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&= \mathbf{\Phi}(t)\mathbf{c}
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Theorem (Fundamental Matrix Solution) *Let* $\mathbf{\Phi}$ *be a fundamental matrix for the homogeneous linear system*

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where $\mathbf{P} : (a, b) \longrightarrow \mathcal{M}_n$ *is continuous. Let* $t_0 \in (a, b)$ *and* $\mathbf{x}^0 \in \mathbb{R}^n$*.*

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Example Solve the IVP $$\mathbf{x}' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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To find general solution (or n linearly independent solutions)

- Elimination method
- Matrix exponential method
- Eigenvalue method

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Solve $\lambda^2 - 2\lambda - 3 = 0$, roots $-1, 3$

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General case

(1) Can we define $\mathrm{e}^{\mathbf{A}t}$?

(2) If yes, how can we *obtain general solution from* $\mathrm{e}^{\mathbf{A}t}$?

Recall $$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

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Let $\mathbf{A}$ be a square matrix of order n.

- Consider the sequence of $n \times n$-matrices $(\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \ldots)$ where

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- If $\mathbf{AB} = \mathbf{BA}$, then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}} \cdot e^{\mathbf{B}}$

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(2) Let $\mathbf{A} \in \mathcal{M}_n$ and $f : (a,b) \longrightarrow \mathbb{R}$ a differentiable function. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t)\mathbf{A} = f'(t)\mathbf{A}$$

Let $\mathbf{A}$ be a square matrix (of order n, with real entries)

- Consider matrix-valued function $t \mapsto \mathrm{e}^{t\mathbf{A}}$ $(\mathbb{R} \longrightarrow \mathcal{M}_n)$

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$(*)$ means $\begin{bmatrix} \dfrac{d}{dt}\mathbf{x}^{(1)}(t) & \dfrac{d}{dt}\mathbf{x}^{(2)}(t) & \cdots & \dfrac{d}{dt}\mathbf{x}^{(n)}(t) \end{bmatrix} =$

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- Consider matrix-valued function $t \mapsto e^{t\mathbf{A}}$ $(\mathbb{R} \longrightarrow \mathcal{M}_n)$

- Differentiate

$$\begin{aligned}
\frac{d}{dt}e^{t\mathbf{A}} &= \frac{d}{dt}\Big(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots\Big) \\
&= \mathbf{A} + t\mathbf{A}^2 + \frac{t^2}{2!}\mathbf{A}^3 + \cdots \\
&= \mathbf{A}\Big(\mathbf{I} + t\mathbf{A} + \frac{t^2}{2!}\mathbf{A}^2 + \frac{t^3}{3!}\mathbf{A}^3 + \cdots\Big) \\
&= \mathbf{A}e^{t\mathbf{A}} \qquad (*)
\end{aligned}$$

Denote $e^{t\mathbf{A}} = \begin{bmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) & \cdots & \mathbf{x}^{(n)}(t) \end{bmatrix}$

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That is, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ ($\mathbb{R}^n$-valued functions) are *solutions* to $\mathbf{x}' = \mathbf{A}\mathbf{x}$

$$W[\mathbf{x}^{(1)} \ \cdots \ \mathbf{x}^{(n)}](0)$$

$$W[\mathbf{x}^{(1)} \;\cdots\; \mathbf{x}^{(n)}](0) \quad = \quad \det\left(\mathrm{e}^{0\mathbf{A}}\right)$$

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Theorem *Consider the following first order homogeneous linear system with constant coefficients*

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- find *eigenvalues*, *eigenvectors and generalized eigenvectors*;
- change to *Jordan form*.