$$E_c = \Big\{ f : [0, \infty) \longrightarrow \mathbb{R} \text{ such that } f \text{ is continuous and of exponential order} \Big\}$$

- $\mathcal{L} : E_c \longrightarrow \mathcal{L}(E_c)$ is injective
- $\mathcal{L}^{-1} : \mathcal{L}(E_c) \longrightarrow E_c$
- $F = \dfrac{P(s)}{\left((s-\alpha)^2 + \beta^2\right)^n}$, where $n \geq 2$, $\deg(P) < 2n$, what is $\mathcal{L}^{-1}(F)$?

Convolution

Definition Let $f, g : [0, \infty) \longrightarrow \mathbb{R}$ be piecewise continuous functions. The *convolution* of f and g is the function $f * g : [0, \infty) \longrightarrow \mathbb{R}$ defined by

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Multiplication of Polynomial/Power Series

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_{n-i}\, b_i\right) x^n$$

Example Let $f(t) = \cos t$. Find $f * 1$.

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$$\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

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Euler Formula $\mathrm{e}^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta, \quad \theta \in \mathbb{R}$

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Example Let $g(t) = \sin t$. Find $g * g$.

Solution

$$\big(g * g\big)(t) \quad = \quad \int_0^t \sin(t - \tau) \sin \tau \, \mathrm{d}\tau$$

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&= \frac{1}{2}\int_0^t \Big(\cos(t-2\tau) - \cos t\Big)\,\mathrm{d}\tau \\
&= \frac{1}{2}\left[\frac{\sin(t-2\tau)}{-2} - \tau\cos t\right]_{\tau=0}^{\tau=t} \\
&= \frac{1}{2}\bigg(\Big(\frac{\sin(-t)}{-2} - t\cos t\Big) - \Big(\frac{\sin t}{-2} - 0\Big)\bigg)
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&= \frac{1}{2}\left[\frac{\sin(t-2\tau)}{-2} - \tau\cos t\right]_{\tau=0}^{\tau=t} \\
&= \frac{1}{2}\left(\Big(\frac{\sin(-t)}{-2} - t\cos t\Big) - \Big(\frac{\sin t}{-2} - 0\Big)\right) \\
&= \frac{1}{2}(\sin t - t\cos t)
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Properties

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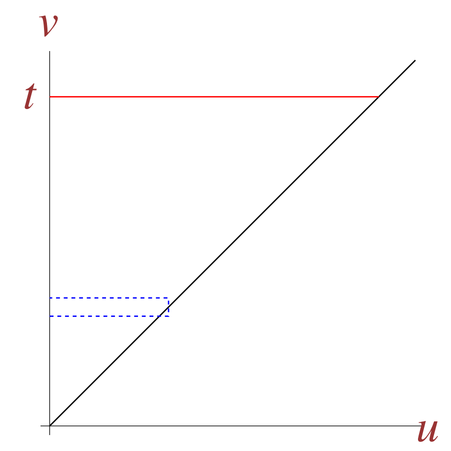

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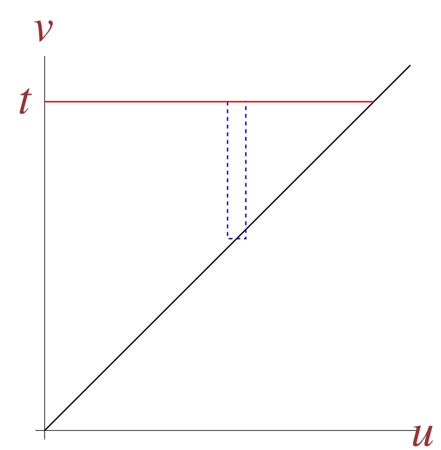

