Chapter 4: Laplace Transform

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If g is piecewise continuous on $[0, \infty)$, modify meaning of solution.

Laplace Transform Method Consider

$$\begin{cases} ay'' + by' + cy = g(t), & t \geq 0, \\ y(0) = y_0, \quad y'(0) = y_1 \end{cases}$$

where g is a continuous/piecewise continuous function on $[0, \infty)$.

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(4) Solve to get $\mathcal{L}(y) =$ a function

(5) Take inverse transform $y = \mathcal{L}^{-1}$(the function)

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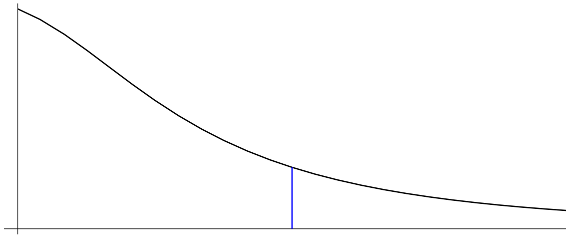

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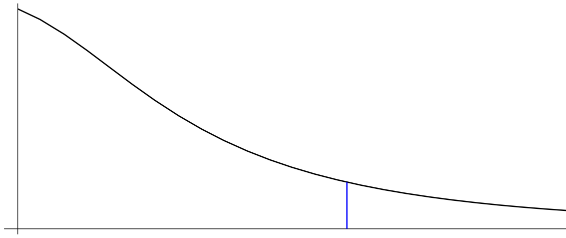

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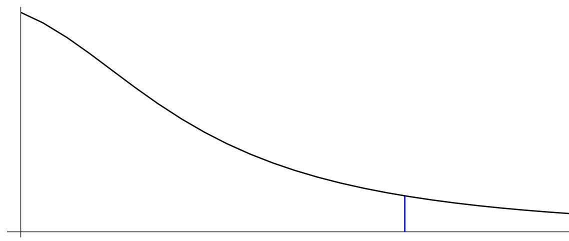

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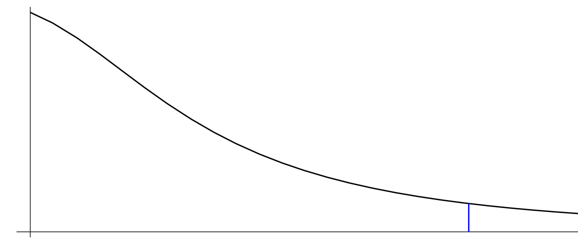

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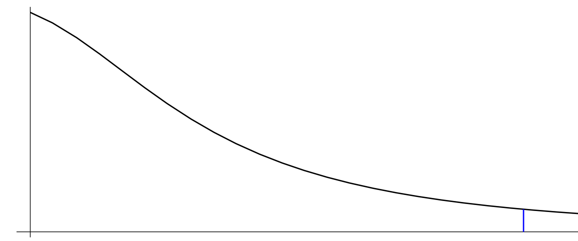

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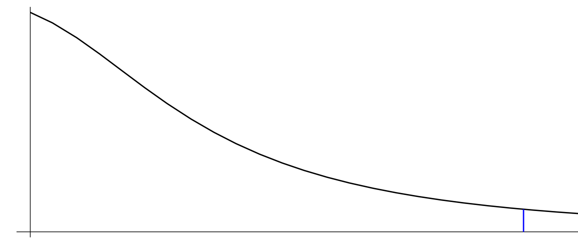

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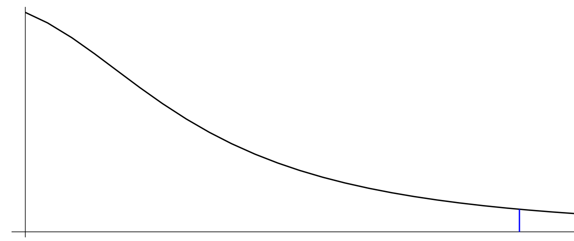

