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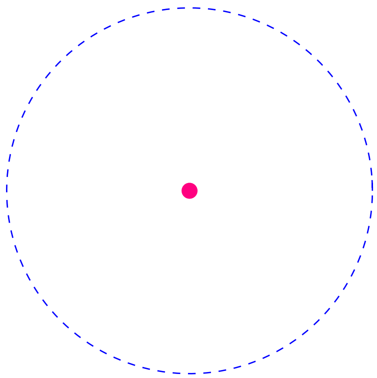

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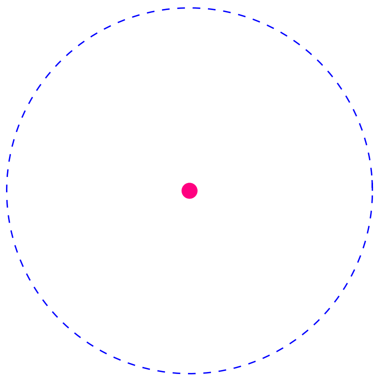

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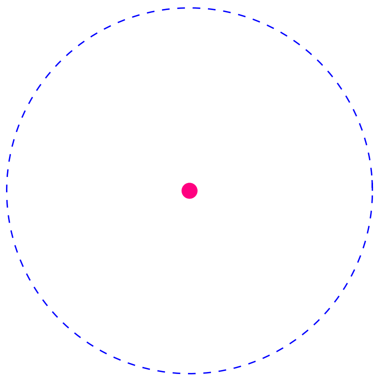

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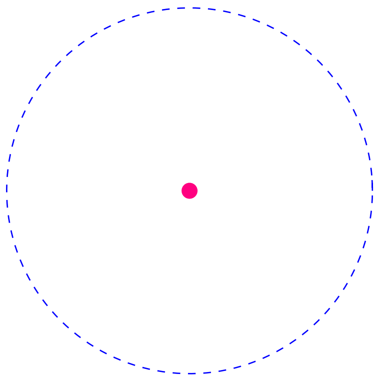

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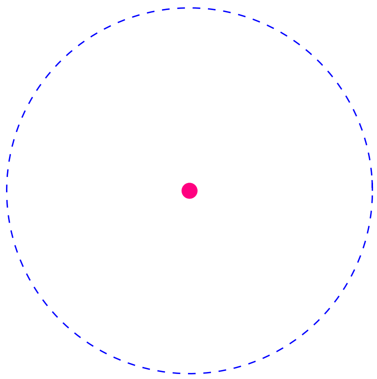

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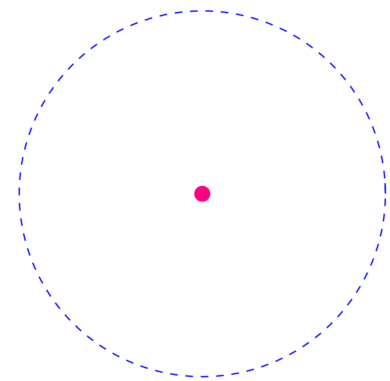

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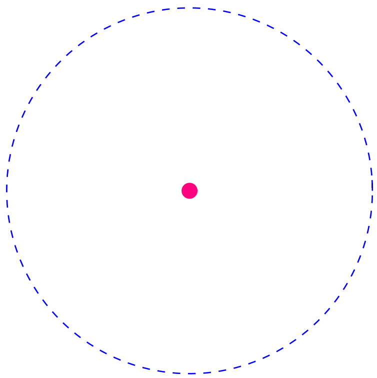

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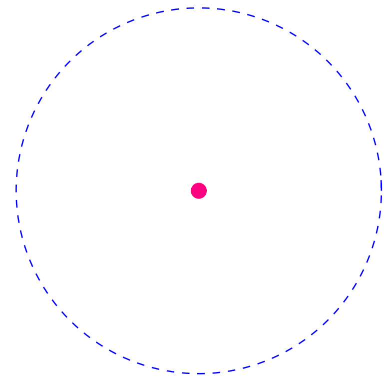

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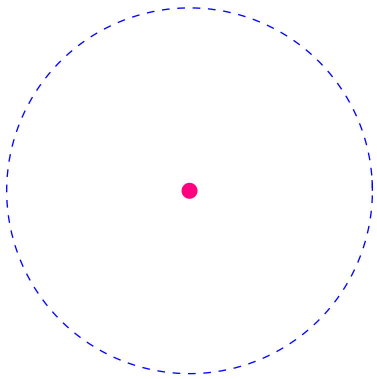

