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Consider 2nd order IVP $\begin{cases} ay'' + by' + cy = g(t), & t \geq 0, \\ y(0) = y_0, & y'(0) = y_1, \end{cases}$

where $g : [0, \infty) \longrightarrow \mathbb{R}$ is piecewise continuous and of exponential order.

Meaning of solution $\varphi : [0, \infty) \longrightarrow \mathbb{R}$ such that

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Remark Need result for "piecewise continuous" f''.

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- In definition above, t_k is also called a *point of discontinuity* of f.

Remark If f and g are piecewise continuous on $[0, \infty)$ and $\mathcal{L}(f)(s) = \mathcal{L}(g)(s)$ for large enough s,

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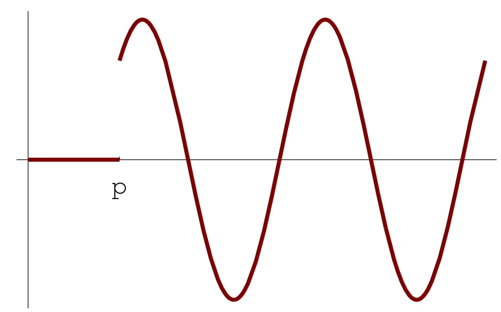

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Proof Similar to proof of Theorem 2. Apply Theorem 1′.

Example Solve $\begin{cases} y'' + y = g(t), & t \geq 0 \\ y(0) = 1, \quad y'(0) = 0, \end{cases}$ where $g(t) = \begin{cases} 0 & \text{if } 0 \leq t < \pi, \\ 1 & \text{if } t \geq \pi. \end{cases}$

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By partial fraction

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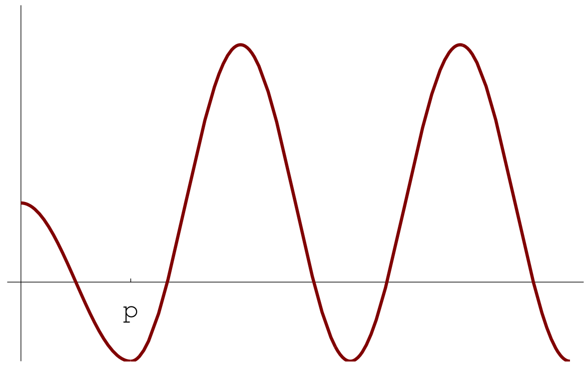

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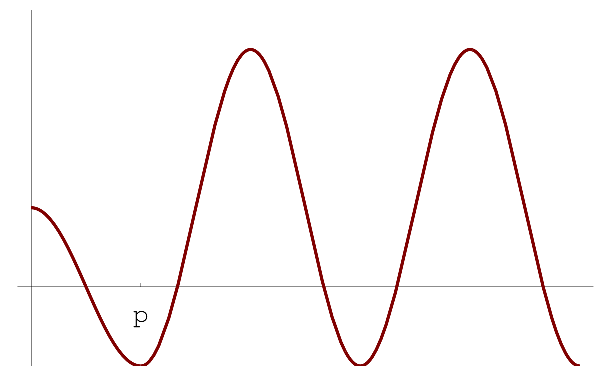

Remark $\dfrac{e^{-\pi s}}{s} \in \mathcal{L}(E_{pc})$,

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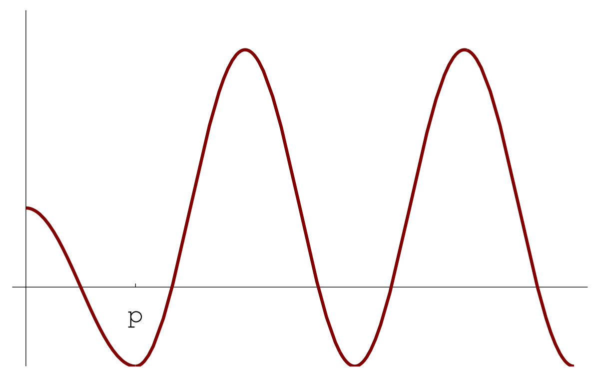

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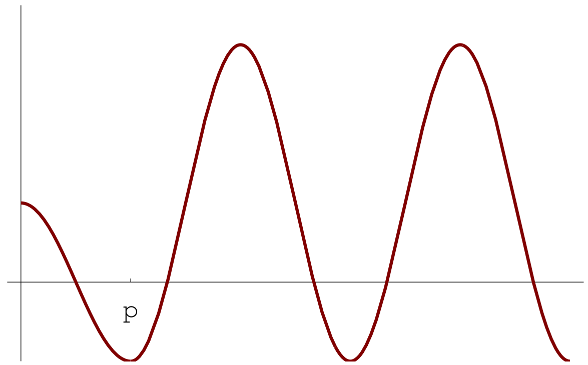

Remark $\frac{e^{-\pi s}}{s} \in \mathcal{L}(E_{pc})$, $\frac{1}{s^2+1} \in \mathcal{L}(E_c)$, $\therefore$ $\frac{e^{-\pi s}}{s(s^2+1)} \in \mathcal{L}(E_c)$

Example Solve $\begin{cases} y'' + 4y = g(t) \\ y(0) = 0, \quad y'(0) = 1, \end{cases}$ where $g(t) = \begin{cases} 1 & \text{if } \pi \leq t < 2\pi \\ 0 & \text{if } 0 \leq t < \pi \text{ or } t \geq 2\pi \end{cases}$

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$$= u_\pi(t) - u_{2\pi}(t)$$

