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$f'' : [0, \infty) \longrightarrow \mathbb{R}$ continuous

From the DE $f''(t) + f(t) = \sin 2t, \qquad t \geq 0$

Take Laplace transform $\mathcal{L}(f'')(s) + \mathcal{L}(f)(s) = \dfrac{2}{s^2 + 4}$ for large enough s

Example Solve the IVP $\begin{cases} x'' + x = \sin 2t, \\ x(0) = 0, \quad x'(0) = 1. \end{cases}$

Solution The solution $f : (-\infty, \infty) \longrightarrow \mathbb{R}$ is in the form

$$f(t) = A\cos t + B\sin t + C\cos 2t + D\sin 2t$$

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By Theorem 2, $\quad s^2\mathcal{L}(f)(s) - sf(0) - f'(0) + \mathcal{L}(f)(s) = \dfrac{2}{s^2+4}$

By initial condition

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Therefore, $\quad f(t) = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t \quad$ for all $t \in \mathbb{R}$.

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- System IVP *See lecture notes.*

Second order linear (const coeff) DE with piecewise continuous forcing functions

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In fact, $\nexists$ $\varphi : (0, \infty) \longrightarrow \mathbb{R}$ satisfying (1).

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Hence range of $f' = \{0, 1\}$, contradicts IVT for derived function.

Consider first order IVP

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where $g : [0, \infty) \longrightarrow \mathbb{R}$ is piecewise continuous.

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Remark The solution is
$$\varphi(t) = \mathrm{e}^{-bt}\left(\int_0^t \mathrm{e}^{bu} g(u)\,\mathrm{d}u + y_0\right)$$

Fundamental Theorem of Calculus

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$$\varphi'(t) \quad = \quad \mathrm{e}^{-bt} \cdot \mathrm{e}^{bt} g(t) + (-b)\mathrm{e}^{-bt} \cdot \left(\int_0^t \mathrm{e}^{bu} g(u)\,\mathrm{d}u + y_0 \right)$$

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Theorem A *Consider the first order IVP*

$$\begin{cases} y' + by = g(t), & t \geq 0, \\ y(0) = y_0, \end{cases}$$

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By Theorem A, φ is of exponential order.

Theorem B *Consider the second order IVP* $$\begin{cases} ay'' + by' + cy = g(t), & t \geq 0, \\ y(0) = y_0, \quad y'(0) = y_1 \end{cases}$$

where $g : [0, \infty) \longrightarrow \mathbb{R}$ *is piecewise continuous.*

Suppose g *is of exponential order, then the derivative of the solution to the IVP is also of exponential order.*

Proof Consider $b^2 - 4ac \geq 0$, with $a = 1$.

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Hence, $\varphi' = \beta\varphi + h$ is of exponential order.

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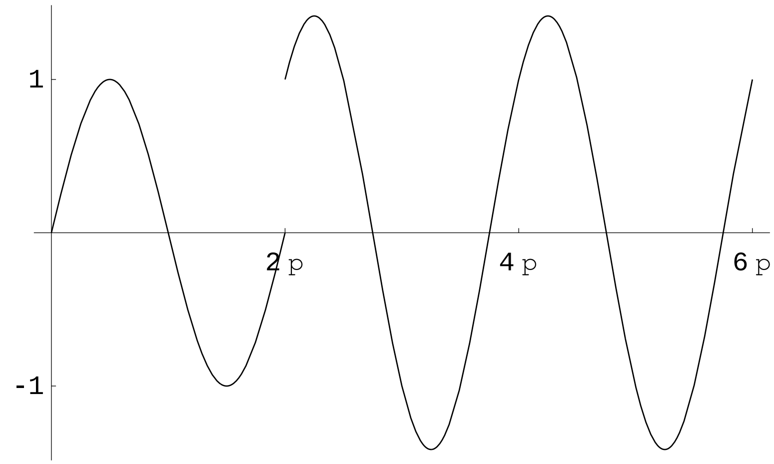

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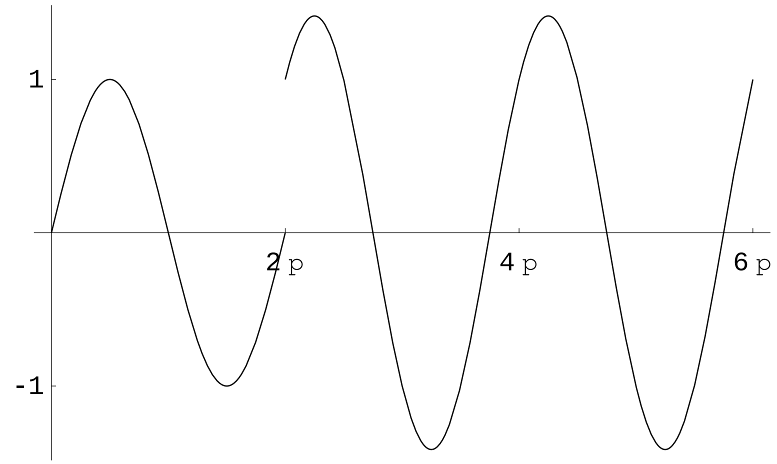