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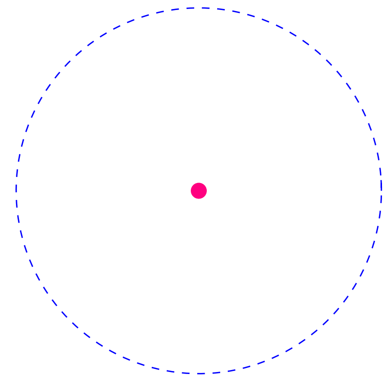

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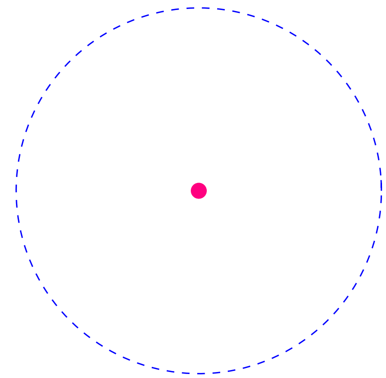

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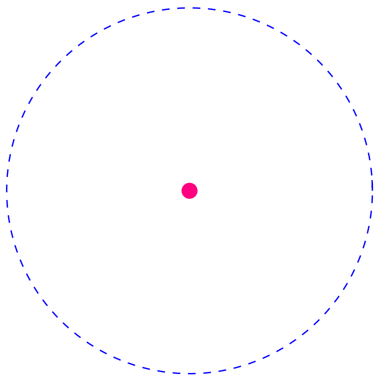

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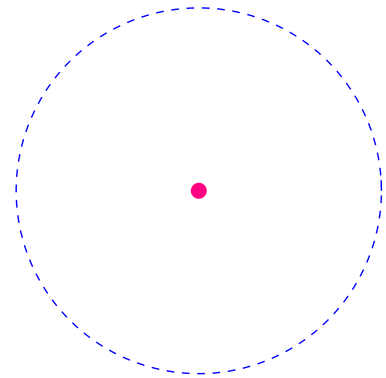

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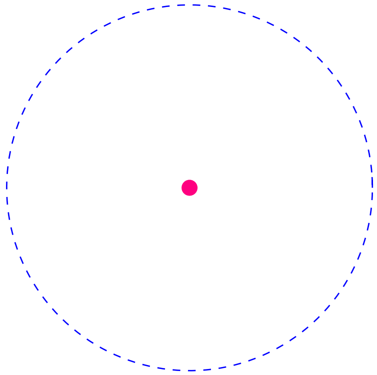

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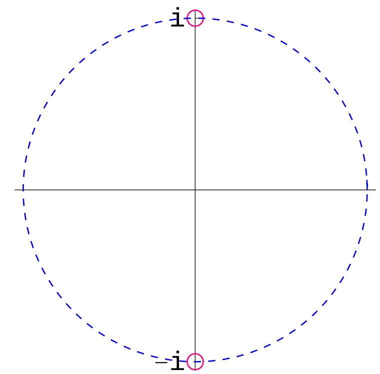

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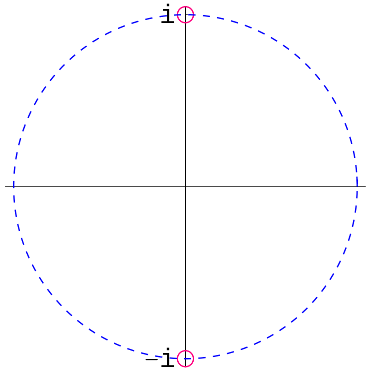

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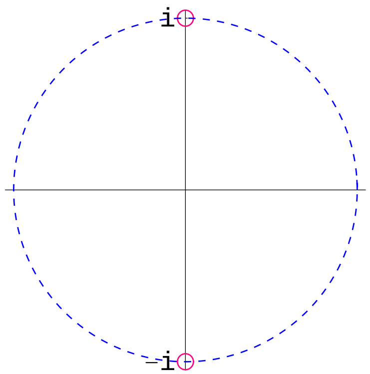

