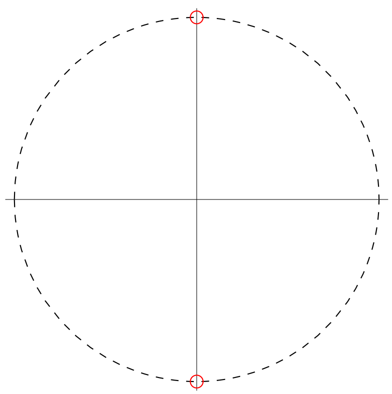
Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

- p and q are analytic at $t = 0$ $\quad \rho(p) = \rho(q) = 1$.

  By Thm, IVP has power series solution, $\quad \rho \geq 1$.



Question φ = power series solution *on* $\mathbb{R}$?

Put $y(t) = \sum_{n=0}^{\infty} a_n t^n$ into the DE

$$(1+2t^2+t^4)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (1+t^2)\sum_{n=1}^{\infty} n a_n t^{n-1} +$$

Example Solve the IVP $\begin{cases} (1+t^2)^2 y'' + (1+t^2)y' + 2(1+t-3t^2)y = 0 \\ y(0) = 1, \quad y'(0) = 0 \end{cases}$

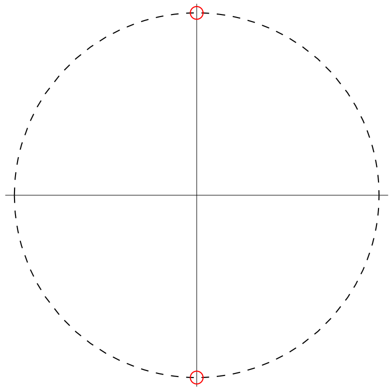
Solution

- $p(t) = \dfrac{1}{1+t^2}, \quad q(t) = \dfrac{2(1+t-3t^2)}{(1+t^2)^2}$

  By E&U Thm for 2nd order linear IVP, unique solution φ exists on $\mathbb{R}$.

- p and q are analytic at $t = 0$ $\quad \rho(p) = \rho(q) = 1$.

  By Thm, IVP has power series solution, $\quad \rho \geq 1$.



Question φ = power series solution *on* $\mathbb{R}$?

Put $y(t) = \sum_{n=0}^{\infty} a_n t^n$ into the DE

$$(1+2t^2+t^4)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (1+t^2)\sum_{n=1}^{\infty} n a_n t^{n-1} + 2(1+t-3t^2)\sum_{n=0}^{\infty} a_n t^n = 0$$

Expand and shift indices

$$\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n t^{n} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}\right) +$$

Expand and shift indices

$$\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n t^{n} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}\right) +$$

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=1}^{\infty} n a_n t^{n+1}\right) +$$

Expand and shift indices

$$\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n t^{n} + \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}\right) +$$

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=1}^{\infty} n a_n t^{n+1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^{n} + \sum_{n=0}^{\infty} 2a_n t^{n+1} - \sum_{n=0}^{\infty} 6a_n t^{n+2}\right) = 0$$

Expand and shift indices

$$\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}\right) +$$

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=1}^{\infty} n a_n t^{n+1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=0}^{\infty} 2a_n t^{n+1} - \sum_{n=0}^{\infty} 6a_n t^{n+2}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=4}^{\infty} (n-2)(n-3)a_{n-2} t^n\right) +$$

Expand and shift indices

$$\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}\right) +$$

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=1}^{\infty} n a_n t^{n+1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=0}^{\infty} 2a_n t^{n+1} - \sum_{n=0}^{\infty} 6a_n t^{n+2}\right) = 0$$

$$\left(\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=4}^{\infty}(n-2)(n-3)a_{n-2}t^n\right) +$$

$$\left(\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=2}^{\infty}(n-1)a_{n-1}t^n\right) +$$

Expand and shift indices

$$\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}\right) +$$

$$\left(\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=1}^{\infty} n a_n t^{n+1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=0}^{\infty} 2a_n t^{n+1} - \sum_{n=0}^{\infty} 6a_n t^{n+2}\right) = 0$$

$$\left(\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2} t^n + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=4}^{\infty}(n-2)(n-3)a_{n-2} t^n\right) +$$

$$\left(\sum_{n=0}^{\infty}(n+1)a_{n+1} t^n + \sum_{n=2}^{\infty}(n-1)a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=1}^{\infty} 2a_{n-1} t^n - \sum_{n=2}^{\infty} 6a_{n-2} t^n\right) = 0$$

Expand and shift indices

$$\left(\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=2}^{\infty} n(n-1)a_n t^{n+2}\right) + \left(\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=1}^{\infty} n a_n t^{n+1}\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=0}^{\infty} 2a_n t^{n+1} - \sum_{n=0}^{\infty} 6a_n t^{n+2}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=2}^{\infty} 2n(n-1)a_n t^n + \sum_{n=4}^{\infty} (n-2)(n-3)a_{n-2} t^n\right) + \left(\sum_{n=0}^{\infty} (n+1)a_{n+1} t^n + \sum_{n=2}^{\infty} (n-1)a_{n-1} t^n\right) + \left(\sum_{n=0}^{\infty} 2a_n t^n + \sum_{n=1}^{\infty} 2a_{n-1} t^n - \sum_{n=2}^{\infty} 6a_{n-2} t^n\right) = 0$$

Compare coefficients

$$2a_2 + a_1 + 2a_0 = 0$$

$$\begin{aligned} 2a_2 + a_1 + 2a_0 &= 0 \\ 6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \end{aligned}$$

$$
\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0
\end{aligned}
$$

$$
\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0 \\
20a_5 + 4a_4 + 14a_3 + 4a_2 - 6a_1 &= 0
\end{aligned}
$$

$$
\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0 \\
20a_5 + 4a_4 + 14a_3 + 4a_2 - 6a_1 &= 0 \\
(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 2(n^2 - n + 1)a_n & \\
+(n+1)a_{n-1} + (n^2 - 5n)a_{n-2} &= 0 \quad (n \geq 4)
\end{aligned}
$$

$$
\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0 \\
20a_5 + 4a_4 + 14a_3 + 4a_2 - 6a_1 &= 0 \\
(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 2(n^2 - n + 1)a_n & \\
+(n+1)a_{n-1} + (n^2 - 5n)a_{n-2} &= 0 \quad (n \geq 4)
\end{aligned}
$$

- Initial conditions: $a_0 = 1, \; a_1 = 0$.

$$
\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0 \\
20a_5 + 4a_4 + 14a_3 + 4a_2 - 6a_1 &= 0 \\
(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 2(n^2 - n + 1)a_n & \\
+(n+1)a_{n-1} + (n^2 - 5n)a_{n-2} &= 0 \quad (n \geq 4)
\end{aligned}
$$

- Initial conditions: $a_0 = 1, \; a_1 = 0.$
- First four equations: $a_2 = -1, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 0$

$$
\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0 \\
20a_5 + 4a_4 + 14a_3 + 4a_2 - 6a_1 &= 0 \\
(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 2(n^2 - n + 1)a_n & \\
+(n+1)a_{n-1} + (n^2 - 5n)a_{n-2} &= 0 \quad (n \geq 4)
\end{aligned}
$$

- Initial conditions: $a_0 = 1, \; a_1 = 0$.
- First four equations: $a_2 = -1, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 0$
- Recurrence formula: $a_{2k} = (-1)^k \qquad a_{2k+1} = 0 \quad (k \geq 0)$

$$\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0 \\
20a_5 + 4a_4 + 14a_3 + 4a_2 - 6a_1 &= 0 \\
(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 2(n^2 - n + 1)a_n & \\
+(n+1)a_{n-1} + (n^2 - 5n)a_{n-2} &= 0 \quad (n \geq 4)
\end{aligned}$$

- Initial conditions: $a_0 = 1, \ a_1 = 0$.
- First four equations: $a_2 = -1, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 0$
- Recurrence formula: $a_{2k} = (-1)^k \qquad a_{2k+1} = 0 \quad (k \geq 0)$
- Power series solution: $y(t) = 1 - t^2 + t^4 - t^6 + \cdots \quad -1 < t < 1$

$$\begin{aligned}
2a_2 + a_1 + 2a_0 &= 0 \\
6a_3 + 2a_2 + 2a_1 + 2a_0 &= 0 \\
12a_4 + 3a_3 + 6a_2 + 3a_1 - 6a_0 &= 0 \\
20a_5 + 4a_4 + 14a_3 + 4a_2 - 6a_1 &= 0 \\
(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + 2(n^2 - n + 1)a_n & \\
+(n+1)a_{n-1} + (n^2 - 5n)a_{n-2} &= 0 \quad (n \geq 4)
\end{aligned}$$

- Initial conditions: $a_0 = 1,\ a_1 = 0$.
- First four equations: $a_2 = -1, \quad a_3 = 0, \quad a_4 = 1, \quad a_5 = 0$
- Recurrence formula: $a_{2k} = (-1)^k \qquad a_{2k+1} = 0 \quad (k \geq 0)$
- Power series solution: $y(t) = 1 - t^2 + t^4 - t^6 + \cdots \quad -1 < t < 1$
- Solution on $\mathbb{R}$: $\varphi(t) = \dfrac{1}{1 + t^2}$

Series Solutions: Singular Point

Consider 2nd order linear homogeneous DE

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Suppose $t = t_0$ is a singular point.

