Non-homogeneous linear systems

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where $\mathbf{A}(t) = \left[a_{ij}(t)\right]_{n\times n}$, $\mathbf{g}(t) = \begin{bmatrix} g_1(t) & \cdots & g_n(t) \end{bmatrix}^T$ *with* $a_{ij}, g_i \in C(a, b)$ *for all* i, j.

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Theorem *The general solution to the non-homogeneous DE*

$$y'' + p(t)y' + q(t)y = g(t)$$

is given by $y(t) = y_p(t) + y_h(t)$

Undetermined coefficients

Consider 1st order non-homogeneous linear systems with *constant coefficients*

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- Write down the form of a particular solution $\mathbf{x}_p$.
- Find the coefficients (which are vectors) by substitution.

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Proof See appendix.

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In $\mathbf{x}_p(t) = \Big(\mathbf{v}_{k+\ell}t^{k+\ell} + \cdots + \mathbf{v}_1 t + \mathbf{v}_0\Big)e^{\alpha t}$, are the coefficients unique ?

Illustration Consider $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{p}_k(t)\mathrm{e}^{\alpha t}$

If α is an eigenvalue of $\mathbf{A}$ with $A.M. = 3$ and $G.M. = 1$

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For any const c, $\left(\mathbf{v}_{k+3}t^{k+3} + \cdots + \mathbf{v}_1 t + \mathbf{v}_0\right) \mathrm{e}^{\alpha t} + c \cdot \left(\mathbf{v}\frac{t^2}{2} + \mathbf{u}t + \mathbf{w}\right)\mathrm{e}^{\alpha t}$ is also soln

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For any const c, $\left(\mathbf{v}_{k+3}t^{k+3} + \cdots + \mathbf{v}_1 t + \mathbf{v}_0\right) e^{\alpha t} + c \cdot \left(\mathbf{v}\frac{t^2}{2} + \mathbf{u}t + \mathbf{w}\right)e^{\alpha t}$ is also soln

$\therefore \quad \mathbf{v}_0, \mathbf{v}_1$ and $\mathbf{v}_2$ are not unique.

Example Solve the non-homogeneous system

$$\mathbf{x}'(t) = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -8t \\ 2t + 3 \end{bmatrix}$$

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Solve Eq. (1) and (3) $\mathbf{a} = \begin{bmatrix} \frac{2}{5} \\ \frac{16}{5} \end{bmatrix}$,

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General soln to system $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} \frac{2}{5} \\ \frac{16}{5} \end{bmatrix} t + \begin{bmatrix} \frac{2}{25} \\ \frac{1}{25} \end{bmatrix}$

Example Find a particular solution to the system

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  b_1, b_2 not unique

Sub into sys

$$\left[a_1 e^{-2t} - 2a_1 t e^{-2t} - 2b_1 e^{-2t} \right.$$

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Compare $\begin{cases} -2a_1 t e^{-2t} + (a_1 - 2b_1) e^{-2t} \quad = \quad (4a_1 + 2a_2) t e^{-2t} + \\ \end{cases}$

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Compare coeff
$$\begin{cases} -2a_1 = 4a_1 + 2a_2 \quad \text{1st entry, } te^{-2t} \text{ term} \quad (1) \end{cases}$$

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$-2a_2 = 3a_1 - a_2$ 2nd entry, te^{-2t} term (3)

Sub into sys
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Solve (1) and (3) $\mathbf{a} = \begin{bmatrix} \frac{-2}{7} \\ \frac{6}{7} \end{bmatrix}$,

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Put $b_1 = 0$, say, obtain a particular soln

Exercise For each of the following case/sub-cases, write down the value of ℓ *(length of longest chain of generalized eigenvectors)*.