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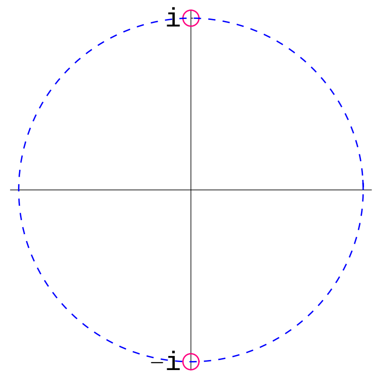

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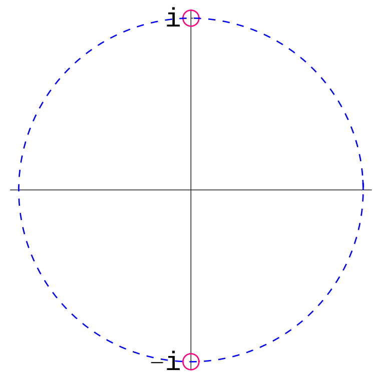

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Remark Simply write $\rho(f)$ is x_0 is understood.

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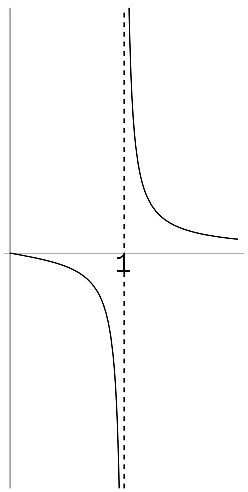

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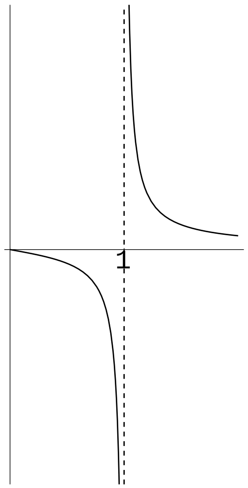

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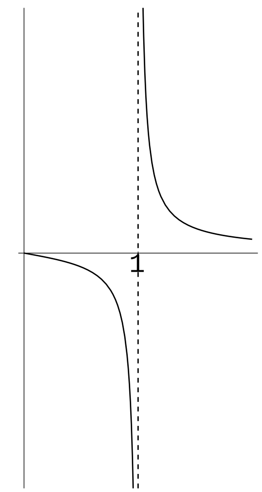

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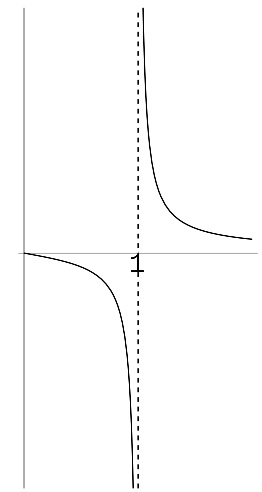

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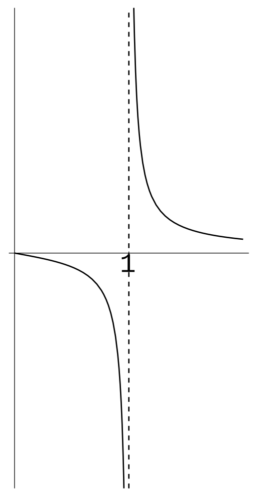

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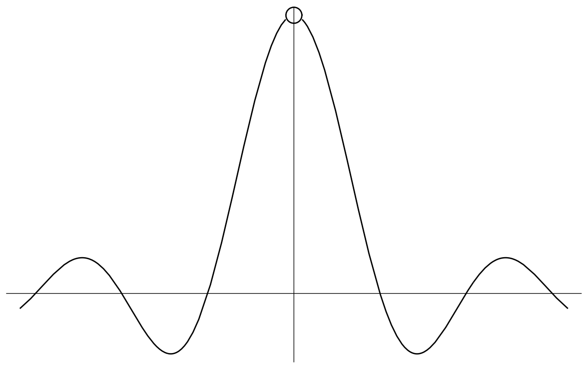

