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Subcase (2,2) Two linearly independent eigenvectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$

Subcase (2,1) Only one linearly independent eigenvector $\mathbf{v}$

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Two linearly independent solutions to system

$$\mathbf{x}^{(1)}(t) = \mathbf{v}^{(1)}\mathrm{e}^{rt}, \qquad \mathbf{x}^{(2)}(t) = \mathbf{v}^{(2)}\mathrm{e}^{rt}$$

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Solve $(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{v}$

Two linearly independent solutions to system

$$\mathbf{x}^{(1)}(t) = \mathbf{v}e^{rt}, \qquad \mathbf{x}^{(2)}(t) = (\mathbf{v}t + \mathbf{u})e^{rt}$$

Sub-case (3,3) $A.M. = 3, \quad G.M. = 3$

- Associated to eigenvalue $r \in \mathbb{R}$, there are 3 linearly independent eigenvectors $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \mathbf{v}^{(3)} \quad \in \mathbb{R}^n$

Sub-case (3,3) $A.M. = 3, \quad G.M. = 3$

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- Get three solutions

$$\begin{aligned} \mathbf{x}^{(1)}(t) &= \mathbf{v}^{(1)} \mathrm{e}^{rt} \\ \mathbf{x}^{(2)}(t) &= \mathbf{v}^{(2)} \mathrm{e}^{rt} \\ \mathbf{x}^{(3)}(t) &= \mathbf{v}^{(3)} \mathrm{e}^{rt} \end{aligned}$$

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$$
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\mathbf{x}^{(1)}(t) &= \mathbf{v}^{(1)}\mathrm{e}^{rt} \\
\mathbf{x}^{(2)}(t) &= \mathbf{v}^{(2)}\mathrm{e}^{rt} \\
\mathbf{x}^{(3)}(t) &= \mathbf{v}^{(3)}\mathrm{e}^{rt}
\end{aligned}
$$

They are linearly independent because $W(0) \neq 0$

Sub-case (3,1) $A.M. = 3$, $G.M. = 1$

Associated to eigenvalue $r \in \mathbb{R}$, *only one* linearly independent eigenvector $\in \mathbb{R}^n$

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  Note that $(\mathbf{A} - r\mathbf{I})^2\mathbf{w} \quad = \quad (\mathbf{A} - r\mathbf{I})\mathbf{u}$

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  Note that
  $$\begin{aligned}(\mathbf{A} - r\mathbf{I})^2\mathbf{w} &= (\mathbf{A} - r\mathbf{I})\mathbf{u} \\ &= \mathbf{v}\end{aligned}$$

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  Note that $(\mathbf{A} - r\mathbf{I})^2\mathbf{w} = (\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{v} \qquad (\mathbf{A} - r\mathbf{I})^3\mathbf{w} = (\mathbf{A} - r\mathbf{I})\mathbf{v}$

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- Solve for $\mathbf{w}$ such that $\boxed{(\mathbf{A} - r\mathbf{I})\mathbf{w} = \mathbf{u}}$

  Note that
$$
\begin{aligned}
(\mathbf{A} - r\mathbf{I})^2\mathbf{w} &= (\mathbf{A} - r\mathbf{I})\mathbf{u} & \qquad (\mathbf{A} - r\mathbf{I})^3\mathbf{w} &= (\mathbf{A} - r\mathbf{I})\mathbf{v} \\
&= \mathbf{v} & &= \mathbf{0}
\end{aligned}
$$

Therefore, $$e^{\mathbf{A}t}\mathbf{w} = e^{rt}\left(\mathbf{I} + t(\mathbf{A} - r\mathbf{I}) + \frac{t^2}{2}(\mathbf{A} - r\mathbf{I})^2 + \frac{t^3}{3!}(\mathbf{A} - r\mathbf{I})^3 + \cdots\right)\mathbf{w}$$