Series Solutions: Singular Point

Consider 2nd order linear homogeneous DE

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Suppose $t = t_0$ is a singular point.

- DE may not have power series solution.

Series Solutions: Singular Point

Consider 2nd order linear homogeneous DE

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Suppose $t = t_0$ is a singular point.

- DE may not have power series solution.
- Consider situation where the singular point is *"not too bad"*

Series Solutions: Singular Point

Consider 2nd order linear homogeneous DE

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Suppose $t = t_0$ is a singular point.

- DE may not have power series solution.
- Consider situation where the singular point is *"not too bad"*

  Have solutions *"similar to"* power series

Series Solutions: Singular Point

Consider 2nd order linear homogeneous DE

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Suppose $t = t_0$ is a singular point.

- DE may not have power series solution.
- Consider situation where the singular point is *"not too bad"*

  Have solutions *"similar to"* power series

$$y(t) = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

Series Solutions: Singular Point

Consider 2nd order linear homogeneous DE

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Suppose $t = t_0$ is a singular point.

- DE may not have power series solution.
- Consider situation where the singular point is *"not too bad"*

  Have solutions *"similar to"* power series

  $$y(t) = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

  called *Frobenius series*.

Series Solutions: Singular Point

Consider 2nd order linear homogeneous DE

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

Suppose $t = t_0$ is a singular point.

- DE may not have power series solution.
- Consider situation where the singular point is *"not too bad"*

  Have solutions *"similar to"* power series

  $$y(t) = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

  called *Frobenius series*.

Definition A singular point $t = t_0$ of the DE $y'' + p(t)y' + q(t)y=0$ is said to be *regular* if both $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ are analytic at t_0

Example Consider the following DE *(Euler equation)*

$$t^2y'' + \alpha ty' + \beta y = 0$$

where α and β are constants.

Example Consider the following DE *(Euler equation)*

$$t^2y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

Example Consider the following DE *(Euler equation)*

$$t^2y'' + \alpha ty' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try

Example Consider the following DE *(Euler equation)*

$$t^2y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Example Consider the following DE *(Euler equation)*

$$t^2y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Direct substitution $\quad t^2 \cdot r(r-1)\, t^{r-2} +$

Example Consider the following DE *(Euler equation)*

$$t^2 y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Direct substitution $t^2 \cdot r(r-1)\, t^{r-2} + \alpha t \cdot r\, t^{r-1} +$

Example Consider the following DE *(Euler equation)*

$$t^2 y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Direct substitution $t^2 \cdot r(r-1)\, t^{r-2} + \alpha t \cdot r\, t^{r-1} + \beta\, t^r = 0$

Example Consider the following DE *(Euler equation)*

$$t^2 y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Direct substitution

$$\begin{aligned} t^2 \cdot r(r-1)\, t^{r-2} + \alpha t \cdot r\, t^{r-1} + \beta\, t^r &= 0 \\ \Big(r(r-1) + \alpha r + \beta \Big) t^r &= 0 \end{aligned}$$

Example Consider the following DE *(Euler equation)*

$$t^2y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Direct substitution

$$\begin{aligned} t^2 \cdot r(r-1)\,t^{r-2} + \alpha t \cdot r\,t^{r-1} + \beta\,t^r &= 0 \\ \big(r(r-1) + \alpha r + \beta\big)\,t^r &= 0 \end{aligned}$$

$y = t^r$ is a solution $\iff$ $r(r-1) + \alpha r + \beta = 0$

Example Consider the following DE *(Euler equation)*

$$t^2y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Direct substitution
$$\begin{aligned} t^2 \cdot r(r-1)\, t^{r-2} + \alpha t \cdot r\, t^{r-1} + \beta\, t^r &= 0 \\ \big(r(r-1) + \alpha r + \beta\big)\, t^r &= 0 \end{aligned}$$

$y = t^r$ is a solution $\Longleftrightarrow$
$$\begin{aligned} r(r-1) + \alpha r + \beta &= 0 \\ r^2 + (\alpha - 1)r + \beta &= 0 \end{aligned}$$

Example Consider the following DE *(Euler equation)*

$$t^2 y'' + \alpha t y' + \beta y = 0$$

where α and β are constants.

$p(t) = \dfrac{\alpha}{t}$ not analytic at $t = 0$ (singular point)

To solve the DE for $t > 0$, try $y = t^r$.

Direct substitution
$$\begin{aligned} t^2 \cdot r(r-1)\, t^{r-2} + \alpha t \cdot r\, t^{r-1} + \beta\, t^r &= 0 \\ \Big(r(r-1) + \alpha r + \beta \Big)\, t^r &= 0 \end{aligned}$$

$y = t^r$ is a solution $\Longleftrightarrow$
$$\begin{aligned} r(r-1) + \alpha r + \beta &= 0 \\ r^2 + (\alpha - 1) r + \beta &= 0 \end{aligned}$$

Roots $r_1 = \dfrac{-(\alpha - 1) + \sqrt{(\alpha - 1)^2 - 4\beta}}{2}, \qquad r_2 = \dfrac{-(\alpha - 1) - \sqrt{(\alpha - 1)^2 - 4\beta}}{2}$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$$r_1 \neq r_2 \in \mathbb{R}$$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$,

Case 1 If $(\alpha-1)^2-4\beta>0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0,\infty)$ $y_1(t)=t^{r_1}$

$y_2(t)=t^{r_2}$

Case 2 If $(\alpha-1)^2-4\beta=0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t)=t^{r_1}$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0,$

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0,$

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order $y = t^{r_1} v(t)$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$
\begin{aligned}
y &= t^{r_1} v(t) \\
y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v
\end{aligned}
$$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$
\begin{aligned}
y &= t^{r_1} v(t) \\
y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v \\
y'' &= t^{r_1} v'' + 2 r_1 t^{r_1 - 1} v' + r_1 (r_1 - 1) t^{r_1 - 2} v
\end{aligned}
$$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$
\begin{aligned}
y &= t^{r_1} v(t) \\
y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v \\
y'' &= t^{r_1} v'' + 2 r_1 t^{r_1 - 1} v' + r_1 (r_1 - 1) t^{r_1 - 2} v
\end{aligned}
$$

Sub. $t^{r_1+2} v'' + 2 r_1 t^{r_1+1} v' + r_1 (r_1 - 1) t^{r_1} v +$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \ \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$$y_2(t) = t^{r_2}$$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$\begin{aligned} y &= t^{r_1} v(t) \\ y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v \\ y'' &= t^{r_1} v'' + 2 r_1 t^{r_1 - 1} v' + r_1 (r_1 - 1) t^{r_1 - 2} v \end{aligned}$$

Sub. $t^{r_1+2} v'' + 2 r_1 t^{r_1+1} v' + r_1 (r_1 - 1) t^{r_1} v + \alpha t^{r_1+1} v' + \alpha r_1 t^{r_1} v +$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$\begin{aligned} y &= t^{r_1} v(t) \\ y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v \\ y'' &= t^{r_1} v'' + 2 r_1 t^{r_1 - 1} v' + r_1 (r_1 - 1) t^{r_1 - 2} v \end{aligned}$$

Sub. $t^{r_1+2} v'' + 2 r_1 t^{r_1+1} v' + r_1 (r_1 - 1) t^{r_1} v + \alpha t^{r_1+1} v' + \alpha r_1 t^{r_1} v + \beta t^{r_1} v \quad = \quad 0$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$\begin{aligned} y &= t^{r_1} v(t) \\ y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v \\ y'' &= t^{r_1} v'' + 2r_1 t^{r_1 - 1} v' + r_1(r_1 - 1) t^{r_1 - 2} v \end{aligned}$$

Sub. $t^{r_1+2} v'' + 2r_1 t^{r_1+1} v' + r_1(r_1 - 1) t^{r_1} v + \alpha t^{r_1+1} v' + \alpha r_1 t^{r_1} v + \beta t^{r_1} v = 0$

$t^{r_1+2} v'' +$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$$y_2(t) = t^{r_2}$$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$\begin{aligned} y &= t^{r_1} v(t) \\ y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v \\ y'' &= t^{r_1} v'' + 2 r_1 t^{r_1 - 1} v' + r_1(r_1 - 1) t^{r_1 - 2} v \end{aligned}$$