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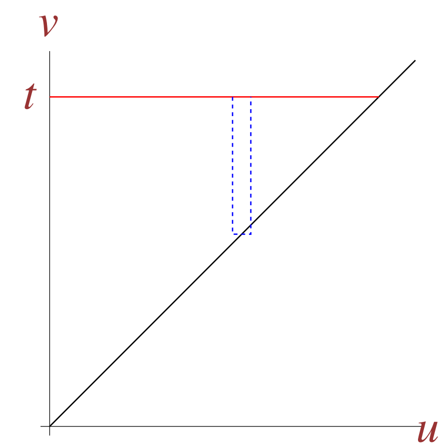
&= \int_0^t \left(\int_u^t f(t-v)g(v-u)h(u)\,\mathrm{d}v\right)\mathrm{d}u \qquad \text{Fubini's Theorem}
\end{aligned}
$$



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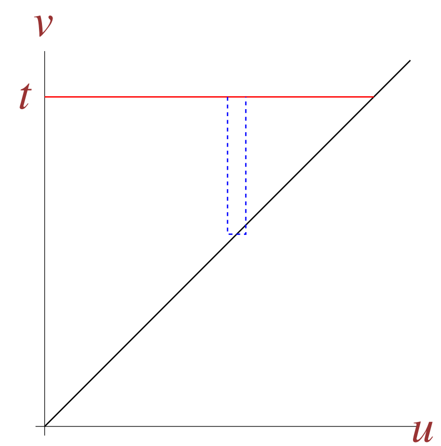
Fubini's Theorem

$$= \int_0^t \left(\int_0^{t-u} f(t-u-w)g(w)\,\mathrm{d}w\right)h(u)\,\mathrm{d}u$$

let $w = v - u$



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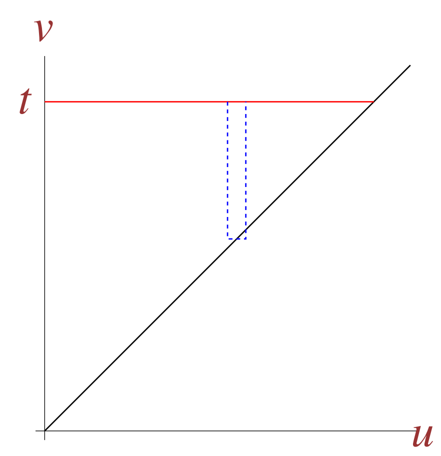

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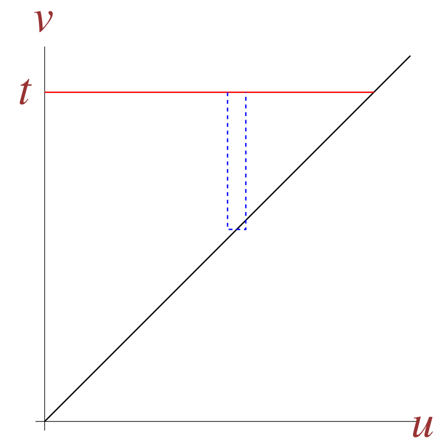
&= \int_0^t \left(\int_u^t f(t-v)g(v-u)h(u)\,\mathrm{d}v\right)\mathrm{d}u \qquad \text{Fubini's Theorem} \\
&= \int_0^t \left(\int_0^{t-u} f(t-u-w)g(w)\,\mathrm{d}w\right)h(u)\,\mathrm{d}u \qquad \text{let } w = v-u \\
&= \int_0^t \big(f * g\big)(t-u)\cdot h(u)\,\mathrm{d}u \\
&= \big((f * g) * h\big)(t)
\end{aligned}$$



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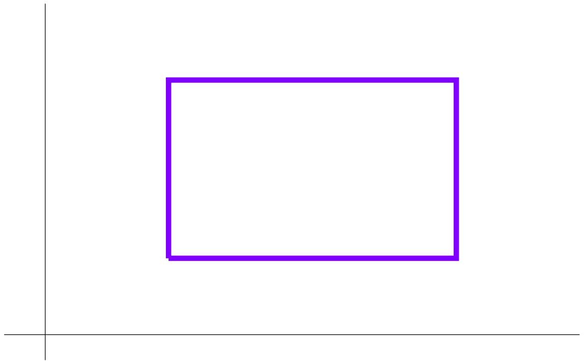