Therefore,

$$
\begin{aligned}
e^{\mathbf{A}t}\mathbf{w} &= e^{rt}\left(\mathbf{I} + t(\mathbf{A} - r\mathbf{I}) + \frac{t^2}{2}(\mathbf{A} - r\mathbf{I})^2 + \frac{t^3}{3!}(\mathbf{A} - r\mathbf{I})^3 + \cdots\right)\mathbf{w} \\
&= e^{rt}\left(\mathbf{w} + t(\mathbf{A} - r\mathbf{I})\mathbf{w} + \frac{t^2}{2}(\mathbf{A} - r\mathbf{I})^2\mathbf{w}\right)
\end{aligned}
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Therefore,

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Third solution $\mathbf{x}^{(3)}(t) = \mathbf{v}\frac{t^2}{2!}e^{rt} + \mathbf{u}te^{rt} + \mathbf{w}e^{rt}$

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Remark $\mathbf{u}$ and $\mathbf{w}$ can be solved because

$$\ker(\mathbf{A} - r\mathbf{I}) \subsetneqq \ker(\mathbf{A} - r\mathbf{I})^2 \subsetneqq \ker(\mathbf{A} - r\mathbf{I})^3$$

Therefore,

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{w} &= e^{rt}\left(\mathbf{I} + t(\mathbf{A}-r\mathbf{I}) + \frac{t^2}{2}(\mathbf{A}-r\mathbf{I})^2 + \frac{t^3}{3!}(\mathbf{A}-r\mathbf{I})^3 + \cdots\right)\mathbf{w} \\ &= e^{rt}\left(\mathbf{w} + t(\mathbf{A}-r\mathbf{I})\mathbf{w} + \frac{t^2}{2}(\mathbf{A}-r\mathbf{I})^2\mathbf{w}\right) \\ &= e^{rt}\left(\mathbf{w} + t\mathbf{u} + \frac{t^2}{2}\mathbf{v}\right) \end{aligned}$$

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Therefore,

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{w} &= e^{rt}\left(\mathbf{I} + t(\mathbf{A} - r\mathbf{I}) + \frac{t^2}{2}(\mathbf{A} - r\mathbf{I})^2 + \frac{t^3}{3!}(\mathbf{A} - r\mathbf{I})^3 + \cdots\right)\mathbf{w} \\ &= e^{rt}\left(\mathbf{w} + t(\mathbf{A} - r\mathbf{I})\mathbf{w} + \frac{t^2}{2}(\mathbf{A} - r\mathbf{I})^2\mathbf{w}\right) \\ &= e^{rt}\left(\mathbf{w} + t\mathbf{u} + \frac{t^2}{2}\mathbf{v}\right) \end{aligned}$$

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$\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$ and $\mathbf{x}^{(3)}(t)$ are linearly independent

Therefore,
$$\begin{aligned} e^{\mathbf{A}t}\mathbf{w} &= e^{rt}\left(\mathbf{I} + t(\mathbf{A}-r\mathbf{I}) + \frac{t^2}{2}(\mathbf{A}-r\mathbf{I})^2 + \frac{t^3}{3!}(\mathbf{A}-r\mathbf{I})^3 + \cdots\right)\mathbf{w} \\ &= e^{rt}\left(\mathbf{w} + t(\mathbf{A}-r\mathbf{I})\mathbf{w} + \frac{t^2}{2}(\mathbf{A}-r\mathbf{I})^2\mathbf{w}\right) \\ &= e^{rt}\left(\mathbf{w} + t\mathbf{u} + \frac{t^2}{2}\mathbf{v}\right) \end{aligned}$$

Third solution $\mathbf{x}^{(3)}(t) = \mathbf{v}\frac{t^2}{2!}e^{rt} + \mathbf{u}te^{rt} + \mathbf{w}e^{rt}$

Remark $\mathbf{u}$ and $\mathbf{w}$ can be solved because

$$\ker(\mathbf{A}-r\mathbf{I}) \subsetneq \ker(\mathbf{A}-r\mathbf{I})^2 \subsetneq \ker(\mathbf{A}-r\mathbf{I})^3 = \ker(\mathbf{A}-r\mathbf{I})^4 = \cdots$$

