First Order Linear Homogeneous Systems

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Guess Solution space dim $= n$ (n = size of system)

subspace of $C\left((a,b), \mathbb{R}^n\right)$

First Order Linear Homogeneous Systems

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Guess Solution space dim $= n$ (n = size of system)
subspace of $C\left((a,b), \mathbb{R}^n\right)$

- Find n solutions.

First Order Linear Homogeneous Systems

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Guess Solution space $\dim = n$ (n = size of system)
subspace of $C\left((a,b),\mathbb{R}^n\right)$

- Find n solutions.
- Show the n solutions form a basis.

First Order Linear Homogeneous Systems

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Guess Solution space dim $= n$ (n = size of system)
subspace of $C\left((a, b), \mathbb{R}^n\right)$

- Find n solutions.
- Show the n solutions form a basis.
  - linearly independent
  - span whole solution space

Lemma *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *be solutions to the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Then TFAE

Lemma *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *be solutions to the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Then TFAE

(1) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ *are linearly dependent for some* $t_0 \in (a, b)$.

Lemma *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *be solutions to the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Then TFAE

(1) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ *are linearly dependent for some* $t_0 \in (a, b)$.

(2) $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ *are linearly dependent for all* $t \in (a, b)$.

Lemma *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *be solutions to the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Then TFAE

(1) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ *are linearly dependent for some* $t_0 \in (a, b)$.

(2) $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ *are linearly dependent for all* $t \in (a, b)$.

(3) $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *are linearly dependent on* (a, b).

Lemma *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *be solutions to the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Then TFAE

(1) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ *are linearly dependent for some* $t_0 \in (a, b)$.

(2) $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ *are linearly dependent for all* $t \in (a, b)$.

(3) $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *are linearly dependent on* (a, b).

Note

- $\mathbf{x}^{(i)}(t_0)$'s are elements in $\mathbb{R}^n$ (n = size of system)

Lemma *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *be solutions to the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Then TFAE

(1) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ *are linearly dependent for some* $t_0 \in (a, b)$.

(2) $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ *are linearly dependent for all* $t \in (a, b)$.

(3) $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *are linearly dependent on* (a, b).

Note

- $\mathbf{x}^{(i)}(t_0)$'s are elements in $\mathbb{R}^n$ (n = size of system)
- $\mathbf{x}^{(i)}$'s are vector-valued functions from (a, b) into $\mathbb{R}^n$.

Lemma *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *be solutions to the homogeneous system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b$$

Then TFAE

(1) $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ *are linearly dependent for some* $t_0 \in (a, b)$.

(2) $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(k)}(t)$ *are linearly dependent for all* $t \in (a, b)$.

(3) $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ *are linearly dependent on* (a, b).

Note

- $\mathbf{x}^{(i)}(t_0)$'s are elements in $\mathbb{R}^n$ (n = size of system)
- $\mathbf{x}^{(i)}$'s are vector-valued functions from (a, b) into $\mathbb{R}^n$.

  $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are linearly dependent on (a, b) means they are linearly dependent elements in $C\big((a, b), \mathbb{R}^n\big)$.

Proof

$(3) \implies (2)$

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0}$$

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$$

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\Big((a, b), \mathbb{R}^n\Big)$$

That is, $\quad \forall\ t \in (a, b), \quad c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0}$

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$$

That is, $\forall\ t \in (a, b), \quad c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0} \quad \in \mathbb{R}^n$

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$$

That is, $\forall\ t \in (a, b), \quad c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0} \quad \in \mathbb{R}^n$

Hence $\forall\ t \in (a, b), \quad \mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ \ldots,\ \mathbf{x}^{(k)}(t)$ are dependent $\in \mathbb{R}^n$

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$$

That is, $\forall\ t \in (a, b), \quad c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0} \quad \in \mathbb{R}^n$

Hence $\forall\ t \in (a, b), \quad \mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ \ldots,\ \mathbf{x}^{(k)}(t)$ are dependent $\in \mathbb{R}^n$

$(2) \Longrightarrow (1)$

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \dots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\Big((a, b), \mathbb{R}^n\Big)$$

That is, $\forall\ t \in (a, b), \quad c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0} \quad \in \mathbb{R}^n$

Hence $\forall\ t \in (a, b), \quad \mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ \dots,\ \mathbf{x}^{(k)}(t)$ are dependent $\in \mathbb{R}^n$

$(2) \Longrightarrow (1)$ Obvious.

Proof

$(3) \Longrightarrow (2)$ Suppose $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(k)}$ are dependent on (a, b).

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all zero*, such that

$$c_1\mathbf{x}^{(1)} + c_2\mathbf{x}^{(2)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$$

That is, $\forall\ t \in (a, b),\quad c_1\mathbf{x}^{(1)}(t) + c_2\mathbf{x}^{(2)}(t) + \cdots + c_k\mathbf{x}^{(k)}(t) = \mathbf{0} \quad \in \mathbb{R}^n$

Hence $\forall\ t \in (a, b),\quad \mathbf{x}^{(1)}(t),\ \mathbf{x}^{(2)}(t),\ \ldots,\ \mathbf{x}^{(k)}(t)$ are dependent $\in \mathbb{R}^n$

$(2) \Longrightarrow (1)$ Obvious.

Remark (3)$\Longrightarrow$ (2) and (2)$\Longrightarrow$ (1)

do not require $\mathbf{x}^{(i)}$'s are solutions to system.