Sub. $t^{r_1+2} v'' + 2 r_1 t^{r_1+1} v' + r_1(r_1 - 1) t^{r_1} v + \alpha t^{r_1+1} v' + \alpha r_1 t^{r_1} v + \beta t^{r_1} v \quad = \quad 0$

$$t^{r_1+2} v'' + (2 r_1 + \alpha) t^{r_1+1} v' +$$

Case 1 If $(\alpha - 1)^2 - 4\beta > 0$,

$r_1 \neq r_2 \in \mathbb{R}$ Two indep solns in $(0, \infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha - 1)^2 - 4\beta = 0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$
\begin{aligned}
y &= t^{r_1} v(t) \\
y' &= t^{r_1} v' + r_1 t^{r_1 - 1} v \\
y'' &= t^{r_1} v'' + 2 r_1 t^{r_1 - 1} v' + r_1 (r_1 - 1) t^{r_1 - 2} v
\end{aligned}
$$

Sub.

$$
\begin{aligned}
t^{r_1+2} v'' + 2 r_1 t^{r_1+1} v' + r_1 (r_1 - 1) t^{r_1} v + \alpha t^{r_1+1} v' + \alpha r_1 t^{r_1} v + \beta t^{r_1} v &= 0 \\
t^{r_1+2} v'' + (2 r_1 + \alpha) t^{r_1+1} v' + \Big(r_1 (r_1 - 1) + \alpha r_1 + \beta \Big) t^{r_1} v &= 0
\end{aligned}
$$

Case 1 If $(\alpha-1)^2-4\beta>0$,

$r_1 \neq r_2 \ \in \mathbb{R}$ Two indep solns in $(0,\infty)$ $y_1(t) = t^{r_1}$

$y_2(t) = t^{r_2}$

Case 2 If $(\alpha-1)^2-4\beta=0$,

$r_1 = r_2 \in \mathbb{R}$, One solution $y_1(t) = t^{r_1}$

Reduction of order

$$\begin{aligned} y &= t^{r_1}v(t) \\ y' &= t^{r_1}v' + r_1t^{r_1-1}v \\ y'' &= t^{r_1}v'' + 2r_1t^{r_1-1}v' + r_1(r_1-1)t^{r_1-2}v \end{aligned}$$

Sub.

$$\begin{aligned} t^{r_1+2}v'' + 2r_1t^{r_1+1}v' + r_1(r_1-1)t^{r_1}v + \alpha t^{r_1+1}v' + \alpha r_1t^{r_1}v + \beta t^{r_1}v &= 0 \\ t^{r_1+2}v'' + (2r_1+\alpha)t^{r_1+1}v' + \Big(r_1(r_1-1) + \alpha r_1 + \beta\Big)t^{r_1}v &= 0 \\ t^{r_1+1}\Big(tv'' + (2r_1+\alpha)v'\Big) &= 0 \end{aligned}$$

$$tv'' + (2r_1 + \alpha)v' = 0$$

$$
\begin{aligned}
tv'' + (2r_1 + \alpha)v' &= 0 \\
tw' + (2r_1 + \alpha)w &= 0 \qquad \text{put } w = v'
\end{aligned}
$$

$$
\begin{aligned}
tv'' + (2r_1 + \alpha)v' &= 0 \\
tw' + (2r_1 + \alpha)w &= 0 \qquad \text{put } w = v' \\
tw' + \Big(-(\alpha - 1) + \alpha\Big)w &= 0
\end{aligned}
$$

$$\begin{aligned} tv'' + (2r_1 + \alpha)v' &= 0 \\ tw' + (2r_1 + \alpha)w &= 0 \qquad \text{put } w = v' \\ tw' + \big(-(\alpha - 1) + \alpha\big)w &= 0 \\ tw' + w &= 0 \end{aligned}$$

$$
\begin{aligned}
tv'' + (2r_1 + \alpha)v' &= 0 \\
tw' + (2r_1 + \alpha)w &= 0 \qquad \text{put } w = v' \\
tw' + \Big(-(\alpha - 1) + \alpha\Big)w &= 0 \\
tw' + w &= 0 \\
\frac{w'}{w} &= \frac{-1}{t}
\end{aligned}
$$

$$
\begin{aligned}
tv'' + (2r_1 + \alpha)v' &= 0 \\
tw' + (2r_1 + \alpha)w &= 0 && \text{put } w = v' \\
tw' + \big(-(\alpha - 1) + \alpha\big)w &= 0 \\
tw' + w &= 0 \\
\frac{w'}{w} &= \frac{-1}{t} \\
\ln w_p &= -\ln t && \text{find part. soln}
\end{aligned}
$$

$$
\begin{aligned}
tv'' + (2r_1 + \alpha)v' &= 0 \\
tw' + (2r_1 + \alpha)w &= 0 && \text{put } w = v' \\
tw' + \big(-(\alpha - 1) + \alpha\big)w &= 0 \\
tw' + w &= 0 \\
\frac{w'}{w} &= \frac{-1}{t} \\
\ln w_p &= -\ln t && \text{find part. soln} \\
v'_p \;=\; w_p &= t^{-1}
\end{aligned}
$$

$$
\begin{aligned}
tv'' + (2r_1 + \alpha)v' &= 0 \\
tw' + (2r_1 + \alpha)w &= 0 && \text{put } w = v' \\
tw' + \big(-(\alpha - 1) + \alpha\big)w &= 0 \\
tw' + w &= 0 \\
\frac{w'}{w} &= \frac{-1}{t} \\
\ln w_p &= -\ln t && \text{find part. soln} \\
v_p' = w_p &= t^{-1} \\
v_p &= \ln t
\end{aligned}
$$

$$
\begin{aligned}
tv'' + (2r_1 + \alpha)v' &= 0 \\
tw' + (2r_1 + \alpha)w &= 0 \qquad \text{put } w = v' \\
tw' + \big(-(\alpha - 1) + \alpha\big)w &= 0 \\
tw' + w &= 0 \\
\frac{w'}{w} &= \frac{-1}{t} \\
\ln w_p &= -\ln t \qquad \text{find part. soln} \\
v_p' = w_p &= t^{-1} \\
v_p &= \ln t
\end{aligned}
$$

Second linearly independent solution $y_2(t) = t^{r_1} \ln t$

Case 3 If $(\alpha - 1)^2 - 4\beta < 0$,

$$r_1 = \lambda + i\mu, \quad r_2 = \lambda - i\mu \quad (\lambda, \mu \in \mathbb{R} \text{ and } \mu \neq 0)$$

Case 3 If $(\alpha - 1)^2 - 4\beta < 0$,

$r_1 = \lambda + \mathrm{i}\mu$, $r_2 = \lambda - \mathrm{i}\mu$ ($\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$)

Two complex-valued solutions

$$z_{1,2}(t) = t^{\lambda \pm \mathrm{i}\mu}$$

Case 3 If $(\alpha - 1)^2 - 4\beta < 0$,

$r_1 = \lambda + \mathrm{i}\mu, \quad r_2 = \lambda - \mathrm{i}\mu$ ($\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$)

Two complex-valued solutions

$$
\begin{aligned}
z_{1,2}(t) &= t^{\lambda \pm \mathrm{i}\mu} \\
&= \mathrm{e}^{(\lambda \pm \mathrm{i}\mu) \ln t}
\end{aligned}
$$

Case 3 If $(\alpha - 1)^2 - 4\beta < 0$,

$r_1 = \lambda + \mathrm{i}\mu, \quad r_2 = \lambda - \mathrm{i}\mu$ ($\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$)

Two complex-valued solutions

$$
\begin{aligned}
z_{1,2}(t) &= t^{\lambda \pm \mathrm{i}\mu} \\
&= \mathrm{e}^{(\lambda \pm \mathrm{i}\mu)\ln t} \\
&= \mathrm{e}^{\lambda \ln t}\Big(\cos(\mu \ln t) \pm \mathrm{i}\sin(\mu \ln t)\Big)
\end{aligned}
$$

Case 3 If $(\alpha - 1)^2 - 4\beta < 0$,

$r_1 = \lambda + \mathrm{i}\mu, \quad r_2 = \lambda - \mathrm{i}\mu$ ($\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0$)

Two complex-valued solutions

$$
\begin{aligned}
z_{1,2}(t) &= t^{\lambda \pm \mathrm{i}\mu} \\
&= \mathrm{e}^{(\lambda \pm \mathrm{i}\mu)\ln t} \\
&= \mathrm{e}^{\lambda \ln t}\Big(\cos(\mu \ln t) \pm \mathrm{i}\sin(\mu \ln t)\Big) \\
&= t^{\lambda}\Big(\cos(\mu \ln t) \pm \mathrm{i}\sin(\mu \ln t)\Big)
\end{aligned}
$$

Case 3 If $(\alpha - 1)^2 - 4\beta < 0$,

$r_1 = \lambda + \mathrm{i}\mu, \quad r_2 = \lambda - \mathrm{i}\mu$ $\quad (\lambda, \mu \in \mathbb{R}$ and $\mu \neq 0)$

Two complex-valued solutions

$$\begin{aligned} z_{1,2}(t) &= t^{\lambda \pm \mathrm{i}\mu} \\ &= \mathrm{e}^{(\lambda \pm \mathrm{i}\mu)\ln t} \\ &= \mathrm{e}^{\lambda \ln t}\Big(\cos(\mu \ln t) \pm \mathrm{i}\sin(\mu \ln t)\Big) \\ &= t^{\lambda}\Big(\cos(\mu \ln t) \pm \mathrm{i}\sin(\mu \ln t)\Big) \end{aligned}$$

Real-valued solutions: take real part and imaginary part.