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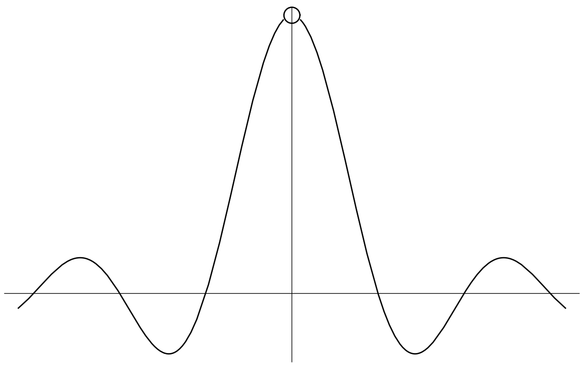

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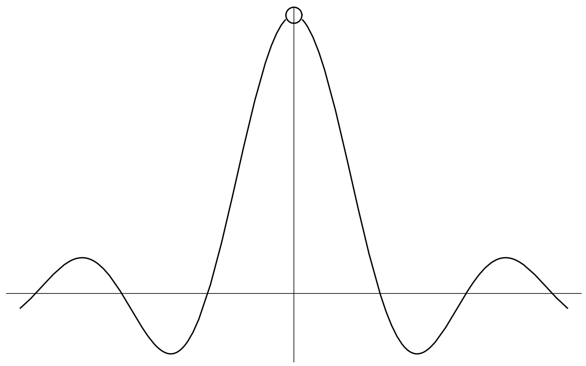

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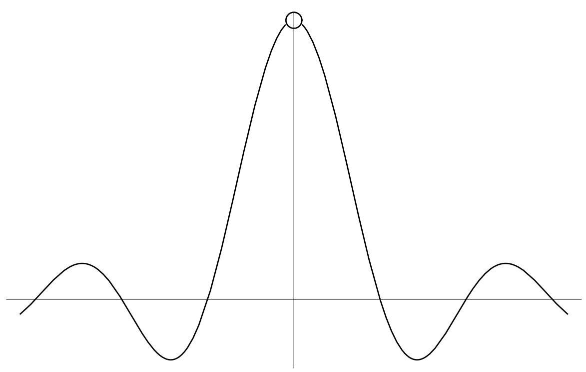

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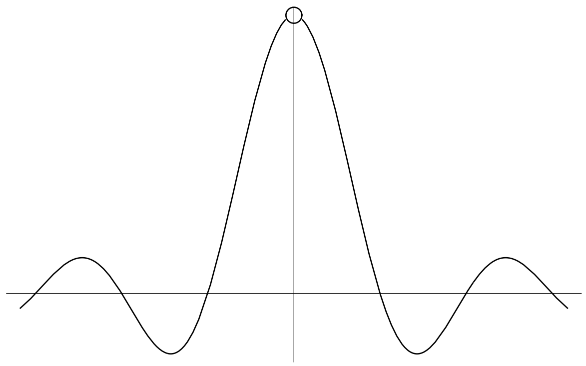

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Theorem *If* t_0 *is an ordinary point of the DE*

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$$y(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n = a_0 y_1(t) + a_1 y_2(t),$$

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Take absolute value in (9)

$$(n+1)(n+2)|a_{n+2}| \quad \leq \quad \sum_{k=0}^{n} \Big((k+1)|p_{n-k}a_{k+1}| + |q_{n-k}a_k| \Big)$$

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- General solution $y(t) = a_0 y_1(t) + a_1 y_2(t)$ where a_0, a_1 are arbitrary.

Example Solve the following DE *(Chebyshev equation)*

$$(1 - t^2)y'' - ty' + \alpha^2 y = 0$$

where α is a constant.

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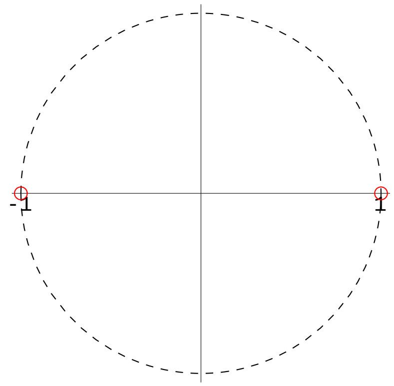

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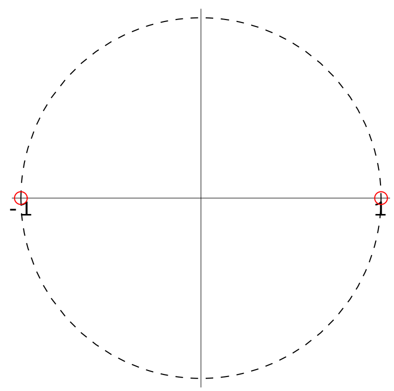