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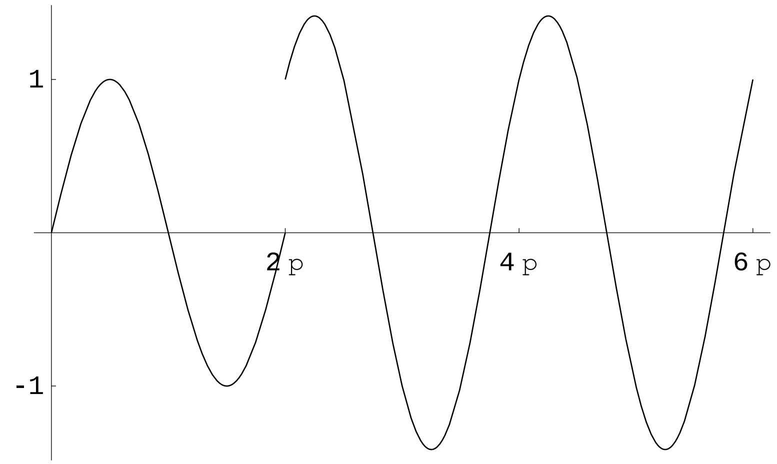

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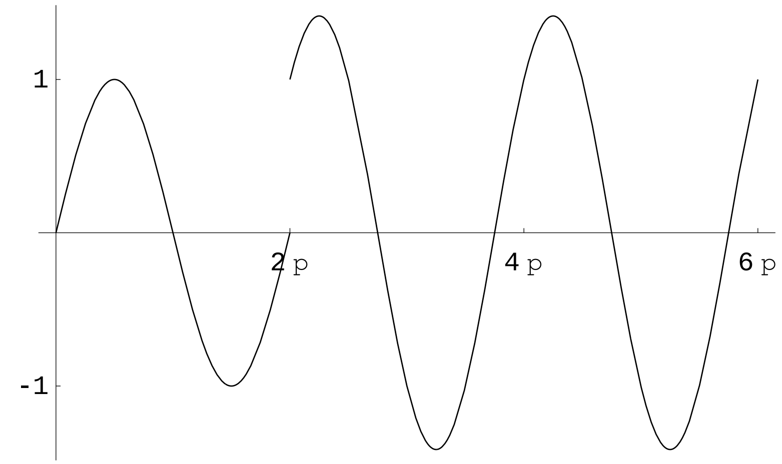

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Definition A *unit step function* is a function $u_c : [0, \infty) \longrightarrow \mathbb{R}$ in the form

$$u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t > c \end{cases}$$

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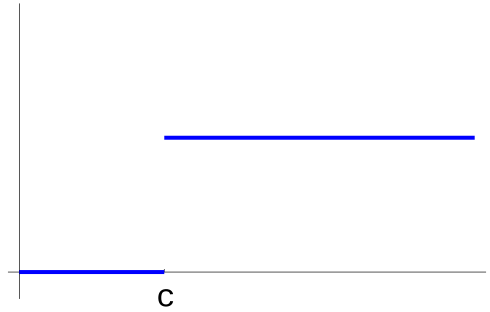

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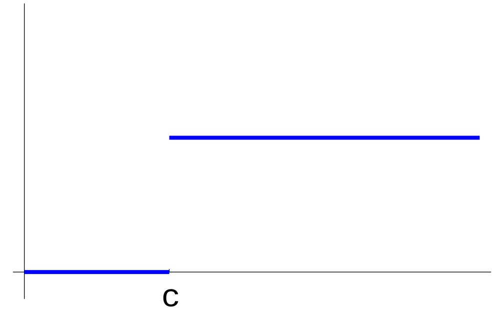

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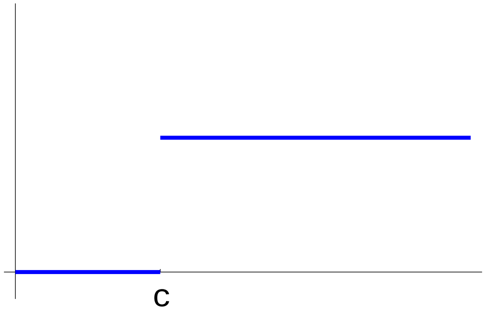