Theorem *Consider the Euler equation (where $\alpha, \beta \in \mathbb{R}$)*

$$t^2 y'' + \alpha t y' + \beta y = 0, \qquad t > 0 \qquad \text{or } t < 0$$

Theorem *Consider the Euler equation (where* $\alpha, \beta \in \mathbb{R}$*)*

$$t^2y'' + \alpha t y' + \beta y = 0, \qquad t > 0 \qquad \text{or } \ t < 0$$

Let r_1, r_2 *be the roots of* $r(r-1) + \alpha r + \beta = 0$ *(indicial equation)*

Theorem *Consider the Euler equation (where $\alpha, \beta \in \mathbb{R}$)*

$$t^2y'' + \alpha t y' + \beta y = 0, \qquad t > 0 \qquad \text{or } \ t < 0$$

Let r_1, r_2 *be the roots of* $r(r-1) + \alpha r + \beta = 0$ *(indicial equation)*

(1) *If* $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$, *the general solution is*

$$y(t) = c_1|t|^{r_1} + c_2|t|^{r_2}$$

Theorem *Consider the Euler equation (where $\alpha, \beta \in \mathbb{R}$)*

$$t^2 y'' + \alpha t y' + \beta y = 0, \qquad t > 0 \qquad \text{or } t < 0$$

Let r_1, r_2 *be the roots of* $r(r-1) + \alpha r + \beta = 0$ *(indicial equation)*

(1) If $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$, the general solution is

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_2}$$

(2) If $r_1 = r_2$, the general solution is

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_1} \ln |t|$$

Theorem *Consider the Euler equation (where $\alpha, \beta \in \mathbb{R}$)*

$$t^2 y'' + \alpha t y' + \beta y = 0, \qquad t > 0 \qquad \text{or } \; t < 0$$

Let r_1, r_2 *be the roots of* $r(r-1) + \alpha r + \beta = 0$ *(indicial equation)*

(1) *If* $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$, *the general solution is*

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_2}$$

(2) *If* $r_1 = r_2$, *the general solution is*

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_1} \ln |t|$$

(3) *If* r_1, r_2 *are complex,* $r_1 = \lambda + \mathrm{i}\mu$, $r_2 = \lambda - \mathrm{i}\mu$ $(\lambda, \mu \in \mathbb{R}$ *and* $\mu \neq 0)$, *the general solution is*

$$y(t) = c_1 |t|^{\lambda} \cos(\mu \ln |t|) + c_2 |t|^{\lambda} \sin(\mu \ln |t|)$$

Theorem *Consider the Euler equation (where $\alpha, \beta \in \mathbb{R}$)*

$$t^2 y'' + \alpha t y' + \beta y = 0, \qquad t > 0 \qquad \text{or } \; t < 0$$

Let r_1, r_2 *be the roots of* $r(r-1) + \alpha r + \beta = 0$ *(indicial equation)*

(1) *If* $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$, *the general solution is*

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_2}$$

(2) *If* $r_1 = r_2$, *the general solution is*

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_1} \ln |t|$$

(3) *If* r_1, r_2 *are complex,* $r_1 = \lambda + \mathrm{i}\mu$, $r_2 = \lambda - \mathrm{i}\mu$ $(\lambda, \mu \in \mathbb{R}$ *and* $\mu \neq 0)$, *the general solution is*

$$y(t) = c_1 |t|^{\lambda} \cos(\mu \ln |t|) + c_2 |t|^{\lambda} \sin(\mu \ln |t|)$$

Proof • $t > 0$, done.

Theorem *Consider the Euler equation (where $\alpha, \beta \in \mathbb{R}$)*

$$t^2 y'' + \alpha t y' + \beta y = 0, \qquad t > 0 \qquad \text{or } t < 0$$

Let r_1, r_2 *be the roots of* $r(r-1) + \alpha r + \beta = 0$ *(indicial equation)*

(1) If $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$, *the general solution is*

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_2}$$

(2) If $r_1 = r_2$, *the general solution is*

$$y(t) = c_1 |t|^{r_1} + c_2 |t|^{r_1} \ln |t|$$

(3) If r_1, r_2 *are complex,* $r_1 = \lambda + \mathrm{i}\mu$, $r_2 = \lambda - \mathrm{i}\mu$ $(\lambda, \mu \in \mathbb{R}$ *and* $\mu \neq 0)$, *the general solution is*

$$y(t) = c_1 |t|^{\lambda} \cos(\mu \ln |t|) + c_2 |t|^{\lambda} \sin(\mu \ln |t|)$$

Proof

- $t > 0$, done.
- $t < 0$, change of variable $t = -s$

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

In Euler Equation, if α and β are replaced by functions of t,

$$t^2 y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form $y = t^r \times$ power series

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form $y = t^r \times$ power series

Consider DE in standard form

$$y'' + p(t)y' + q(t)y = 0$$

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form $y = t^r \times$ power series

Consider DE in standard form

$$y'' + p(t)y' + q(t)y = 0$$

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0 \qquad (**)$$

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form $y = t^r \times$ power series

Consider DE in standard form

$$\begin{aligned} y'' + p(t)y' + q(t)y &= 0 \\ t^2y'' + t \cdot tp(t)y' + t^2q(t)y &= 0 \qquad (**) \end{aligned}$$

Compare $(*)$ and $(**)$,

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form $y = t^r \times$ power series

Consider DE in standard form

$$y'' + p(t)y' + q(t)y = 0$$

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0 \qquad (**)$$

Compare $(*)$ and $(**)$, want $tp(t) = \alpha(t)$ and

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form $y = t^r \times$ power series

Consider DE in standard form

$$\begin{aligned} y'' + p(t)y' + q(t)y &= 0 \\ t^2y'' + t \cdot tp(t)y' + t^2q(t)y &= 0 \qquad (**) \end{aligned}$$

Compare $(*)$ and $(**)$, want $tp(t) = \alpha(t)$ and

$$t^2q(t) = \beta(t)$$

In Euler Equation, if α and β are replaced by functions of t,

$$t^2y'' + t\alpha(t)y' + \beta(t)y = 0 \qquad (*)$$

Guess if $\alpha(t)$ and $\beta(t)$ are analytic at $t = 0$,

there are solutions in the form $y = t^r \times$ power series

Consider DE in standard form

$$\begin{aligned} y'' + p(t)y' + q(t)y &= 0 \\ t^2y'' + t \cdot tp(t)y' + t^2q(t)y &= 0 \qquad (**) \end{aligned}$$

Compare $(*)$ and $(**)$, want $tp(t) = \alpha(t)$ and

$t^2q(t) = \beta(t)$ analytic at $t = 0$

Definition Consider (1) $$P(t)y'' + Q(t)y' + R(t)y = 0$$

Definition Consider (1) $P(t)y'' + Q(t)y' + R(t)y = 0$ or

(2) $y'' + p(t)y' + q(t)y = 0$

Definition Consider (1) $P(t)y'' + Q(t)y' + R(t)y = 0$ or

(2) $y'' + p(t)y' + q(t)y = 0$

where $p(t) = \dfrac{Q(t)}{P(t)}$ and $q(t) = \dfrac{R(t)}{P(t)}$

Definition Consider

$$(1) \qquad P(t)y'' + Q(t)y' + R(t)y = 0$$

or

$$(2) \qquad y'' + p(t)y' + q(t)y = 0$$

where $p(t) = \dfrac{Q(t)}{P(t)}$ and $q(t) = \dfrac{R(t)}{P(t)}$

A *singular point* $t = t_0$ of (1) or (2) is said to be

- *regular*

Definition Consider (1) $P(t)y'' + Q(t)y' + R(t)y = 0$ or

(2) $y'' + p(t)y' + q(t)y = 0$

where $p(t) = \dfrac{Q(t)}{P(t)}$ and $q(t) = \dfrac{R(t)}{P(t)}$

A *singular point* $t = t_0$ of (1) or (2) is said to be

- *regular* if both $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ (have extensions that) are analytic at t_0

Definition Consider

$$(1) \qquad P(t)y'' + Q(t)y' + R(t)y = 0 \qquad \text{or}$$

$$(2) \qquad y'' + p(t)y' + q(t)y = 0$$

where $p(t) = \dfrac{Q(t)}{P(t)}$ and $q(t) = \dfrac{R(t)}{P(t)}$

A *singular point* $t = t_0$ of (1) or (2) is said to be

- *regular* if both $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ (have extensions that) are analytic at t_0
- *irregular* if it not regular.