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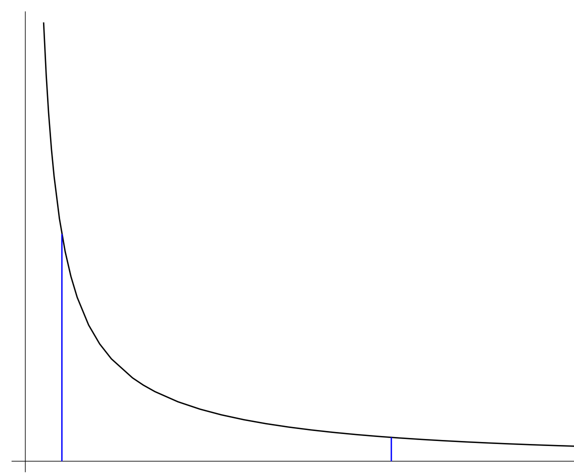

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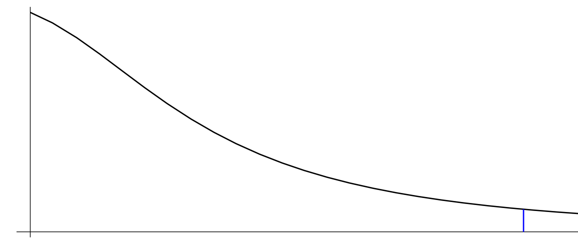

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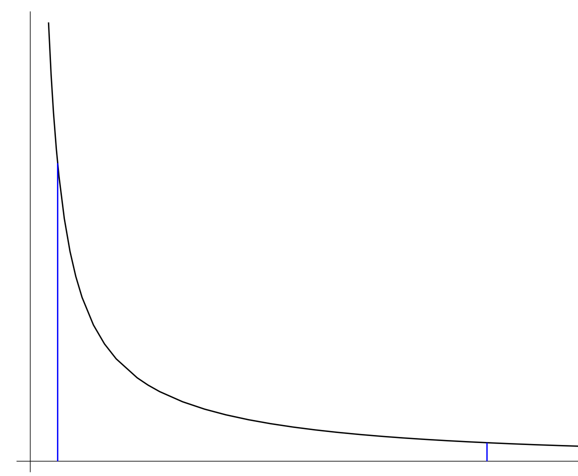

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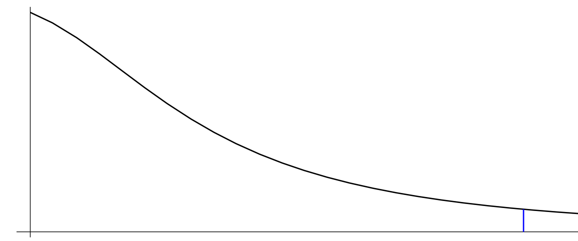

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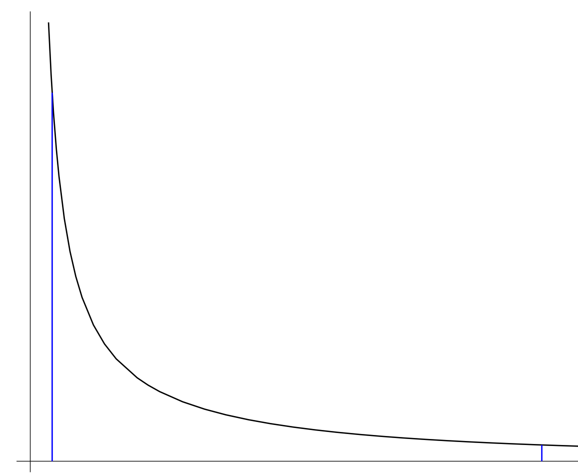

Example

$f(t)$	$\mathcal{L}(f)(s)$
e^{at}	$\frac{1}{s-a}, \quad s > a$
$t^n, \quad n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
$\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
$\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$

Proof

(1) $f(t) = \mathrm{e}^{at}$ $$\mathcal{L}(f)(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{at}\,\mathrm{d}t$$