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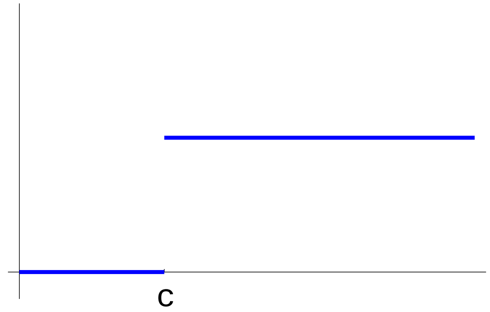

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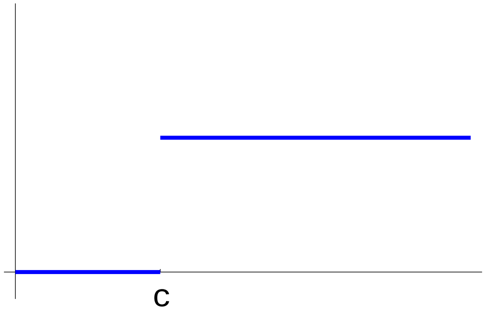

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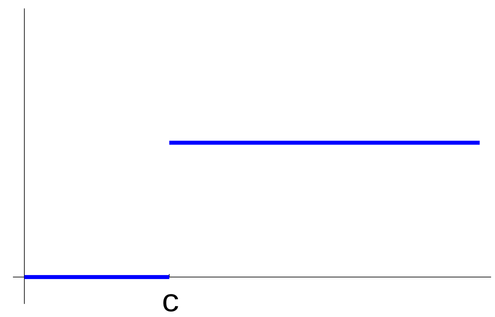

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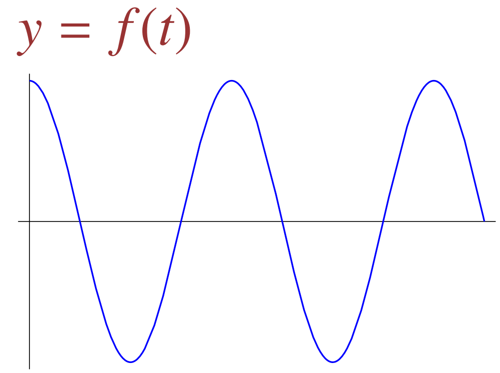

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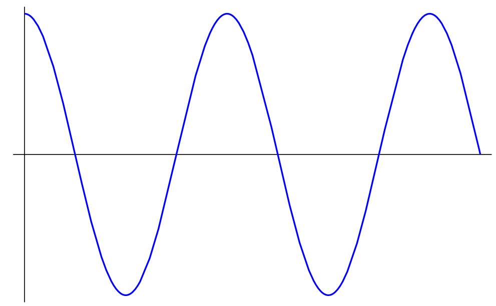

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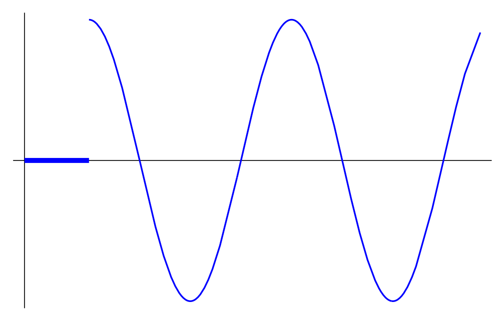

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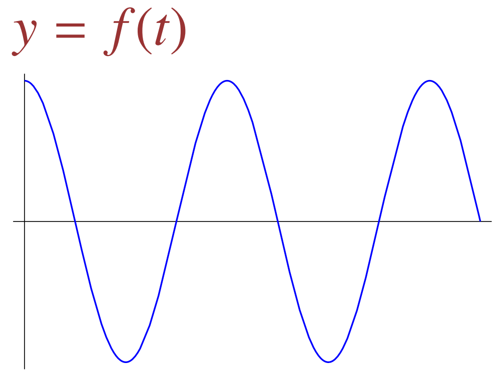

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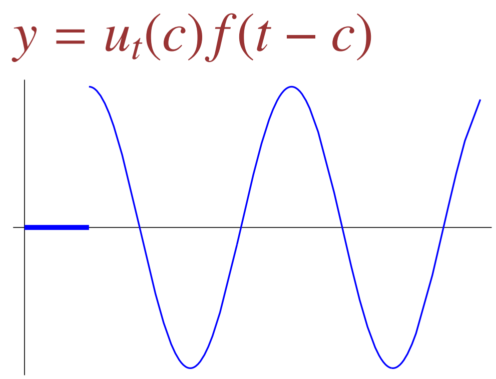

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