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  Can solutions be expressed as power series ?

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**Definition** A *power series about* $x_0 \in \mathbb{R}$ is an expression of the form

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**Example** $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \cdots \qquad \rho = 1$$

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For (1), $$a_n = \begin{cases} \frac{(-1)^k}{(2k)!} & \text{if } n = 2k \text{ is even} \\ 0 & \text{if } n = 2k+1 \text{ is odd} \end{cases}$$

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Power series $(*)$ converges for all $t \in \mathbb{R}$.

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**Proposition** *Suppose* $\sum a_n(x - x_0)^n$ *and* $\sum b_n(x - x_0)^n$ *converge for* $|x - x_0| < \rho$*, where* $\rho > 0$*. Then*

**Proposition** *Suppose* $\sum a_n(x-x_0)^n$ *and* $\sum b_n(x-x_0)^n$ *converge for* $|x-x_0|<\rho$, *where* $\rho>0$. *Then*

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$$\begin{aligned} c_0 &= a_0b_0 \\ c_1 &= a_0b_1+a_1b_0 \end{aligned}$$

**Proposition** *Suppose* $\sum a_n(x-x_0)^n$ *and* $\sum b_n(x-x_0)^n$ *converge for* $|x-x_0|<\rho$, *where* $\rho>0$. *Then*

$$(1) \qquad \sum_{n=0}^{\infty} a_n(x-x_0)^n + \sum_{n=0}^{\infty} b_n(x-x_0)^n = \sum_{n=0}^{\infty}(a_n+b_n)(x-x_0)^n \qquad \text{for } |x-x_0|<\rho$$

$$(2) \qquad \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right)\cdot\left(\sum_{n=0}^{\infty} b_n(x-x_0)^n\right) = \sum_{n=0}^{\infty} c_n(x-x_0)^n \qquad \text{for } |x-x_0|<\rho$$

*where* $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

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check that $\dfrac{f^{(n)}(0)}{n!} = a_n$

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*Meaning* $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}$ is a primitive for $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ in $(x_0 - \rho, x_0 + \rho)$

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- *analytic on an open subset $U$ of $\mathbb{R}$* if it is analytic at each $x_0 \in U$.

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$$P(1) + P'(1)(x-1) + \frac{P''(1)}{2!}(x-1)^2$$

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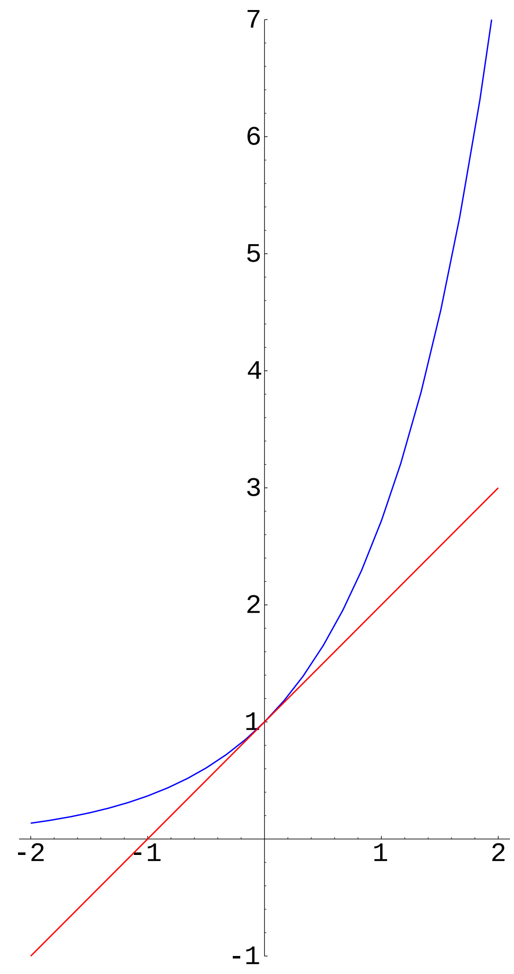