Definition Consider

$$(1) \qquad P(t)y'' + Q(t)y' + R(t)y = 0 \qquad \text{or}$$

$$(2) \qquad y'' + p(t)y' + q(t)y = 0$$

where $p(t) = \dfrac{Q(t)}{P(t)}$ and $q(t) = \dfrac{R(t)}{P(t)}$

A *singular point* $t = t_0$ of (1) or (2) is said to be

- *regular* if both $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ (have extensions that) are analytic at t_0
- *irregular* if it not regular.

Fact Suppose P, Q and R are analytic at t_0 and t_0 is a singular point.

- If $\lim_{t \to t_0} (t - t_0)p(t)$ and $\lim_{t \to t_0} (t - t_0)^2 q(t)$ exist, then t_0 is regular.

Definition Consider

(1) $$P(t)y'' + Q(t)y' + R(t)y = 0$$ or

(2) $$y'' + p(t)y' + q(t)y = 0$$

where $p(t) = \dfrac{Q(t)}{P(t)}$ and $q(t) = \dfrac{R(t)}{P(t)}$

A *singular point* $t = t_0$ of (1) or (2) is said to be

- *regular* if both $(t - t_0)p(t)$ and $(t - t_0)^2 q(t)$ (have extensions that) are analytic at t_0
- *irregular* if it not regular.

Fact Suppose P, Q and R are analytic at t_0 and t_0 is a singular point.

- If $\lim_{t \to t_0} (t - t_0)p(t)$ and $\lim_{t \to t_0} (t - t_0)^2 q(t)$ exist, then t_0 is regular.
- If $\lim_{t \to t_0} (t - t_0)p(t) = \pm\infty$ or $\lim_{t \to t_0} (t - t_0)^2 q(t) = \pm\infty$, then t_0 is irregular.

Example Find and determine the nature of the singular points of

$$t^2y'' + 2ty' + 3y = 0$$

Example Find and determine the nature of the singular points of

$$t^2 y'' + 2ty' + 3y = 0$$

Solution $p(t) = \dfrac{2}{t}, \qquad q(t) = \dfrac{3}{t^2}$

Example Find and determine the nature of the singular points of

$$t^2y'' + 2ty' + 3y = 0$$

Solution $$p(t) = \frac{2}{t}, \qquad q(t) = \frac{3}{t^2}$$

$t = 0$ is the only singular point

Example Find and determine the nature of the singular points of

$$t^2 y'' + 2ty' + 3y = 0$$

Solution $p(t) = \dfrac{2}{t}, \qquad q(t) = \dfrac{3}{t^2}$

$t = 0$ is the only singular point

$t \cdot p(t) = 2$ and

Example Find and determine the nature of the singular points of

$$t^2 y'' + 2ty' + 3y = 0$$

Solution $p(t) = \frac{2}{t}, \qquad q(t) = \frac{3}{t^2}$

$t = 0$ is the only singular point

$t \cdot p(t) = 2$ and

$t^2 \cdot q(t) = 3$

Example Find and determine the nature of the singular points of

$$t^2y'' + 2ty' + 3y = 0$$

Solution $p(t) = \frac{2}{t}, \qquad q(t) = \frac{3}{t^2}$

$t = 0$ is the only singular point

$t \cdot p(t) = 2$ and

$t^2 \cdot q(t) = 3$ are analytic at $t = 0$

Example Find and determine the nature of the singular points of

$$t^2y'' + 2ty' + 3y = 0$$

Solution $p(t) = \dfrac{2}{t}, \qquad q(t) = \dfrac{3}{t^2}$

$t = 0$ is the only singular point

$t \cdot p(t) = 2$ and

$t^2 \cdot q(t) = 3$ are analytic at $t = 0$

Therefore, the singular point $t = 0$ is *regular*

Example Find and determine the nature of the singular points of

$$t^3y'' + 2ty' + 3y = 0$$

Example Find and determine the nature of the singular points of

$$t^3y'' + 2ty' + 3y = 0$$

Solution $$p(t) = \frac{2}{t^2}, \quad q(t) = \frac{3}{t^3}$$

Example Find and determine the nature of the singular points of

$$t^3y'' + 2ty' + 3y = 0$$

Solution $p(t) = \frac{2}{t^2}, \quad q(t) = \frac{3}{t^3}$

$t = 0$ is the only singular point

Example Find and determine the nature of the singular points of

$$t^3 y'' + 2ty' + 3y = 0$$

Solution $p(t) = \dfrac{2}{t^2}, \quad q(t) = \dfrac{3}{t^3}$

$t = 0$ is the only singular point

$t \cdot p(t) = \dfrac{2}{t}$ is not analytic at $t = 0$

Example Find and determine the nature of the singular points of

$$t^3 y'' + 2ty' + 3y = 0$$

Solution $p(t) = \dfrac{2}{t^2}, \quad q(t) = \dfrac{3}{t^3}$

$t = 0$ is the only singular point

$t \cdot p(t) = \dfrac{2}{t}$ is not analytic at $t = 0$

The singular point $t = 0$ is *irregular*

Example Find and determine the nature of the singular points of

$$t^3 y'' + (\sin t) y = 0$$

Example Find and determine the nature of the singular points of

$$t^3 y'' + (\sin t)y = 0$$

Solution $p(t) = 0, \quad q(t) = \dfrac{\sin t}{t^3}$

Example Find and determine the nature of the singular points of

$$t^3 y'' + (\sin t)y = 0$$

Solution $p(t) = 0, \quad q(t) = \dfrac{\sin t}{t^3}$

$t = 0$ is the only singular points

Example Find and determine the nature of the singular points of

$$t^3 y'' + (\sin t) y = 0$$

Solution $p(t) = 0, \quad q(t) = \dfrac{\sin t}{t^3}$

$t = 0$ is the only singular points

$t \cdot p(t) = 0$ and

Example Find and determine the nature of the singular points of

$$t^3 y'' + (\sin t)y = 0$$

Solution $p(t) = 0, \quad q(t) = \dfrac{\sin t}{t^3}$

$t = 0$ is the only singular points

$t \cdot p(t) = 0$ and

$$t^2 \cdot q(t) = \frac{\sin t}{t}$$

Example Find and determine the nature of the singular points of

$$t^3 y'' + (\sin t)y = 0$$

Solution $p(t) = 0, \quad q(t) = \dfrac{\sin t}{t^3}$

$t = 0$ is the only singular points

$t \cdot p(t) = 0$ and

$t^2 \cdot q(t) = \dfrac{\sin t}{t}$ are analytic at 0,

Example Find and determine the nature of the singular points of

$$t^3y'' + (\sin t)y = 0$$

Solution $p(t) = 0,\quad q(t) = \dfrac{\sin t}{t^3}$

$t = 0$ is the only singular points

$t \cdot p(t) = 0$ and

$t^2 \cdot q(t) = \dfrac{\sin t}{t}$ are analytic at 0,

The singular point $t = 0$ is *regular*

Frobenius Theorem *Consider the DE*

$$y'' + p(t)y' + q(t)y = 0, \qquad t > 0$$

Suppose $t = 0$ *is a regular singular point.*

Frobenius Theorem *Consider the DE*

$$y'' + p(t)y' + q(t)y = 0, \qquad t > 0$$

Suppose $t = 0$ *is a regular singular point.*

Let the Taylor series expansions at $t = 0$ *for* $tp(t)$ *and* $t^2q(t)$ *be*

$$tp(t) = \sum_{n=0}^{\infty} p_n t^n, \qquad t^2q(t) = \sum_{n=0}^{\infty} q_n t^n$$

Frobenius Theorem *Consider the DE*

$$y'' + p(t)y' + q(t)y = 0, \qquad t > 0$$

Suppose $t = 0$ *is a regular singular point.*

Let the Taylor series expansions at $t = 0$ *for* $tp(t)$ *and* $t^2q(t)$ *be*

$$tp(t) = \sum_{n=0}^{\infty} p_n t^n, \qquad t^2q(t) = \sum_{n=0}^{\infty} q_n t^n$$

for $|t| < \rho$ *where* $\rho > 0$*.*

Frobenius Theorem *Consider the DE*

$$y'' + p(t)y' + q(t)y = 0, \qquad t > 0$$

Suppose $t = 0$ *is a regular singular point.*

Let the Taylor series expansions at $t = 0$ *for* $tp(t)$ *and* $t^2q(t)$ *be*

$$tp(t) = \sum_{n=0}^{\infty} p_n t^n, \qquad t^2q(t) = \sum_{n=0}^{\infty} q_n t^n$$

for $|t| < \rho$ *where* $\rho > 0$.

Let r_1, r_2 *be the roots of the equation (called the indicial equation)*

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

Frobenius Theorem *Consider the DE*

$$y'' + p(t)y' + q(t)y = 0, \qquad t > 0$$

Suppose $t = 0$ *is a regular singular point.*

Let the Taylor series expansions at $t = 0$ *for* $tp(t)$ *and* $t^2q(t)$ *be*

$$tp(t) = \sum_{n=0}^{\infty} p_n t^n, \qquad t^2q(t) = \sum_{n=0}^{\infty} q_n t^n$$

for $|t| < \rho$ *where* $\rho > 0$.