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$$= \lim_{R\to\infty}\int_0^R \mathrm{e}^{(a-s)t}\,\mathrm{d}t$$

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$$\begin{aligned} \mathcal{L}(f)(s) &= \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{at}\,\mathrm{d}t \\ &= \lim_{R\to\infty}\int_0^R \mathrm{e}^{(a-s)t}\,\mathrm{d}t \\ &= \lim_{R\to\infty}\left[\frac{\mathrm{e}^{(a-s)t}}{a-s}\right]_0^R \end{aligned}$$

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&= \lim_{R\to\infty}\left[\frac{\mathrm{e}^{(a-s)t}}{a-s}\right]_0^R \\
&= \lim_{R\to\infty}\left(\frac{\mathrm{e}^{(a-s)R}}{a-s} - \frac{1}{a-s}\right)
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Remark $\mathcal{L}(1)(s) = \dfrac{1}{s}$ for $s > 0$.

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&= \frac{n}{s} \mathcal{L}(t^{n-1})(s)
\end{aligned}
$$

(2) $f(t) = t^n$ **Use induction**

- $n = 0$ Done.
- Suppose $\mathcal{L}\left(t^{n-1}\right)(s) = \dfrac{(n-1)!}{s^n}$ for $s > 0$

$$
\begin{aligned}
\mathcal{L}\left(t^n\right)(s) &= \int_0^\infty e^{-st} t^n \, dt = \lim_{R\to\infty} \int_0^R t^n \, d\frac{e^{-st}}{-s} \\
&= \lim_{R\to\infty} \left(\left[\frac{t^n e^{-st}}{-s} \right]_0^R - \int_0^R \frac{e^{-st}}{-s} n t^{n-1} \, dt \right) \\
&= \lim_{R\to\infty} \frac{R^n e^{-sR}}{-s} - 0 + \lim_{R\to\infty} \int_0^R \frac{n}{s} e^{-st} t^{n-1} \, dt \\
&= 0 + \frac{n}{s} \int_0^\infty e^{-st} t^{n-1} \, dt \qquad \because \lim_{R\to\infty} \frac{R^n}{e^{sR}} = 0, \text{ for } s > 0 \\
&= \frac{n}{s} \mathcal{L}(t^{n-1})(s) \\
&= \frac{n}{s} \cdot \frac{(n-1)!}{s^n} = \frac{n!}{s^{n+1}} \qquad \text{for } s > 0
\end{aligned}
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$$\begin{aligned}
\mathcal{L}(f)(s) &= \int_0^\infty e^{-st}\cos at\,dt \qquad \text{integration by parts twice} \\
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&= \frac{s^2}{s^2+a^2}\left(\lim_{R\to\infty}\left(\frac{e^{-sR}\cos aR}{-s}+\frac{ae^{-sR}\sin aR}{s^2}\right)-\frac{1}{-s}\right)
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$$\mathrm{codom}\,(\mathcal{L}) = \left\{ \text{all functions } F \text{ such that } F(s) \text{ is defined if } s \text{ is large enough} \right\}$$

Example

- $$\mathcal{L}(\cosh at)(s) \quad = \quad \mathcal{L}\left(\frac{\mathrm{e}^{at} + \mathrm{e}^{-at}}{2}\right)(s)$$

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Definition The Laplace transform of a function $f : [0, \infty) \to \mathbb{C}$, denoted by $\mathcal{L}(f)$, is defined *formally* by

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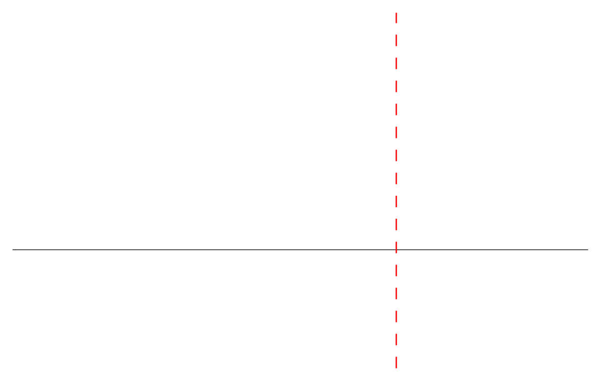