In general,

$$h(t) = \begin{cases} h_1(t) & \text{if } 0 < t < a \\ h_2(t) & \text{if } a < t < b \\ h_3(t) & \text{if } t > b \end{cases}$$

Example Solve $\begin{cases} y'' + 4y = g(t) \\ y(0) = 0, \quad y'(0) = 1, \end{cases}$ where $g(t) = \begin{cases} 1 & \text{if } \pi \le t < 2\pi \\ 0 & \text{if } 0 \le t < \pi \text{ or } t \ge 2\pi \end{cases}$

Note that

$$g(t) = \begin{cases} 0 & \text{if } 0 \le t < \pi, \\ 1 & \text{if } \pi \le t < 2\pi, \\ 0 & \text{if } t \ge 2\pi \end{cases}$$

$$= u_\pi(t) - u_{2\pi}(t)$$

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$$= h_1(t) + u_a(t)\Big(h_2(t) - h_1(t)\Big) + u_b(t)\Big(h_3(t) - h_2(t)\Big)$$

Solution Take Laplace transform, denote $Y = \mathcal{L}(y)$

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\begin{aligned}
s^2Y(s) - sy(0) - y'(0) + 4Y(s) &= \mathcal{L}(u_\pi)(s) - \mathcal{L}(u_{2\pi})(s) \\
s^2Y(s) - 1 + 4Y(s) &=
\end{aligned}
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(s^2 + 4)Y(s) &= 1 + e^{-\pi s}\frac{1}{s} - e^{-2\pi s}\frac{1}{s}
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Take inverse Laplace transform $y(t) = \frac{1}{2}\sin 2t + u_\pi(t)f(t-\pi)$

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$$
f(t) = \mathcal{L}^{-1}\Big(\frac{1}{s(s^2+4)}\Big)(t)
$$

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$\vdots$ similar to last example, use partial fraction

Example Solve $\begin{cases} x'' + x = h(t) \\ x(0) = 1, \quad x'(0) = 1 \end{cases}$ where $h(t) = \begin{cases} 0 & \text{if } 0 \leq t < \frac{\pi}{2} \\ \sin t & \text{if } t \geq \frac{\pi}{2} \end{cases}$

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Solution

$$\begin{aligned} h(t) &= u_{\frac{\pi}{2}}(t)\sin t = u_{\frac{\pi}{2}}(t)\cos(\tfrac{\pi}{2} - t) \\ &= u_{\frac{\pi}{2}}(t)\cos(t - \tfrac{\pi}{2}) \end{aligned}$$

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Take Laplace transform, denote $X = \mathcal{L}(x)$

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$$\begin{aligned} s^2 X(s) - s x(0) - x'(0) + X(s) &= \mathcal{L}(h)(s) \\ s^2 X(s) - s - 1 + X(s) &= \end{aligned}$$

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Take inverse Laplace transform $x(t) =$

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Take inverse Laplace transform $x(t) \;=\; \cos t + \sin t + u_{\frac{\pi}{2}}(t) f(t - \frac{\pi}{2})$ where

Take inverse Laplace transform $x(t) = \cos t + \sin t + u_{\frac{\pi}{2}}(t) f(t - \frac{\pi}{2})$ where

$$f(t) = \mathcal{L}^{-1}\Big(\frac{s}{s^2+1} \cdot \frac{1}{s^2+1}\Big)(t)$$

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Therefore, $x(t) = \cos t + \sin t + u_{\frac{\pi}{2}}(t) \cdot \frac{1}{2}(t - \frac{\pi}{2}) \sin(t - \frac{\pi}{2})$

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&= \begin{cases} \cos t + \sin t & \text{if } t < \frac{\pi}{2} \\ & \end{cases}
\end{aligned}
$$

Take inverse Laplace transform $x(t) = \cos t + \sin t + u_{\frac{\pi}{2}}(t) f(t - \frac{\pi}{2})$ where

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&= \cos t + \sin t - u_{\frac{\pi}{2}}(t) \cdot \tfrac{1}{2}(t - \tfrac{\pi}{2}) \cos t \\
&= \begin{cases} \cos t + \sin t & \text{if } t < \frac{\pi}{2} \\ \cos t + \sin t - \frac{1}{2}(t - \frac{\pi}{2}) \cos t & \text{if } t \geq \frac{\pi}{2} \end{cases}
\end{aligned}
$$

Laplace transform method without Laplace transform

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Example Solve the IVP
$$\begin{cases} y'' + y = e^{-t^2}, \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$$

Express the answer in the form $y(t) = f(t) + \int_0^t g(t, s)\, ds$

Laplace transform method without Laplace transform

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Solution

$$\begin{aligned} s^2\mathcal{L}(y)(s) - sy(0) - y'(0) + \mathcal{L}(y)(s) &= \mathcal{L}(e^{-t^2})(s) \\ s^2\mathcal{L}(y)(s) - s + \mathcal{L}(y)(s) &= \mathcal{L}(e^{-t^2})(s) \\ (s^2+1)\mathcal{L}(y)(s) &= s + \mathcal{L}(e^{-t^2})(s) \end{aligned}$$

Laplace transform method without Laplace transform

Example Solve the IVP $\begin{cases} y'' + y = e^{-t^2}, \\ y(0) = 1, \quad y'(0) = 0. \end{cases}$

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\end{aligned}
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Take inverse Laplace transform $\quad y(t) = \cos t + \sin t * \mathrm{e}^{-t^2}$

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\end{aligned}$$

Take inverse Laplace transform

$$\begin{aligned}
y(t) &= \cos t + \sin t * e^{-t^2} \\
&= \cos t + \int_0^t \sin(t-\tau)e^{-\tau^2}\,d\tau
\end{aligned}$$