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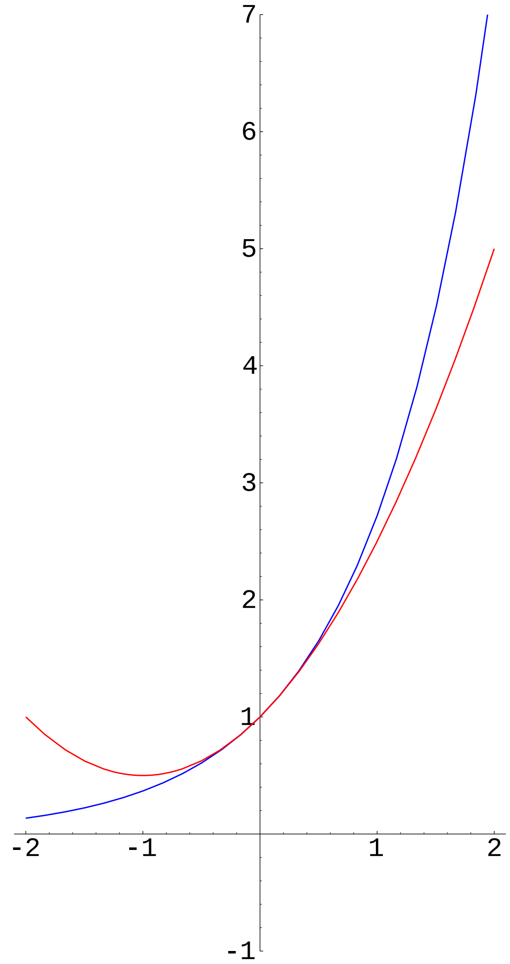

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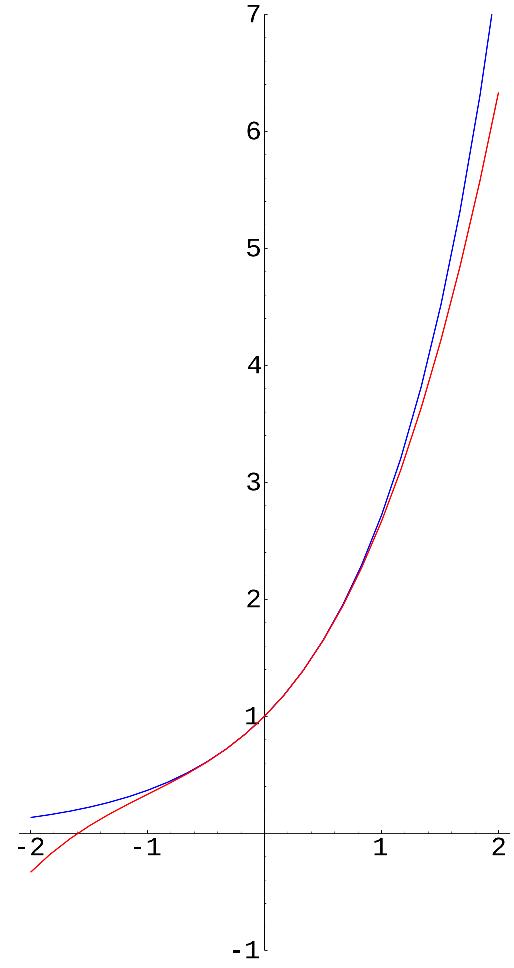

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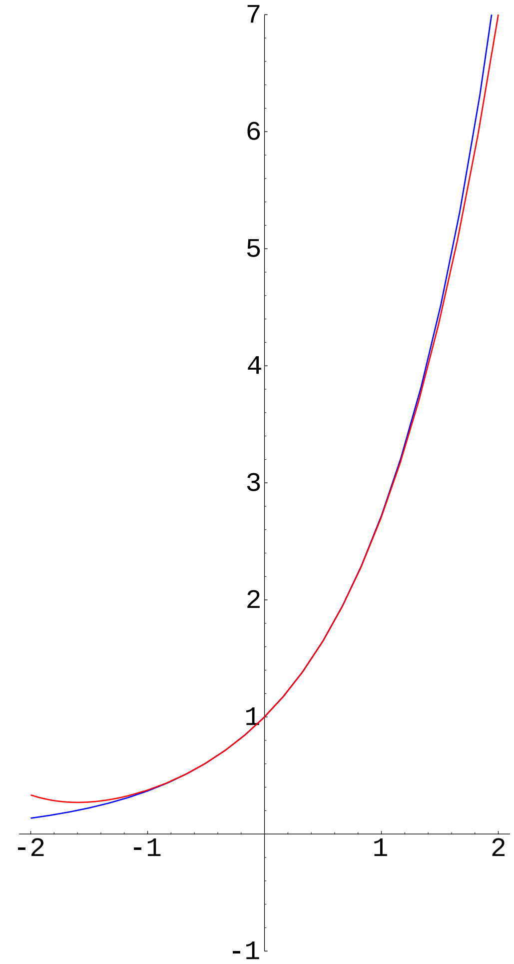