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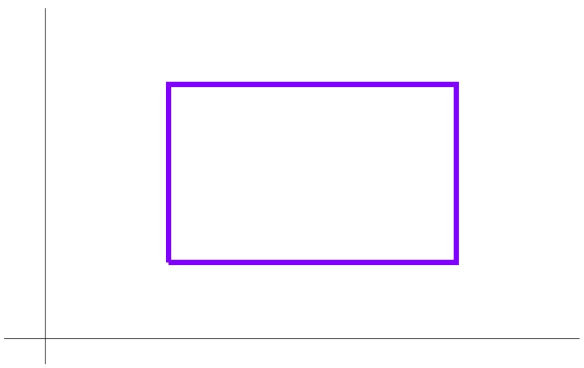

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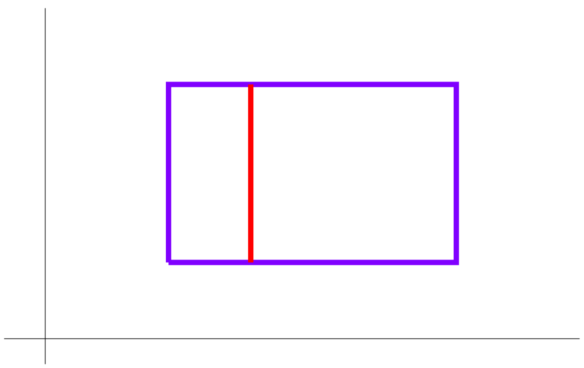

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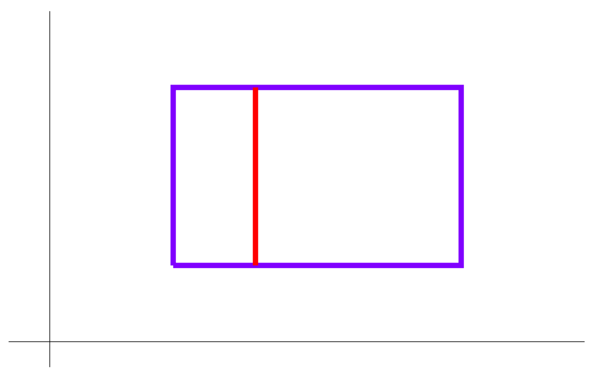

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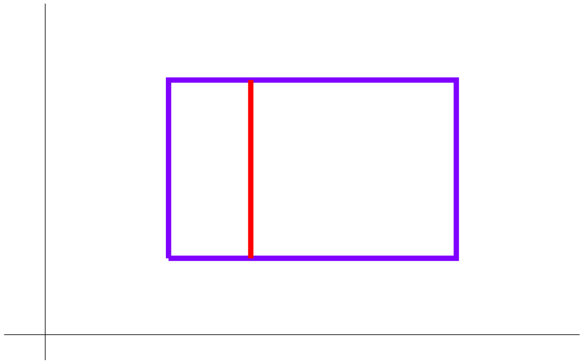

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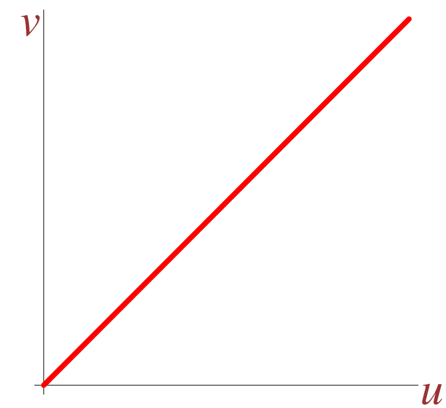