	Value of ℓ
Case "simple"	
Subcase $(2, 1)$	
Subcase $(2, 2)$	
Subcase $(3, 1)$	
Subcase $(3, 2)$	
Subcase $(3, 3)$	

Variation of parameters

Consider the following non-homogeneous system

$$\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{g}(t), \qquad a < t < b$$

where $\mathbf{A}(t) = \left[a_{ij}(t)\right]$, $\mathbf{g}(t) = \left[g_1(t) \quad \cdots \quad g_n(t)\right]^T$, $a_{ij}, g_i \in C(a, b)$ for all i, j.

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  $$\mathbf{x}_h(t) = \mathbf{\Phi}(t)\mathbf{c}$$

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- To find general solution to non-homogeneous system, try

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}(t)$$

  where $\mathbf{c}(t) = \begin{bmatrix} c_1(t) & \dots & c_n(t) \end{bmatrix}^T$, $c_i(t)$ to be determined

- **Product Rule** Suppose $\mathbf{F}(t) = \left[f_{ij}(t)\right]_{m\times n}$ and $\mathbf{G}(t) = \left[g_{ij}(t)\right]_{n\times p}$ where f_{ij}, g_{jk} differentiable. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(t)\mathbf{G}(t) =$$

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&= \mathbf{A}(t)\left[\mathbf{x}^{(1)}(t) \quad \cdots \quad \mathbf{x}^{(n)}(t)\right] \\
&= \mathbf{A}(t)\boldsymbol{\Phi}(t) \qquad\qquad (*)
\end{aligned}$$

Put $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{c}(t)$ into system $\mathbf{x}'(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{g}(t)$

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\mathbf{\Phi}'(t)\mathbf{c}(t) + \mathbf{\Phi}(t)\mathbf{c}'(t) &= \mathbf{A}(t)\mathbf{\Phi}(t)\mathbf{c}(t) + \mathbf{g}(t) && \text{Product Rule} \\
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\mathbf{\Phi}(t)\mathbf{c}'(t) &= \mathbf{g}(t) && \\
\mathbf{c}'(t) &= \mathbf{\Phi}(t)^{-1}\mathbf{g}(t) && \mathbf{\Phi}(t) \text{ non-singular}
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Example Solve the system $\mathbf{x}'(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}, \quad t > 0.$

Solution

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  Sub into system

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1'(t)$$

Example Solve the system $\mathbf{x}'(t) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}, \quad t > 0.$

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$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1'(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} \left(c_2'(t) e^{-5t} - 5c_2(t) e^{-5t} \right)$$

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To find $c_1'(t)$ and $c_2'(t)$

$$\begin{bmatrix}1\\2\end{bmatrix}c_1'(t)+\begin{bmatrix}-2\\1\end{bmatrix}c_2'(t)e^{-5t} = \begin{bmatrix}\frac{1}{t}\\ \frac{2}{t}+4\end{bmatrix}$$

To find $c_1'(t)$ and $c_2'(t)$

- System of linear equation

- Matrix equation

$$\begin{bmatrix}1\\2\end{bmatrix}c_1'(t) + \begin{bmatrix}-2\\1\end{bmatrix}c_2'(t)e^{-5t} \quad = \quad \begin{bmatrix}\frac{1}{t}\\ \frac{2}{t}+4\end{bmatrix}$$

To find $c_1'(t)$ and $c_2'(t)$

- System of linear equation $\left\{ \begin{array}{l} c_1'(t) - 2c_2'(t)e^{-5t} \quad = \quad \frac{1}{t} \\ \\ \end{array} \right.$

- Matrix equation

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1'(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} c_2'(t)\mathrm{e}^{-5t} = \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

To find $c_1'(t)$ and $c_2'(t)$

- System of linear equation
$$\begin{cases} c_1'(t) - 2c_2'(t)\mathrm{e}^{-5t} = \frac{1}{t} \\ 2c_1'(t) + c_2'(t)\mathrm{e}^{-5t} = \frac{2}{t} + 4 \end{cases}$$

- Matrix equation

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1'(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} c_2'(t) e^{-5t} = \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