Let r_1, r_2 *be the roots of the equation (called the indicial equation)*

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

where $r_1 \geq r_2$ *if* $r_1, r_2 \in \mathbb{R}$.

Frobenius Theorem *Consider the DE*

$$y'' + p(t)y' + q(t)y = 0, \qquad t > 0$$

Suppose $t = 0$ *is a regular singular point.*

Let the Taylor series expansions at $t = 0$ *for* $tp(t)$ *and* $t^2q(t)$ *be*

$$tp(t) = \sum_{n=0}^{\infty} p_n t^n, \qquad t^2q(t) = \sum_{n=0}^{\infty} q_n t^n$$

for $|t| < \rho$ *where* $\rho > 0$.

Let r_1, r_2 *be the roots of the equation (called the indicial equation)*

$$F(r) = r(r-1) + p_0 r + q_0 = 0$$

where $r_1 \geq r_2$ *if* $r_1, r_2 \in \mathbb{R}$.

Then in the interval $(0, \rho)$, *there exist two linearly independent solutions given by*

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

$$y_1(t) \;=\; t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) \;=\; t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1c) $r_1, r_2 \notin \mathbb{R}$

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1c) $r_1, r_2 \notin \mathbb{R}$ $y_i(t)$ *obtained in Case (1) are complex-valued functions.*

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1c) $r_1, r_2 \notin \mathbb{R}$ $y_i(t)$ *obtained in Case (1) are complex-valued functions.*

To get real-valued solutions, take real- and imaginary parts.

Case (1) $r_1, r_2 \in \mathbb{R}, \;\; r_1 - r_2 \notin \mathbb{Z},$

$$y_1(t) \;=\; t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) \;=\; t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1c) $r_1, r_2 \notin \mathbb{R}$ $\quad y_i(t)$ *obtained in Case (1) are complex-valued functions.*

To get real-valued solutions, take real- and imaginary parts.

Case (2) $r_1 = r_2$ $\quad y_1(t) \;=\; t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1c) $r_1, r_2 \notin \mathbb{R}$ $y_i(t)$ *obtained in Case (1) are complex-valued functions.*

To get real-valued solutions, take real- and imaginary parts.

Case (2) $r_1 = r_2$

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$$

$$y_2(t) = y_1(t)\ln t + t^{r_1}\sum_{n=1}^{\infty} b_n(r_1)t^n$$

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1c) $r_1, r_2 \notin \mathbb{R}$ $\quad y_i(t)$ *obtained in Case (1) are complex-valued functions.*

To get real-valued solutions, take real- and imaginary parts.

Case (2) $r_1 = r_2$

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$$

$$y_2(t) = y_1(t)\ln t + t^{r_1}\sum_{n=1}^{\infty} b_n(r_1)t^n$$

Case (3) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \in \mathbb{Z}^+$

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$$

Case (1) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \notin \mathbb{Z}$,

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right) \qquad y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$$

Case (1c) $r_1, r_2 \notin \mathbb{R}$ $\quad y_i(t)$ *obtained in Case (1) are complex-valued functions.*

To get real-valued solutions, take real- and imaginary parts.

Case (2) $r_1 = r_2$

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$$

$$y_2(t) = y_1(t)\ln t + t^{r_1}\sum_{n=1}^{\infty} b_n(r_1)t^n$$

Case (3) $r_1, r_2 \in \mathbb{R}$, $r_1 - r_2 \in \mathbb{Z}^+$

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$$

$$y_2(t) = Ay_1(t)\ln t + t^{r_2}\left(1 + \sum_{n=1}^{\infty} c_n(r_2)t^n\right)$$

Proof

Rewrite DE $$t^2y'' + t \cdot tp(t)y' + t^2q(t)y \quad = \quad 0$$

Proof

Rewrite DE

$$
\begin{aligned}
t^2y'' + t\cdot tp(t)y' + t^2q(t)y &= 0 \\
t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y &= 0
\end{aligned}
$$

Proof

Rewrite DE

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0$$

$$t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y = 0$$

Try Frobenius series solution

$$y = t^r \sum_{n=0}^{\infty} a_n t^n$$

Proof

Rewrite DE

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0$$

$$t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y = 0$$

Try Frobenius series solution

$$y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{r+n} = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

Proof

Rewrite DE

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0$$

$$t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y = 0$$

Try Frobenius series solution

$$y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{r+n} = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

where a_0 is non-zero

Proof

Rewrite DE

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y \;=\; 0$$

$$t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y \;=\; 0$$

Try Frobenius series solution

$$y \;=\; t^r \sum_{n=0}^{\infty} a_n t^n \;=\; \sum_{n=0}^{\infty} a_n t^{r+n} \;=\; a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

where a_0 is non-zero *May assume* $a_0 = 1$

Proof

Rewrite DE

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0$$

$$t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y = 0$$

Try Frobenius series solution

$$y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{r+n} = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

where a_0 is non-zero *May assume* $a_0 = 1$

Substitute into DE

$$y = \sum_{n=0}^{\infty} a_n t^{r+n}$$

Proof

Rewrite DE

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0$$

$$t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y = 0$$

Try Frobenius series solution

$$y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{r+n} = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

where a_0 is non-zero *May assume* $a_0 = 1$

Substitute into DE

$$y = \sum_{n=0}^{\infty} a_n t^{r+n}$$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n t^{r+n-1}$$

Proof

Rewrite DE

$$t^2y'' + t \cdot tp(t)y' + t^2q(t)y = 0$$

$$t^2y'' + t\left(\sum_{n=0}^{\infty} p_n t^n\right)y' + \left(\sum_{n=0}^{\infty} q_n t^n\right)y = 0$$

Try Frobenius series solution

$$y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{r+n} = a_0 t^r + a_1 t^{r+1} + a_2 t^{r+2} + \cdots$$

where a_0 is non-zero *May assume* $a_0 = 1$

Substitute into DE

$$y = \sum_{n=0}^{\infty} a_n t^{r+n}$$

$$y' = \sum_{n=0}^{\infty} (r+n) a_n t^{r+n-1}$$

$$y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1) a_n t^{r+n-2}$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} +$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right) \cdot \left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right)$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right) + \left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right)$$

$$+\left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) \quad = \quad 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} +$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right) + \left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)a_k p_{n-k}\right)t^{r+n}$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right) + \left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)a_k p_{n-k}\right)t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k q_{n-k}\right)t^{r+n} = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right) + \left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)a_k p_{n-k}\right)t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k q_{n-k}\right)t^{r+n} = 0$$

Compare coefficients

$$a_0\big(r(r-1) + p_0 r + q_0\big) = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right)$$

$$+\left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) \quad = \quad 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)a_k p_{n-k}\right)t^{r+n}$$

$$+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k q_{n-k}\right)t^{r+n} \quad = \quad 0$$

Compare coefficients

$$a_0\big(r(r-1) + p_0 r + q_0\big) \quad = \quad 0$$

$$(r+n)(r+n-1)a_n +$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right)$$

$$+\left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)a_k p_{n-k}\right)t^{r+n}$$

$$+\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k q_{n-k}\right)t^{r+n} = 0$$

Compare coefficients

$$a_0\big(r(r-1)+p_0 r+q_0\big) = 0$$

$$(r+n)(r+n-1)a_n + \left(\sum_{k=0}^{n-1}(r+k)a_k p_{n-k} + (r+n)a_n p_0\right) +$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \left(\sum_{n=0}^{\infty} p_n t^n\right)\cdot\left(\sum_{n=0}^{\infty}(r+n)a_n t^{r+n}\right) + \left(\sum_{n=0}^{\infty} q_n t^n\right)\cdot\left(\sum_{n=0}^{\infty} a_n t^{r+n}\right) = 0$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(r+k)a_k p_{n-k}\right)t^{r+n} + \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k q_{n-k}\right)t^{r+n} = 0$$

Compare coefficients

$$a_0\big(r(r-1)+p_0 r+q_0\big) = 0$$

$$(r+n)(r+n-1)a_n + \left(\sum_{k=0}^{n-1}(r+k)a_k p_{n-k} + (r+n)a_n p_0\right) + \left(\sum_{k=0}^{n-1} a_k q_{n-k} + a_n q_0\right) = 0 \quad (n \geq 1)$$

Put $F(r) = r(r-1) + p_0 r + q_0$

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \tag{1}$$

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) \quad = \quad 0 \qquad (1)$$

$$F(r+n)a_n +$$

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \qquad (1)$$

$$F(r+n)a_n + \sum_{k=0}^{n-1} \Big((r+k)p_{n-k} + q_{n-k}\Big)a_k = 0 \qquad (n \geq 1) \qquad (2)$$

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \qquad (1)$$

$$F(r+n)a_n + \sum_{k=0}^{n-1} \Big((r+k)p_{n-k} + q_{n-k}\Big)a_k = 0 \qquad (n \geq 1) \qquad (2)$$

Let r_1, r_2 be the roots of (1).