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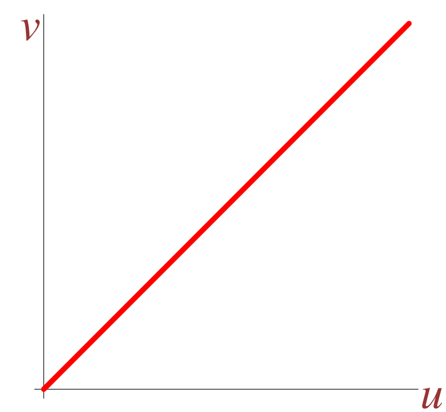

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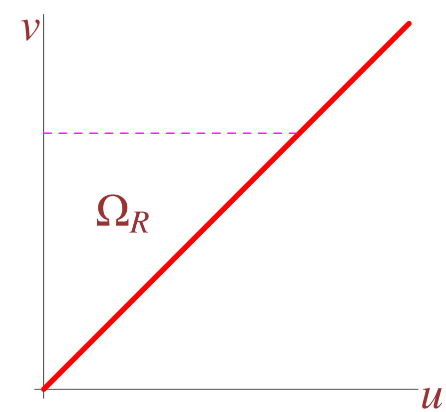

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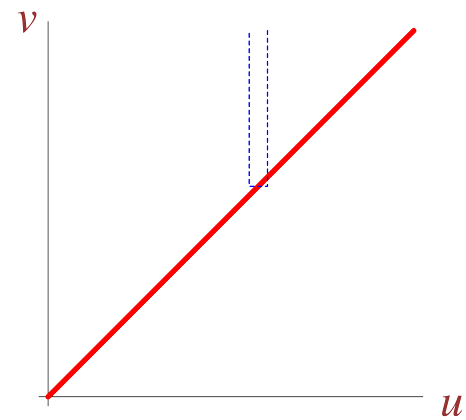

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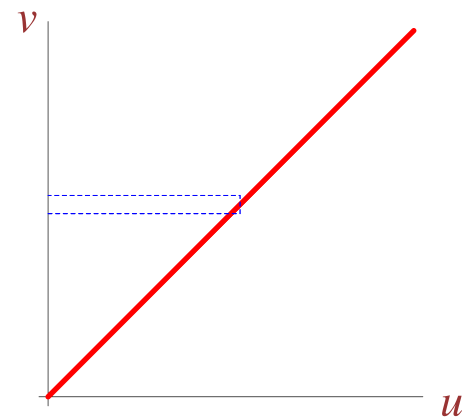

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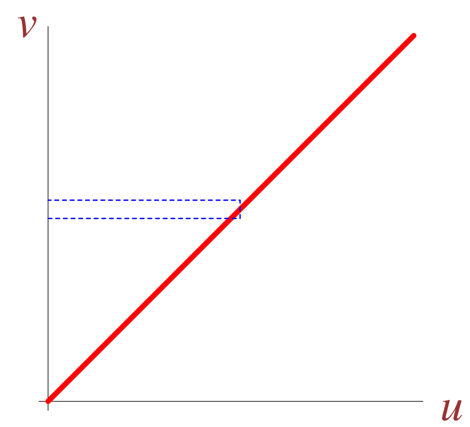

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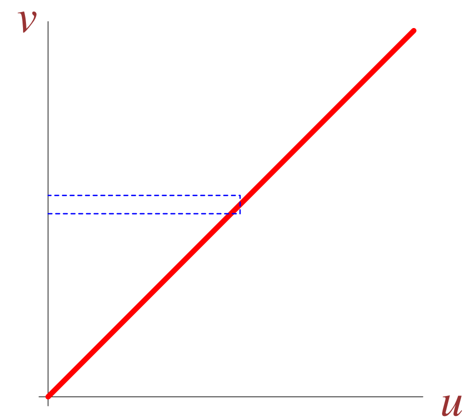

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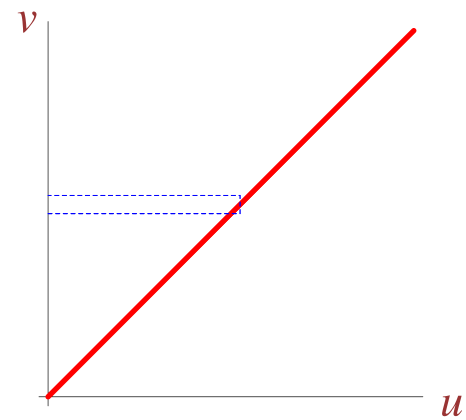