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  - By elimination

- Matrix equation

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} c_1'(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} c_2'(t)e^{-5t} = \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

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- System of linear equation $\begin{cases} c_1'(t) - 2c_2'(t)e^{-5t} = \frac{1}{t} \\ 2c_1'(t) + c_2'(t)e^{-5t} = \frac{2}{t} + 4 \end{cases}$
  - By elimination
  - Apply Cramer's Rule   Det of coefficient matrix $\neq 0$
- Matrix equation

$$\begin{bmatrix}1\\2\end{bmatrix}c_1'(t)+\begin{bmatrix}-2\\1\end{bmatrix}c_2'(t)e^{-5t} = \begin{bmatrix}\frac{1}{t}\\ \frac{2}{t}+4\end{bmatrix}$$

To find $c_1'(t)$ and $c_2'(t)$

- System of linear equation $\begin{cases} c_1'(t)-2c_2'(t)e^{-5t} = \frac{1}{t} \\ 2c_1'(t)+c_2'(t)e^{-5t} = \frac{2}{t}+4 \end{cases}$
  - By elimination
  - Apply Cramer's Rule   Det of coefficient matrix $\neq 0$
- Matrix equation

$$\begin{bmatrix}1 & -2e^{-5t}\\2 & e^{-5t}\end{bmatrix}\begin{bmatrix}c_1'(t)\\c_2'(t)\end{bmatrix} = \begin{bmatrix}\frac{1}{t}\\ \frac{2}{t}+4\end{bmatrix}$$

$$\begin{bmatrix}1\\2\end{bmatrix} c_1'(t) + \begin{bmatrix}-2\\1\end{bmatrix} c_2'(t)e^{-5t} = \begin{bmatrix}\frac{1}{t}\\ \frac{2}{t}+4\end{bmatrix}$$

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$$\begin{bmatrix}c_1'(t)\\c_2'(t)\end{bmatrix} = \begin{bmatrix}1 & -2e^{-5t}\\2 & e^{-5t}\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{t}\\ \frac{2}{t}+4\end{bmatrix}$$

$$\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

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$$= \frac{1}{5\,e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

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$$= \frac{1}{5\,e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{t} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}$$

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Therefore $$c_1(t) = \ln t + \frac{8}{5}t + A, \qquad t > 0$$

$$\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2\mathrm{e}^{-5t} \\ 2 & \mathrm{e}^{-5t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

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Therefore

$$c_1(t) = \ln t + \frac{8}{5}t + A, \qquad t > 0$$

$$c_2(t) = \frac{4}{25}\mathrm{e}^{5t} + B$$

$$\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

$$= \frac{1}{5\,e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{t} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}$$

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General solution $$\mathbf{x}(t) = \left(\ln t + \frac{8}{5}t + A\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(\frac{4}{25}e^{5t} + B\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t}$$

$$\begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & -2e^{-5t} \\ 2 & e^{-5t} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix}$$

$$= \frac{1}{5\,e^{-5t}} \begin{bmatrix} e^{-5t} & 2e^{-5t} \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{t} \\ \frac{2}{t} + 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{t} + \frac{8}{5} \\ \frac{4}{5}e^{5t} \end{bmatrix}$$

Therefore $c_1(t) = \ln t + \frac{8}{5}t + A, \qquad t > 0$

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General solution $\mathbf{x}(t) = \left(\ln t + \frac{8}{5}t + A\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \left(\frac{4}{25}e^{5t} + B\right)\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t}$

$$= \left(\ln t + \frac{8}{5}t\right)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \frac{4}{25}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + A\begin{bmatrix} 1 \\ 2 \end{bmatrix} + B\begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-5t}$$

Variation of Parameters for n-th order DE

Illustration $n = 3$

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$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = -r(t)x_1 - q(t)x_2 - p(t)x_3 + g(t) \end{cases}$$

That is

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -r(t) & -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ g(t) \end{bmatrix} \qquad (**)$$

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