$\mathbf{x}^{(1)}(t)$, $\mathbf{x}^{(2)}(t)$ and $\mathbf{x}^{(3)}(t)$ are linearly independent

$$\because \quad W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)}](0) = \det\begin{bmatrix}\mathbf{v} & \mathbf{u} & \mathbf{w}\end{bmatrix}$$

Therefore,

$$\begin{aligned} e^{\mathbf{A}t}\mathbf{w} &= e^{rt}\left(\mathbf{I} + t(\mathbf{A}-r\mathbf{I}) + \frac{t^2}{2}(\mathbf{A}-r\mathbf{I})^2 + \frac{t^3}{3!}(\mathbf{A}-r\mathbf{I})^3 + \cdots\right)\mathbf{w} \\ &= e^{rt}\left(\mathbf{w} + t(\mathbf{A}-r\mathbf{I})\mathbf{w} + \frac{t^2}{2}(\mathbf{A}-r\mathbf{I})^2\mathbf{w}\right) \\ &= e^{rt}\left(\mathbf{w} + t\mathbf{u} + \frac{t^2}{2}\mathbf{v}\right) \end{aligned}$$

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$$\begin{aligned} \because \quad W[\mathbf{x}^{(1)},\mathbf{x}^{(2)},\mathbf{x}^{(3)}](0) &= \det\begin{bmatrix}\mathbf{v} & \mathbf{u} & \mathbf{w}\end{bmatrix} \\ &\neq 0 \end{aligned}$$

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof Suppose $a\mathbf{v} + b\mathbf{u} + c\mathbf{w} = \mathbf{0}$ (1)

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof Suppose $a\mathbf{v} + b\mathbf{u} + c\mathbf{w} = \mathbf{0}$ (1)

Apply $(\mathbf{A} - r\mathbf{I})$ $a(\mathbf{A} - r\mathbf{I})\mathbf{v} + b(\mathbf{A} - r\mathbf{I})\mathbf{u} + c(\mathbf{A} - r\mathbf{I})\mathbf{w} = \mathbf{0}$

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof Suppose $a\mathbf{v} + b\mathbf{u} + c\mathbf{w} = \mathbf{0}$ (1)

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By construction $b\mathbf{v} + c\mathbf{u} = \mathbf{0}$ (2)

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

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Apply $(\mathbf{A} - r\mathbf{I})$ $b(\mathbf{A} - r\mathbf{I})\mathbf{v} + c(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}$

By construction $c\mathbf{v} = \mathbf{0}$ (3)

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof Suppose $a\mathbf{v} + b\mathbf{u} + c\mathbf{w} = \mathbf{0}$ (1)

Apply $(\mathbf{A} - r\mathbf{I})$ $\quad a(\mathbf{A} - r\mathbf{I})\mathbf{v} + b(\mathbf{A} - r\mathbf{I})\mathbf{u} + c(\mathbf{A} - r\mathbf{I})\mathbf{w} = \mathbf{0}$

By construction $\quad b\mathbf{v} + c\mathbf{u} = \mathbf{0}$ (2)

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By construction $\quad c\mathbf{v} = \mathbf{0}$ (3)

From (3), (2) and (1), get $c = 0$, $b = 0$ and $a = 0$

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof Suppose $a\mathbf{v} + b\mathbf{u} + c\mathbf{w} = \mathbf{0}$ (1)

Apply $(\mathbf{A} - r\mathbf{I})$ $\quad a(\mathbf{A} - r\mathbf{I})\mathbf{v} + b(\mathbf{A} - r\mathbf{I})\mathbf{u} + c(\mathbf{A} - r\mathbf{I})\mathbf{w} = \mathbf{0}$

By construction $\quad b\mathbf{v} + c\mathbf{u} = \mathbf{0}$ (2)

Apply $(\mathbf{A} - r\mathbf{I})$ $\quad b(\mathbf{A} - r\mathbf{I})\mathbf{v} + c(\mathbf{A} - r\mathbf{I})\mathbf{u} = \mathbf{0}$

By construction $\quad c\mathbf{v} = \mathbf{0}$ (3)

From (3), (2) and (1), get $c = 0$, $b = 0$ and $a = 0$

Terminology $\mathbf{v}$ is an eigenvector and $\mathbf{u}$ and $\mathbf{w}$ are *generalized eigenvectors*.