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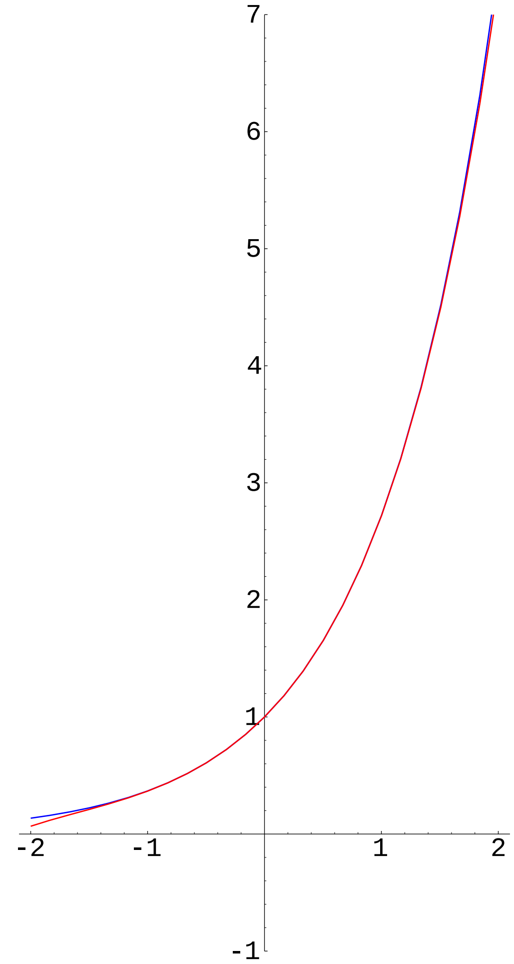

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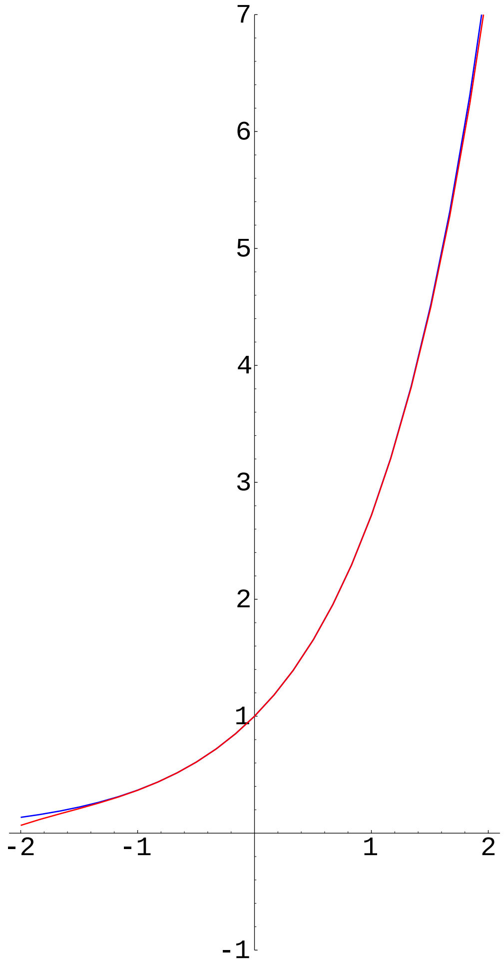

