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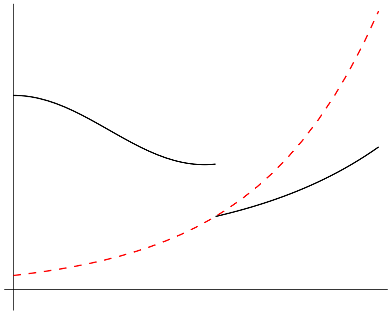

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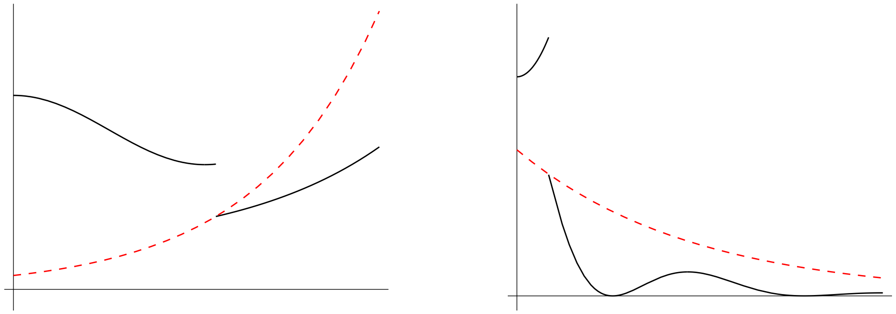

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Therefore, $\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$ exists for all $s > a$.

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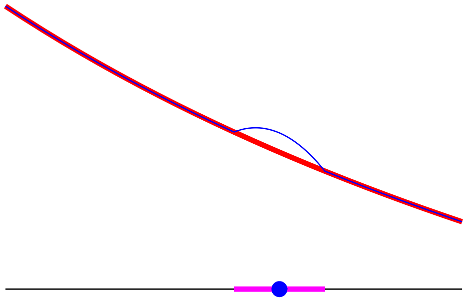

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- $\mathcal{L} : E_{pc} \longrightarrow \mathcal{L}(E_{pc})$ is not injective
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Remark In considering $ay'' + by' + cy = g(t)$,

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To find p and q, use (1) or (2)

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Remark Can find inverse Laplace transform for $F(s) = \dfrac{As + B}{as^2 + bs + c}$ $\dfrac{\text{linear}}{\text{quadratic}}$

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What is the inverse Laplace transform of $F(s)$?

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??? $$\mathcal{L}^{-1}(F)(t) = \left(\frac{1}{\mu}\cdot \mathrm{e}^{\lambda t}\sin\mu t\right)^2$$

??? $$\mathcal{L}^{-1}(G\cdot H) = \mathcal{L}^{-1}(G)\cdot\mathcal{L}^{-1}(H)$$

- $$F(s) = \frac{1}{\left((s-\lambda)^2+\mu^2\right)^2}$$

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