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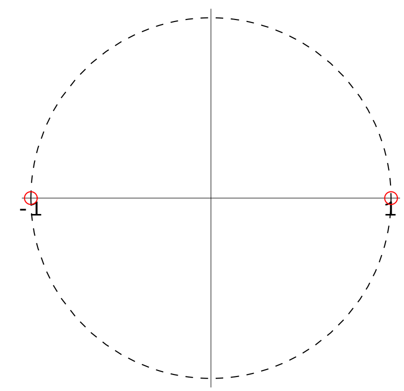

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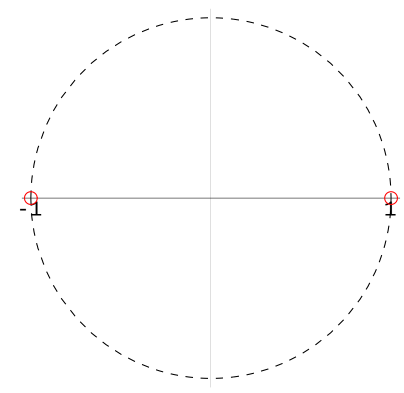

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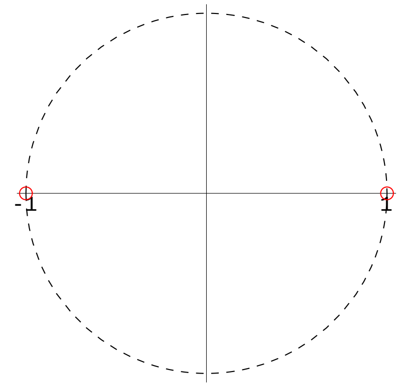

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$$(1-t^2)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - t\sum_{n=1}^{\infty} n a_n t^{n-1} + \alpha^2 \sum_{n=0}^{\infty} a_n t^n \quad = \quad 0$$

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- Thus for $n \geq 0$, $$a_{n+2} = \frac{-(\alpha^2 - n^2)}{(n+2)(n+1)} a_n$$

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a_{2n} &= \frac{(-1)^n \, \alpha(\alpha - 2) \cdots \big(\alpha - (2n - 2)\big) \, \alpha(\alpha + 2) \cdots \big(\alpha + (2n - 2)\big)}{(2n)!} a_0
\end{aligned}
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$$a_{n+2} = \frac{-(\alpha^2 - n^2)}{(n+2)(n+1)} a_n$$

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- General series solution to the DE for $-1 < t < 1$

$$y = \left(a_0 + a_2 t^2 + a_4 t^4 + \cdots\right) + \left(a_1 t + a_3 t^3 + a_5 t^5 + \cdots\right)$$

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$$= \Big(a_0 + \sum_{n=1}^{\infty} a_{2n} t^{2n}\Big) + \Big(a_1 + \sum_{n=1}^{\infty} a_{2n+1} t^{2n+1}\Big)$$

$$y(t) \quad = \quad a_0\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, \alpha(\alpha-2)\cdots\Big(\alpha-(2n-2)\Big)\,\alpha(\alpha+2)\cdots\Big(\alpha+(2n-2)\Big)}{(2n)!}t^{2n}\right)$$

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Remark

- If $\alpha \in \mathbb{Z}$ is even, first series is polynomial

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- If $\alpha \notin \mathbb{Z}$, both are infinite series.

$$
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& +a_1\left(t+\sum_{n=1}^{\infty}\frac{(-1)^n\,(\alpha-1)\cdots\big(\alpha-(2n-1)\big)\,(\alpha+1)\cdots\big(\alpha+(2n-1)\big)}{(2n+1)!}t^{2n+1}\right)
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Remark

- If $\alpha \in \mathbb{Z}$ is even, first series is polynomial

  $\rho(y_1) = \infty$

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- If $\alpha \in \mathbb{Z}$ is odd, second series is polynomial

  $\rho(y_1) = 1 \qquad \rho(y_2) = \infty$

- If $\alpha \notin \mathbb{Z}$, both are infinite series. $\rho(y_1) = 1, \qquad \rho(y_2) = 1$