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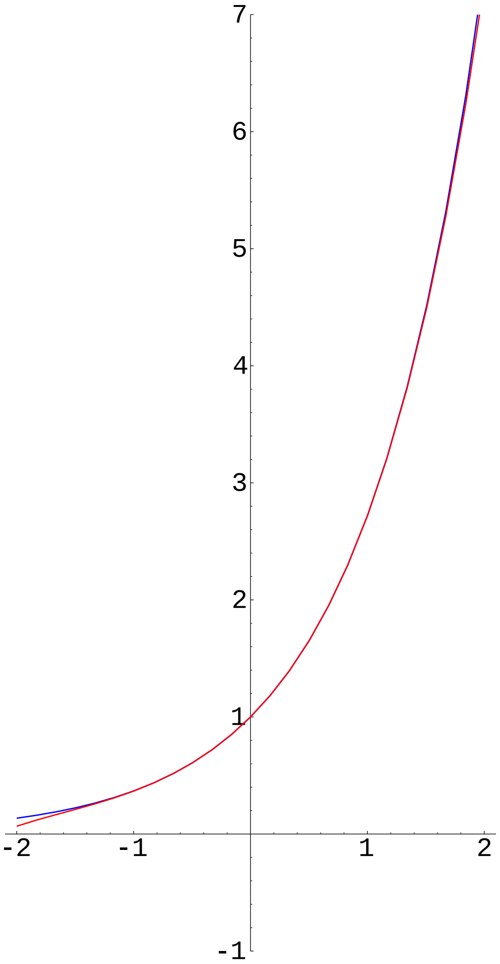

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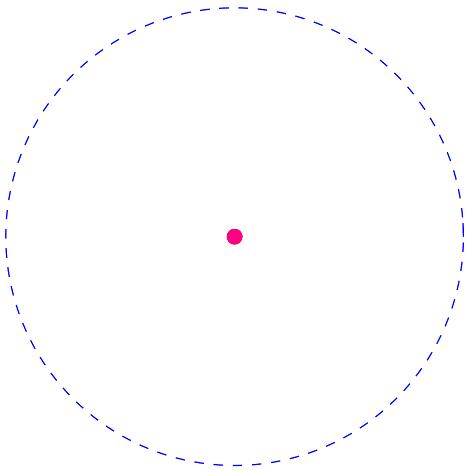

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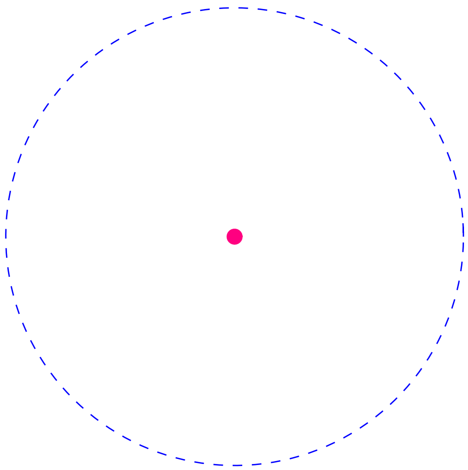

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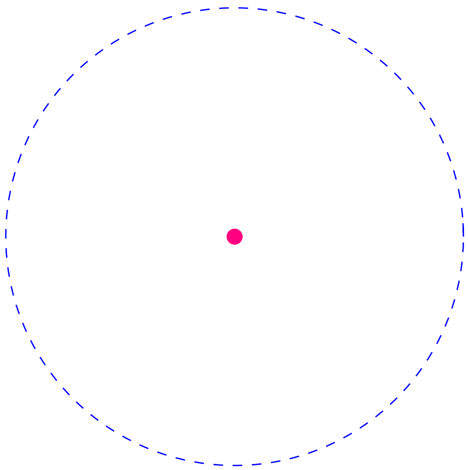

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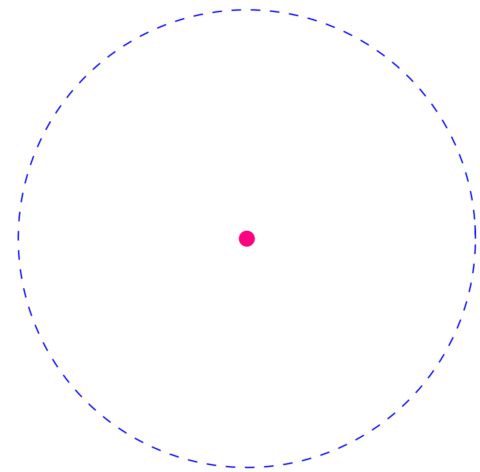

