First Order Linear Homogeneous Systems with Constants Coefficients

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- If not, consider $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v}$

Eigenvalue Method

Definition Let $\mathbf{A}$ be a square matrix of order n (entries can be complex)

A number λ (can be complex) is said to be an *eigenvalue* of $\mathbf{A}$ if there exists a non-zero vector $\mathbf{v} \in \mathbb{C}^n$ such that

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Reason λ is an eigenvalue of $\mathbf{A}$ iff $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$ has non-trivial solution.

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Linear Algebra Result If $r \in \mathbb{R}$ is an eigenvalue of $\mathbf{A}$ where $\mathbf{A} \in \mathcal{M}_n$, then there is an associated eigenvector $\mathbf{v} \in \mathbb{R}^n$

Theorem

$\mathbf{x}(t) = \mathbf{v}e^{rt}$ *is non-trivial solution to* $\mathbf{x}' = \mathbf{A}\mathbf{x} \iff$ $r \in \mathbb{R}$ *is an* eigenvalue *of* $\mathbf{A}$ *and* $\mathbf{v} \in \mathbb{R}^n$ *an associated* eigenvector

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Non-trivial means $\mathbf{v} \neq \mathbf{0}$

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General solution $\mathbf{x}(t) = c_1 \mathbf{v}^{(1)} \mathrm{e}^{r_1 t} + \cdots + c_n \mathbf{v}^{(n)} \mathrm{e}^{r_n t}$

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Note complex roots occur in conjugate pairs.

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$$\begin{bmatrix} 1 - 3\,\mathrm{i} & 2 \\ -5 & -1 - 3\,\mathrm{i} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 + 3\,\mathrm{i} & 2 \\ -5 & -1 + 3\,\mathrm{i} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

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Example Solve the system $\mathbf{x}' = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \mathbf{x}$

Solution Denote $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix}$

Solve charac eqt $\det(\mathbf{A} - r\mathbf{I}) = 0$ $\begin{vmatrix} 1 - r & 2 \\ -5 & -1 - r \end{vmatrix} = 0$

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General solution $\quad \mathbf{x}(t) = c_1 \begin{bmatrix} -\cos 3t + 3\sin 3t \\ 5\cos 3t \end{bmatrix} + c_2 \begin{bmatrix} -3\cos 3t - \sin 3t \\ 5\sin 3t \end{bmatrix}$

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  Fact $G.M. \leq A.M.$

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$$

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Linear algebra result Above equation has solution $\mathbf{u} \in \mathbb{R}^n$

$$
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e^{\mathbf{A}t}\mathbf{u} &= e^{rt}\left(\mathbf{I} + t(\mathbf{A} - r\mathbf{I}) + \frac{t^2}{2}(\mathbf{A} - r\mathbf{I})^2 + \frac{t^3}{3!}(\mathbf{A} - r\mathbf{I})^3 + \cdots\right)\mathbf{u} \\
&= e^{rt}(\mathbf{u} + t\mathbf{v})
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Second solution $\mathbf{x}^{(2)}(t) = \mathbf{v}te^{rt} + \mathbf{u}e^{rt}$

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Summary for Subcase (2,1)

- Solve for $\mathbf{v}$ and $\mathbf{u}$ $$\left\{\begin{aligned} (\mathbf{A} - r\mathbf{I})\mathbf{v} &= \mathbf{0} \\ (\mathbf{A} - r\mathbf{I})\mathbf{u} &= \mathbf{v} \end{aligned}\right.$$

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\end{cases}
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- Two independent solutions
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\end{cases}
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Definition Let λ be an eigenvalue of a square matrix $\mathbf{A}$.

Suppose $\mathbf{u}$ is a *non-zero vector* such that

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Remark Eigenvector: special case where $k = 1$

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