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \tag{1}$$

$$F(r+n)a_n + \sum_{k=0}^{n-1}\Big((r+k)p_{n-k} + q_{n-k}\Big)a_k = 0 \qquad (n \geq 1) \tag{2}$$

Let r_1, r_2 be the roots of (1).

Below, consider cases where $r_1 \geq r_2$ are real.

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$\begin{aligned} F(r) &= 0 && (1) \\ F(r+n)a_n + \sum_{k=0}^{n-1} \Big((r+k)p_{n-k} + q_{n-k}\Big)a_k &= 0 \qquad (n \geq 1) && (2) \end{aligned}$$

Let r_1, r_2 be the roots of (1).

Below, consider cases where $r_1 \geq r_2$ are real.

From (2) if $F(r+1), F(r+2), \ldots$ are all non-zero

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \qquad (1)$$

$$F(r+n)a_n + \sum_{k=0}^{n-1} \Big((r+k)p_{n-k} + q_{n-k}\Big)a_k = 0 \qquad (n \geq 1) \qquad (2)$$

Let r_1, r_2 be the roots of (1).

Below, consider cases where $r_1 \geq r_2$ are real.

From (2) if $F(r+1), F(r+2), \ldots$ are all non-zero

coefficients $a_1, a_2, \ldots$ can be computed (in terms of p_n, q_n and r)

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \qquad (1)$$

$$F(r+n)a_n + \sum_{k=0}^{n-1} \Big((r+k)p_{n-k} + q_{n-k}\Big)a_k = 0 \qquad (n \geq 1) \qquad (2)$$

Let r_1, r_2 be the roots of (1).

Below, consider cases where $r_1 \geq r_2$ are real.

From (2) if $F(r+1), F(r+2), \ldots$ are all non-zero

coefficients $a_1, a_2, \ldots$ can be computed (in terms of p_n, q_n and r)

Since $F(r_1 + n) \neq 0$ for all $n \geq 1$,

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \tag{1}$$

$$F(r+n)a_n + \sum_{k=0}^{n-1} \Big((r+k)p_{n-k} + q_{n-k}\Big)a_k = 0 \qquad (n \geq 1) \tag{2}$$

Let r_1, r_2 be the roots of (1).

Below, consider cases where $r_1 \geq r_2$ are real.

From (2) if $F(r+1), F(r+2), \ldots$ are all non-zero

coefficients $a_1, a_2, \ldots$ can be computed (in terms of p_n, q_n and r)

Since $F(r_1 + n) \neq 0$ for all $n \geq 1$, can always find a solution

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$$

Put $F(r) = r(r-1) + p_0 r + q_0$

Above equations reduce to

$$F(r) = 0 \tag{1}$$

$$F(r+n)a_n + \sum_{k=0}^{n-1}\Big((r+k)p_{n-k} + q_{n-k}\Big)a_k = 0 \qquad (n \geq 1) \tag{2}$$

Let r_1, r_2 be the roots of (1).

Below, consider cases where $r_1 \geq r_2$ are real.

From (2) if $F(r+1), F(r+2), \ldots$ are all non-zero

coefficients $a_1, a_2, \ldots$ can be computed (in terms of p_n, q_n and r)

Since $F(r_1 + n) \neq 0$ for all $n \geq 1$, can always find a solution

$$y_1(t) = t^{r_1}\left(1 + \sum_{n=1}^{\infty} a_n(r_1)t^n\right)$$

where $a_n(r_1)$ means *the coefficients are determined by* (2) *with* $r = r_1$

To get a second linearly independent solution

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

Case 2 $r_1 = r_2$

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

Case 2 $r_1 = r_2$

Case 3 $r_1 - r_2$ a positive integer.

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

- $F(r_2) = 0$

Case 2 $r_1 = r_2$

Case 3 $r_1 - r_2$ a positive integer.

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

- $F(r_2) = 0$
- Since $F(r_2 + 1) \neq 0, \quad F(r_2 + 2) \neq 0, \quad \ldots$

Case 2 $r_1 = r_2$

Case 3 $r_1 - r_2$ a positive integer.

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

- $F(r_2) = 0$
- Since $F(r_2 + 1) \neq 0, \quad F(r_2 + 2) \neq 0, \quad \ldots$

  can use (2) to solve for $a_1, a_2, \ldots$ (with $r = r_2$)

Case 2 $r_1 = r_2$

Case 3 $r_1 - r_2$ a positive integer.

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

- $F(r_2) = 0$

- Since $F(r_2 + 1) \neq 0, \quad F(r_2 + 2) \neq 0, \quad \dots$

  can use (2) to solve for $a_1, a_2, \dots$ (with $r = r_2$)

Obtain second solution $y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$

Case 2 $r_1 = r_2$

Case 3 $r_1 - r_2$ a positive integer.

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

- $F(r_2) = 0$
- Since $F(r_2+1) \neq 0, \quad F(r_2+2) \neq 0, \quad \ldots$

  can use (2) to solve for $a_1, a_2, \ldots$ (with $r = r_2$)

Obtain second solution $y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$

Remark

- Power series in y_1 and y_2 converges for $|t| < \rho$

Case 2 $r_1 = r_2$

Case 3 $r_1 - r_2$ a positive integer.

To get a second linearly independent solution

Case 1 $r_1 - r_2 \notin \mathbb{Z}$

- $F(r_2) = 0$
- Since $F(r_2 + 1) \neq 0, \quad F(r_2 + 2) \neq 0, \quad \ldots$

  can use (2) to solve for $a_1, a_2, \ldots$ (with $r = r_2$)

Obtain second solution $y_2(t) = t^{r_2}\left(1 + \sum_{n=1}^{\infty} a_n(r_2)t^n\right)$

Remark

- Power series in y_1 and y_2 converges for $|t| < \rho$
- y_1 and y_2 are linearly independent on $(0, \rho)$.

Case 2 $r_1 = r_2$

Case 3 $r_1 - r_2$ a positive integer.

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1 - 2t)y' - 2y = 0 \qquad t > 0.$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1 - 2t)y' - 2y = 0 \qquad t > 0.$$

Solution $p(t) = \dfrac{1 - 2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1 - 2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1 - 2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1 - 2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1 - 2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1 - 2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r},$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1 - 2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1 - 2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1 - 2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\therefore$ regular $\rho = \infty$

Try Frobenius series $y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-2} +$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\therefore$ regular $\rho = \infty$

Try Frobenius series $y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2} + (1-2t) \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} -$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2} + (1-2t) \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} - 2 \sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2} + (1-2t) \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} - 2 \sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n t^{n+r-1} +$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-2} + (1-2t) \sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} - 2\sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n t^{n+r-1} + \left(\sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)a_n t^{n+r} \right)$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n t^{n+r-2} + (1-2t) \sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} - 2 \sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) a_n t^{n+r-1} + \left(\sum_{n=0}^{\infty} (n+r) a_n t^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r) a_n t^{n+r} \right) - \sum_{n=0}^{\infty} 2 a_n t^{n+r} = 0$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-2} + (1-2t) \sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} - 2 \sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n t^{n+r-1} + \left(\sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)a_n t^{n+r} \right) - \sum_{n=0}^{\infty} 2a_n t^{n+r} = 0$$

$$\sum_{n=-1}^{\infty} 2(n+r+1)(n+r)a_{n+1} t^{n+r}$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* 0 $\quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n t^{n+r-2} + (1-2t)\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1} - 2\sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n t^{n+r-1} + \left(\sum_{n=0}^{\infty}(n+r)a_n t^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)a_n t^{n+r}\right) - \sum_{n=0}^{\infty} 2a_n t^{n+r} = 0$$

$$\sum_{n=-1}^{\infty} 2(n+r+1)(n+r)a_{n+1} t^{n+r} + \sum_{n=-1}^{\infty}(n+r+1)a_{n+1} t^{n+r}$$

Example Find two linearly independent (Frobenius) series solutions to the DE

$$2ty'' + (1-2t)y' - 2y = 0 \qquad t > 0.$$

Solution $\quad p(t) = \dfrac{1-2t}{2t}, \quad q(t) = \dfrac{-1}{t}$

$t = 0$ is a singular point

$t \cdot p(t) = \dfrac{1-2t}{2}$ *and* $t^2 \cdot q(t) = -t$ *are analytic at* $0 \quad \therefore$ regular $\quad \rho = \infty$

Try Frobenius series $\quad y = t^r \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_n t^{n+r}$, where $a_0 = 1$

$$2t \sum_{n=0}^{\infty} (n+r)(n+r-1)a_n t^{n+r-2} + (1-2t) \sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} - 2\sum_{n=0}^{\infty} a_n t^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n t^{n+r-1} + \left(\sum_{n=0}^{\infty} (n+r)a_n t^{n+r-1} - \sum_{n=0}^{\infty} 2(n+r)a_n t^{n+r} \right) - \sum_{n=0}^{\infty} 2a_n t^{n+r} = 0$$

$$\sum_{n=-1}^{\infty} 2(n+r+1)(n+r)a_{n+1} t^{n+r} + \sum_{n=-1}^{\infty} (n+r+1)a_{n+1} t^{n+r} - \sum_{n=0}^{\infty} 2(n+r)a_n t^{n+r} - \sum_{n=0}^{\infty} 2a_n t^{n+r} = 0$$