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof Suppose $a\mathbf{v} + b\mathbf{u} + c\mathbf{w} = \mathbf{0}$ (1)

$$\begin{aligned}
&\text{Apply } (\mathbf{A} - r\mathbf{I}) & a(\mathbf{A} - r\mathbf{I})\mathbf{v} + b(\mathbf{A} - r\mathbf{I})\mathbf{u} + c(\mathbf{A} - r\mathbf{I})\mathbf{w} &= \mathbf{0} \\
&\text{By construction} & b\mathbf{v} + c\mathbf{u} &= \mathbf{0} \qquad (2) \\
&\text{Apply } (\mathbf{A} - r\mathbf{I}) & b(\mathbf{A} - r\mathbf{I})\mathbf{v} + c(\mathbf{A} - r\mathbf{I})\mathbf{u} &= \mathbf{0} \\
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\end{aligned}$$

From (3), (2) and (1), get $c = 0$, $b = 0$ and $a = 0$

Terminology $\mathbf{v}$ is an eigenvector and $\mathbf{u}$ and $\mathbf{w}$ are *generalized eigenvectors*.

$(\mathbf{v}, \mathbf{u}, \mathbf{w})$ is called a *chain of generalized eigenvectors*

The vectors $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{w}$ are linearly independent.

Proof Suppose $a\mathbf{v} + b\mathbf{u} + c\mathbf{w} = \mathbf{0}$ (1)

Apply $(\mathbf{A} - r\mathbf{I})$ $a(\mathbf{A} - r\mathbf{I})\mathbf{v} + b(\mathbf{A} - r\mathbf{I})\mathbf{u} + c(\mathbf{A} - r\mathbf{I})\mathbf{w} = \mathbf{0}$

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Solutions to system using chain of generalized eigenvectors

Consider the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t)$

(1) $\mathbf{x}(t) = \mathbf{v}e^{rt}$ is a solution $\Longleftrightarrow (\mathbf{A} - r\mathbf{I})\mathbf{v} = \mathbf{0}$ $\quad r \in \mathbb{R}, \; \mathbf{v} \in \mathbb{R}^n$

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Sub-case (3,2) $A.M. = 3, \quad G.M. = 2$

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That is $\langle \mathbf{v}, \mathbf{y} \rangle = 0$ for all $\mathbf{y}$ satisfying $(\mathbf{A}^T - r\mathbf{I})\mathbf{y} = \mathbf{0}$

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$\mathbf{v}$ is a non-trivial linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ $\quad \mathbf{v} = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

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$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c_2 \\ c_1 \\ -c_2 \end{bmatrix}$$

u_1, u_2, u_3 can be found $\Longleftrightarrow$ $c_1 = -c_2$.

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Third solution $\mathbf{x}^{(3)}(t) = \begin{bmatrix} 1 & -1 & -1 \end{bmatrix}^T t\mathrm{e}^t + \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \mathrm{e}^t$

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Use Theorem $\langle \mathbf{v}, \mathbf{y} \rangle = 0$ *for all* $\mathbf{y}$ *belonging to the kernel of* $(\mathbf{A}^T - \mathbf{I})$

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Continue ...

Remark

- Subcase (3,1) to find chain of generalized eigenvectors $(\mathbf{v}, \mathbf{u}, \mathbf{w})$, solve

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- Subcase (3,2) two chain of generalized eigenvectors $(\mathbf{v}^{(1)}), (\mathbf{v}, \mathbf{u})$

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