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Re $s > 0$

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Theorem *Suppose* $\mathcal{L}(f)(s)$ *exists for* $s > a$. *Then for any* $c \in \mathbb{R}$, $\mathcal{L}\left(e^{ct} f(t)\right)(s)$ *exists for* $s > a + c$ *and*

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Proof For $s > a + c$,

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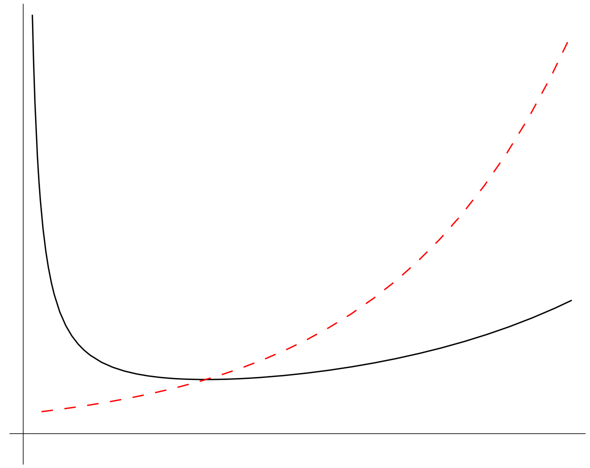

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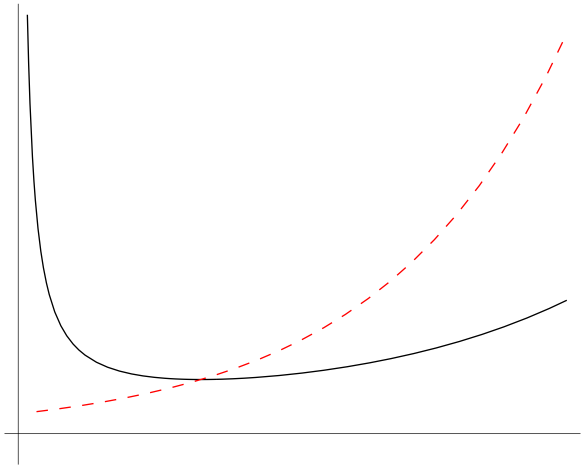

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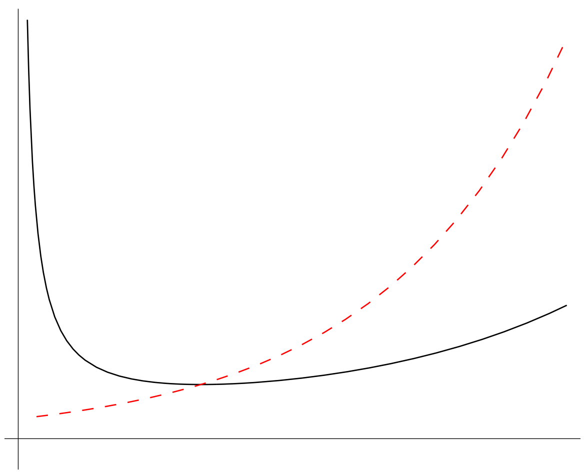

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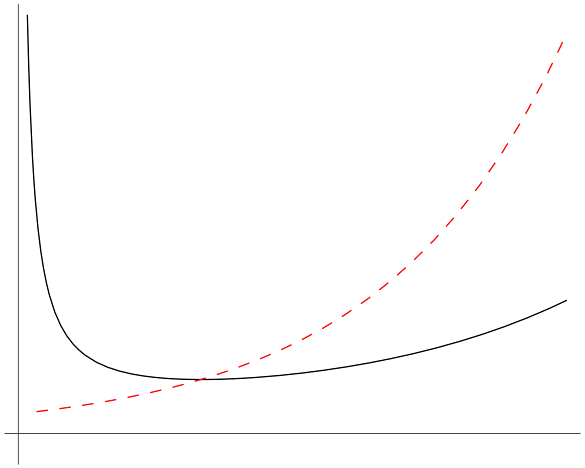