Comparing coefficients

$$2r(r-1)+r \quad = \quad 0$$

Comparing coefficients

$$2r(r-1) + r = 0$$

$$(n + r + 1)(2n + 2r + 1)a_{n+1}$$

Comparing coefficients

$$\begin{aligned} 2r(r-1)+r &= 0 \\ (n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Comparing coefficients

$$\begin{aligned} 2r(r-1)+r &= 0 \\ (n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Indicial roots $\quad r_1 = \dfrac{1}{2}, \quad r_2 = 0.$

Comparing coefficients

$$
\begin{aligned}
2r(r-1)+r &= 0 \\
(n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0
\end{aligned}
$$

Indicial roots $\quad r_1 = \dfrac{1}{2}, \quad r_2 = 0.$

- First solution put $r_1 = \dfrac{1}{2}$ into recurrence formula

Comparing coefficients

$$
\begin{aligned}
2r(r-1) + r &= 0 \\
(n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0
\end{aligned}
$$

Indicial roots $\quad r_1 = \frac{1}{2}, \quad r_2 = 0.$

- First solution put $r_1 = \frac{1}{2}$ into recurrence formula

$$
\left(n + \frac{1}{2} + 1\right)(2n+2)a_{n+1} - 2\left(n + \frac{1}{2} + 1\right)a_n = 0 \qquad n \geq 0
$$

Comparing coefficients

$$
\begin{aligned}
2r(r-1)+r &= 0 \\
(n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0
\end{aligned}
$$

Indicial roots $\quad r_1 = \dfrac{1}{2}, \quad r_2 = 0.$

- First solution put $r_1 = \dfrac{1}{2}$ into recurrence formula

$$
\begin{aligned}
\left(n+\frac{1}{2}+1\right)(2n+2)a_{n+1} - 2\left(n+\frac{1}{2}+1\right)a_n &= 0 \qquad n \geq 0 \\
(n+1)a_{n+1} - a_n &= 0 \qquad n \geq 0
\end{aligned}
$$

Comparing coefficients

$$
\begin{aligned}
2r(r-1)+r &= 0 \\
(n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0
\end{aligned}
$$

Indicial roots $\quad r_1 = \frac{1}{2}, \quad r_2 = 0.$

- First solution put $r_1 = \frac{1}{2}$ into recurrence formula

$$
\begin{aligned}
\left(n+\frac{1}{2}+1\right)(2n+2)a_{n+1} - 2\left(n+\frac{1}{2}+1\right)a_n &= 0 \qquad n \geq 0 \\
(n+1)a_{n+1} - a_n &= 0 \qquad n \geq 0
\end{aligned}
$$

Solve for a_n $\quad a_n = \frac{1}{n}a_{n-1} \qquad n \geq 1$

Comparing coefficients

$$\begin{aligned} 2r(r-1)+r &= 0 \\ (n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Indicial roots $\quad r_1 = \dfrac{1}{2}, \quad r_2 = 0.$

- First solution put $r_1 = \dfrac{1}{2}$ into recurrence formula

$$\begin{aligned} \left(n+\frac{1}{2}+1\right)(2n+2)a_{n+1} - 2\left(n+\frac{1}{2}+1\right)a_n &= 0 \qquad n \geq 0 \\ (n+1)a_{n+1} - a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Solve for a_n

$$\begin{aligned} a_n &= \frac{1}{n}a_{n-1} \qquad n \geq 1 \\ &= \frac{1}{n}\cdot\frac{1}{n-1}a_{n-2} \end{aligned}$$

Comparing coefficients

$$\begin{aligned} 2r(r-1)+r &= 0 \\ (n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Indicial roots $\quad r_1 = \frac{1}{2}, \quad r_2 = 0.$

- First solution put $r_1 = \frac{1}{2}$ into recurrence formula

$$\begin{aligned} \left(n+\frac{1}{2}+1\right)(2n+2)a_{n+1} - 2\left(n+\frac{1}{2}+1\right)a_n &= 0 \qquad n \geq 0 \\ (n+1)a_{n+1} - a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Solve for a_n

$$\begin{aligned} a_n &= \frac{1}{n}a_{n-1} \qquad n \geq 1 \\ &= \frac{1}{n} \cdot \frac{1}{n-1}a_{n-2} = \cdots = \frac{1}{n!}a_0 \end{aligned}$$

Comparing coefficients

$$\begin{aligned} 2r(r-1)+r &= 0 \\ (n+r+1)(2n+2r+1)a_{n+1} - 2(n+r+1)a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Indicial roots $\quad r_1 = \frac{1}{2}, \quad r_2 = 0.$

- First solution put $r_1 = \frac{1}{2}$ into recurrence formula

$$\begin{aligned} \left(n+\frac{1}{2}+1\right)(2n+2)a_{n+1} - 2\left(n+\frac{1}{2}+1\right)a_n &= 0 \qquad n \geq 0 \\ (n+1)a_{n+1} - a_n &= 0 \qquad n \geq 0 \end{aligned}$$

Solve for a_n

$$\begin{aligned} a_n &= \frac{1}{n}a_{n-1} \qquad n \geq 1 \\ &= \frac{1}{n} \cdot \frac{1}{n-1}a_{n-2} = \cdots = \frac{1}{n!}a_0 \qquad n \geq 0 \end{aligned}$$

Therefore, $$y_1(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n$$

Therefore, $$y_1(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n = t^{\frac{1}{2}} \mathrm{e}^t$$

Therefore, $$y_1(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n = t^{\frac{1}{2}} e^t$$

- Second solution put $r_2 = 0$ into recurrence formula

Therefore, $y_1(t) \ = \ t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \ = \ t^{\frac{1}{2}} \mathrm{e}^t$

- Second solution put $r_2 = 0$ into recurrence formula

$$(n+1)(2n+1)a_{n+1} - 2(n+1)a_n \quad = \quad 0$$

Therefore, $y_1(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n = t^{\frac{1}{2}} \mathrm{e}^t$

- Second solution put $r_2 = 0$ into recurrence formula

$$(n+1)(2n+1)a_{n+1} - 2(n+1)a_n = 0$$

$$(2n+1)a_{n+1} - 2a_n = 0 \qquad n \geq 0$$

Therefore, $y_1(t) \;=\; t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n \;=\; t^{\frac{1}{2}} \mathrm{e}^t$

- Second solution put $r_2 = 0$ into recurrence formula

$$(n+1)(2n+1)a_{n+1} - 2(n+1)a_n \;=\; 0$$

$$(2n+1)a_{n+1} - 2a_n \;=\; 0 \qquad n \geq 0$$

Solve for a_n $\quad a_n \;=\; \frac{2}{2n-1} a_{n-1}$

Therefore, $y_1(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n = t^{\frac{1}{2}} e^t$

- Second solution put $r_2 = 0$ into recurrence formula

$$(n+1)(2n+1)a_{n+1} - 2(n+1)a_n = 0$$

$$(2n+1)a_{n+1} - 2a_n = 0 \qquad n \geq 0$$

Solve for a_n

$$\begin{aligned} a_n &= \frac{2}{2n-1} a_{n-1} \\ &= \frac{2}{2n-1} \cdot \frac{2}{2n-3} a_{n-2} \end{aligned}$$

Therefore, $$y_1(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n = t^{\frac{1}{2}} e^t$$

- Second solution put $r_2 = 0$ into recurrence formula

$$\begin{aligned}(n+1)(2n+1)a_{n+1} - 2(n+1)a_n &= 0 \\ (2n+1)a_{n+1} - 2a_n &= 0 \qquad n \geq 0\end{aligned}$$

Solve for a_n $$\begin{aligned} a_n &= \frac{2}{2n-1} a_{n-1} \\ &= \frac{2}{2n-1} \cdot \frac{2}{2n-3} a_{n-2} = \cdots = \frac{2^n}{(2n-1)!!} a_0 \qquad n \geq 1 \end{aligned}$$

Therefore, $$y_1(t) = t^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{1}{n!} t^n = t^{\frac{1}{2}} e^t$$

- Second solution put $r_2 = 0$ into recurrence formula

$$(n+1)(2n+1)a_{n+1} - 2(n+1)a_n = 0$$

$$(2n+1)a_{n+1} - 2a_n = 0 \qquad n \geq 0$$

Solve for a_n $$a_n = \frac{2}{2n-1} a_{n-1}$$

$$= \frac{2}{2n-1} \cdot \frac{2}{2n-3} a_{n-2} = \cdots = \frac{2^n}{(2n-1)!!} a_0 \qquad n \geq 1$$

Therefore, $$y_2(t) = 1 + \sum_{n=1}^{\infty} \frac{2^n}{(2n-1)!!} t^n$$