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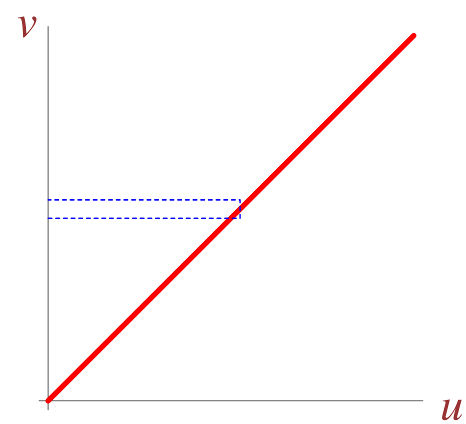

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By Convolution Theorem, $F \cdot G = \mathcal{L}(f * g) \in \mathcal{L}(E_c)$

Convolution Theorem (Corollary) *Let $f, g : [0, \infty) \longrightarrow \mathbb{R}$ be functions that are piecewise continuous and of exponential order. Then*

$$\mathcal{L}^{-1}\Big(F \cdot G\Big) = f * g$$

where $F = \mathcal{L}(f)$ and $G = \mathcal{L}(g)$.

Note f, g piecewise continuous on $[0, \infty) \Longrightarrow f * g$ continuous on $[0, \infty)$

f, g of exponential order on $[0, \infty) \Longrightarrow f * g$ of exponential order on $[0, \infty)$

Therefore, $f * g \in E_c$

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Take inverse $\mathcal{L}^{-1} : \mathcal{L}(E_c) \longrightarrow E_c$

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Method 1 $F(s) = \dfrac{As+B}{s^2+1} + \dfrac{Cs+D}{s^2+9}$ Partial Fraction

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$$\mathcal{L}^{-1}(F)(t) \quad = \quad \frac{1}{3} \cdot \cos t * \sin 3t$$

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Remark Can find $\mathcal{L}^{-1}\left(\dfrac{P}{Q}\right)$, where $\deg P(s) < \deg Q(s)$

Theorem (Partial Fractions Decomposition) Let

$$R(x) = \frac{P(x)}{(x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\cdots(x-\alpha_n)^{m_n}}$$

where $\deg P(x) < (m_1 + \cdots + m_n)$ and $\alpha_1, \ldots \alpha_n \in \mathbb{R}$ are distinct.

Then there exist unique $c_{ij} \in \mathbb{R}$ $(i = 1, \ldots n,\ \ j = 1, \ldots, m_i)$ such that

$$R(x) = \sum_{i=1}^{n} \left(\frac{c_{i1}}{x-\alpha_i} + \frac{c_{i2}}{(x-\alpha_i)^2} + \cdots + \frac{c_{im_i}}{(x-\alpha_i)^{m_i}} \right)$$

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Remark Similar result holds for $R(x) = \dfrac{P(x)}{\prod_{i=1}^{n}(x-\alpha_i)^{m_i} \times \prod_{i=1}^{n'} \left[(x-\lambda_i)^2 + \mu_i^2)\right]^{m_i'}}$

Formula for $\mathcal{L}^{-1}(F)$

If $F \in \mathcal{L}(E_c)$, then

- $$\mathcal{L}^{-1}(F)(t) = \frac{1}{2\pi \mathrm{i}} \int_{\sigma - \mathrm{i}\infty}^{\sigma + \mathrm{i}\infty} \mathrm{e}^{st} F(s)\,\mathrm{d}s$$ where $\sigma > s_c$

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- $$\mathcal{L}^{-1}(F)(t) = \lim_{n \to \infty} \frac{(-1)^n}{n!} \left(\frac{n}{t}\right)^{k+1} F^{(n)}\left(\frac{n}{t}\right)$$