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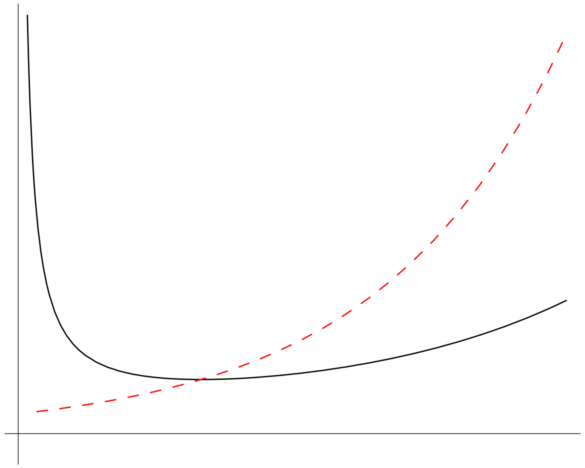

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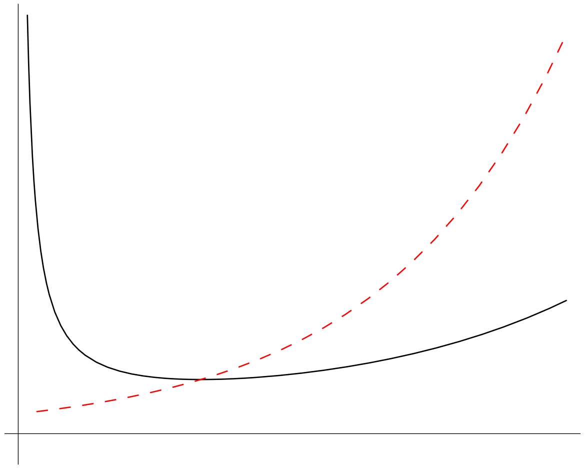

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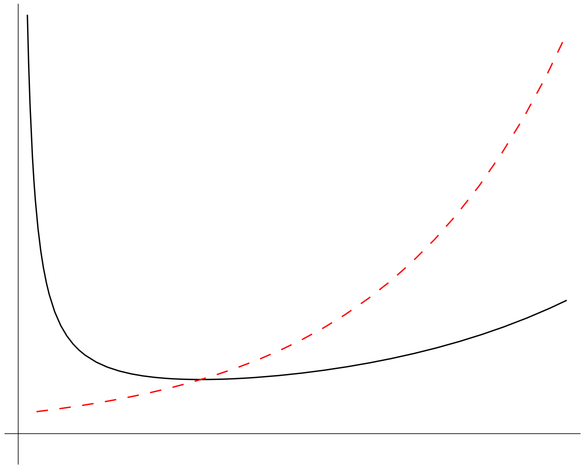

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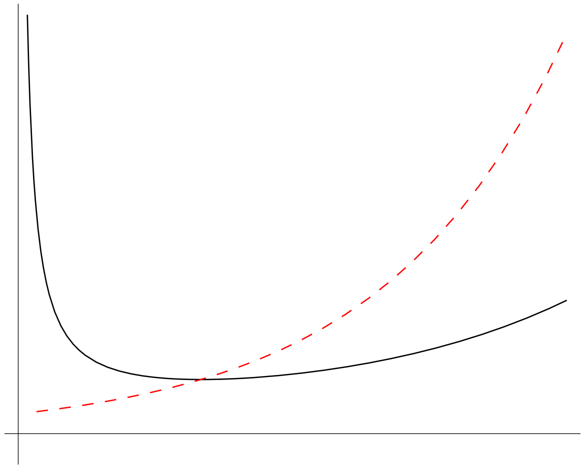