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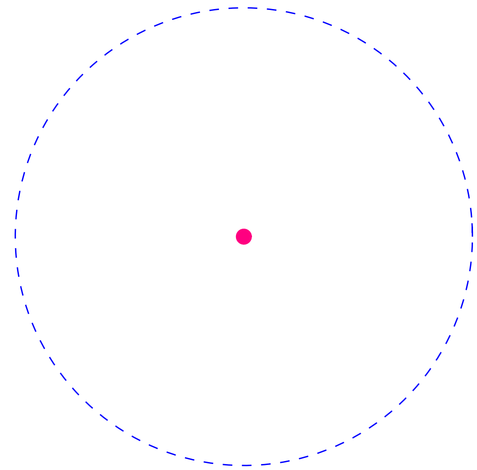

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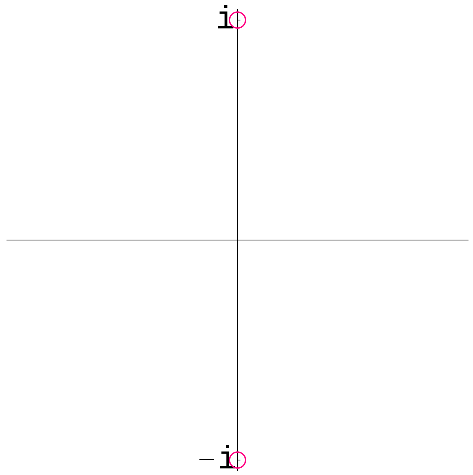

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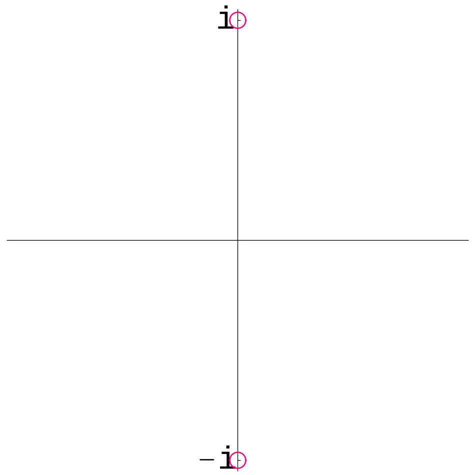


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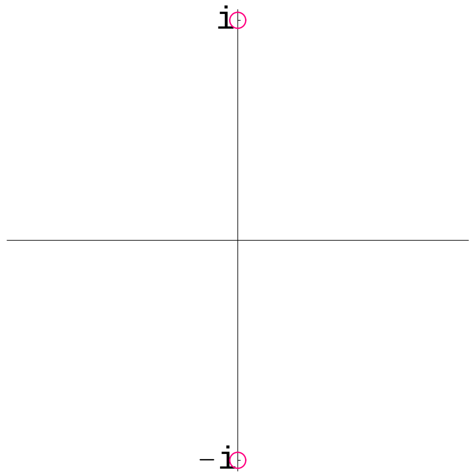


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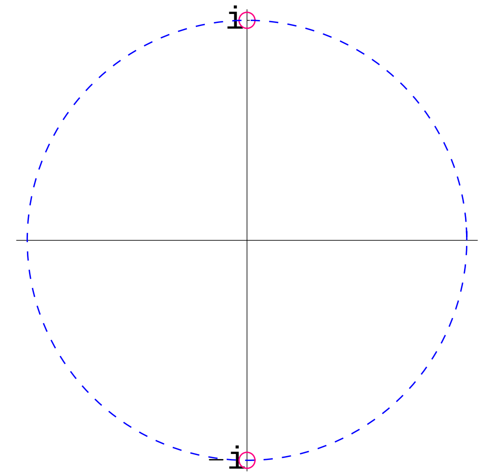


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- adding power series $\sum a_{n+*}x^{n+\triangle} + \sum b_{n+?}x^{n+\square}$
- comparing coefficients: $\sum a_{n+*}x^{n+\triangle} = \sum b_{n+?}x^{n+\square}$

**Example** Rewrite the power series $\sum_{n=1}^{\infty} na_n x^{n-1}$ in the form $\sum b_n x^n$.

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Power series: $a_1 + 2a_2x + 3a_3x^2 + \cdots$

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$$= -a_0 x + a_4 x^2 + \sum_{n=3}^{\infty}\Big((n-2)a_{n-1} + a_{n+2}\Big)x^n$$