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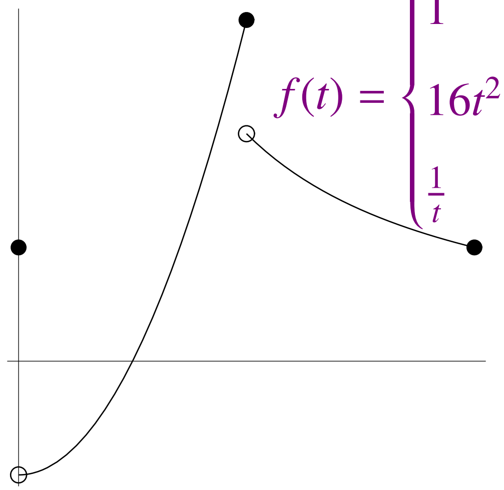

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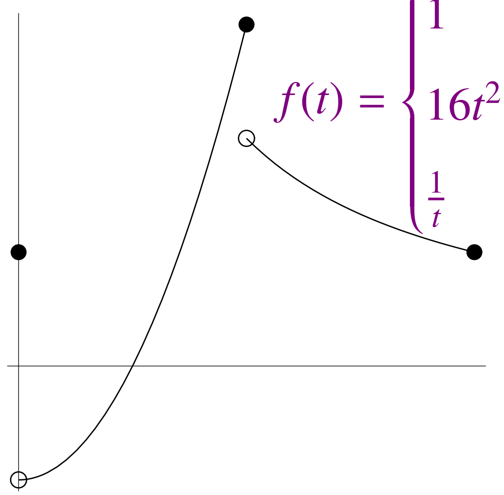

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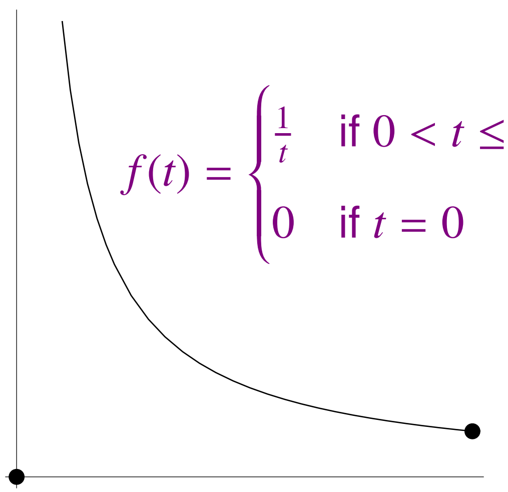

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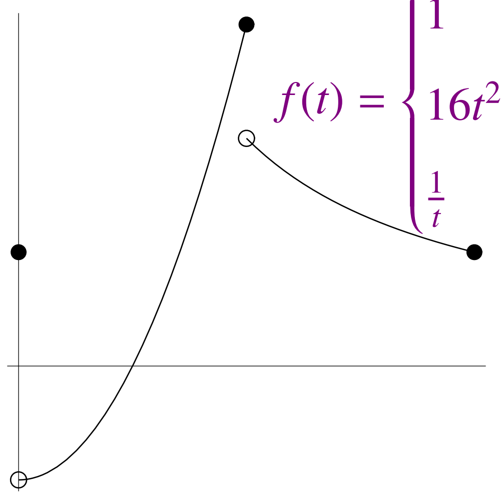

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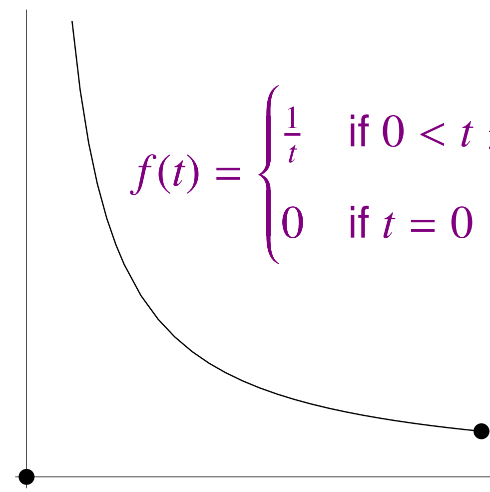

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Compare For series: *absolute convergence $\Longrightarrow$ convergence*