$(1) \Longrightarrow (3)$

(1) $\Longrightarrow$ (3)   Suppose $\exists\ t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0), \ \ldots, \ \mathbf{x}^{(k)}(t_0)$ are dependent

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0),\ \ldots,\ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0}$$

$(1) \Longrightarrow (3)$ Suppose $\exists\ t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0),\ \ldots,\ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

$(1) \Longrightarrow (3)$ Suppose $\exists\ t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0),\ \ldots,\ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0}$

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0), \ \ldots, \ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $\quad c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0), \ \ldots, \ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$

Denote $\boldsymbol{\varphi} = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0), \ \ldots, \ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$

Denote $\varphi = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP

$(1) \Longrightarrow (3)$ Suppose $\exists\ t_0 \in (a,b)$, such that $\mathbf{x}^{(1)}(t_0),\ \ldots,\ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a,b), \mathbb{R}^n\big)$

Denote $\varphi = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b \\ \end{cases}$

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a,b)$, such that $\mathbf{x}^{(1)}(t_0), \; \ldots, \; \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $\quad c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a,b), \mathbb{R}^n\big)$

Denote $\quad \varphi = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b \\ \mathbf{x}(t_0) = \mathbf{0} \end{cases}$

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $\quad c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$

Denote $\quad \boldsymbol{\varphi} = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP $\quad \begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b \\ \mathbf{x}(t_0) = \mathbf{0} \end{cases}$

- $\boldsymbol{\varphi}$ is a solution

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a,b)$, such that $\mathbf{x}^{(1)}(t_0),\ \ldots,\ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a,b), \mathbb{R}^n\big)$

Denote $\boldsymbol{\varphi} = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b \\ \mathbf{x}(t_0) = \mathbf{0} \end{cases}$

- $\boldsymbol{\varphi}$ is a solution *satisfies system* ($\because$ system is linear homogeneous)

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0), \ \ldots, \ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$

Denote $\boldsymbol{\varphi} = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b \\ \mathbf{x}(t_0) = \mathbf{0} \end{cases}$

- $\boldsymbol{\varphi}$ is a solution *satisfies system* ($\because$ system is linear homogeneous)

  satisfies initial condition (by construction)

$(1) \Longrightarrow (3)$ Suppose $\exists\, t_0 \in (a,b)$, such that $\mathbf{x}^{(1)}(t_0), \ \ldots, \ \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a,b), \mathbb{R}^n\big)$

Denote $\boldsymbol{\varphi} = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b \\ \mathbf{x}(t_0) = \mathbf{0} \end{cases}$

- $\boldsymbol{\varphi}$ is a solution *satisfies system* ($\because$ system is linear homogeneous)

  satisfies initial condition (by construction)
- $\mathbf{0}$ is a solution

$(1) \Longrightarrow (3)$ Suppose $\exists\ t_0 \in (a, b)$, such that $\mathbf{x}^{(1)}(t_0), \ldots, \mathbf{x}^{(k)}(t_0)$ are dependent

That is, $\exists$ scalars $c_1, \ldots, c_k$, *not all 0*, such that

$$c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_k\mathbf{x}^{(k)}(t_0) = \mathbf{0} \quad \in \mathbb{R}^n$$

Want to show $c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)} = \mathbf{0} \quad \in C\big((a, b), \mathbb{R}^n\big)$

Denote $\varphi = c_1\mathbf{x}^{(1)} + \cdots + c_k\mathbf{x}^{(k)}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b \\ \mathbf{x}(t_0) = \mathbf{0} \end{cases}$

- φ is a solution *satisfies system* ($\because$ system is linear homogeneous)

  satisfies initial condition (by construction)
- $\mathbf{0}$ is a solution

By E&U Theorem, $\varphi = \mathbf{0}$.

Definition Let $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$ be functions from (a, b) into $\mathbb{R}^n$.

The *Wronskian* of $\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}$, denoted by $W[\mathbf{x}^{(1)}, \dots, \mathbf{x}^{(n)}]$ or simply W, is the function $W : (a, b) \longrightarrow \mathbb{R}$, defined by

Definition Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be functions from (a, b) into $\mathbb{R}^n$.

The *Wronskian* of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$, denoted by $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ or simply W, is the function $W : (a, b) \longrightarrow \mathbb{R}$, defined by

$$W(t) = \det\begin{bmatrix}\mathbf{x}^{(1)}(t) & \cdots & \mathbf{x}^{(n)}(t)\end{bmatrix}$$

Definition Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be functions from (a, b) into $\mathbb{R}^n$.

The *Wronskian* of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$, denoted by $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ or simply W, is the function $W : (a, b) \longrightarrow \mathbb{R}$, defined by

$$W(t) = \det\left[\mathbf{x}^{(1)}(t) \quad \cdots \quad \mathbf{x}^{(n)}(t)\right]$$

where

$$\left[\mathbf{x}^{(1)}(t) \quad \ldots \quad \mathbf{x}^{(n)}(t)\right] = \begin{bmatrix} \mathbf{x}_1^{(1)}(t) & \mathbf{x}_1^{(2)}(t) & \cdots & \mathbf{x}_1^{(n)}(t) \\ \mathbf{x}_2^{(1)}(t) & \mathbf{x}_2^{(2)}(t) & \cdots & \mathbf{x}_2^{(n)}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{x}_n^{(1)}(t) & \mathbf{x}_n^{(2)}(t) & \cdots & \mathbf{x}_n^{(n)}(t) \end{bmatrix}$$

Definition Let $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ be functions from (a, b) into $\mathbb{R}^n$.

The *Wronskian* of $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$, denoted by $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ or simply W, is the function $W : (a, b) \longrightarrow \mathbb{R}$, defined by

$$W(t) = \det \left[\mathbf{x}^{(1)}(t) \quad \cdots \quad \mathbf{x}^{(n)}(t) \right]$$

where

$$\left[\mathbf{x}^{(1)}(t) \quad \ldots \quad \mathbf{x}^{(n)}(t) \right] = \begin{bmatrix} \mathbf{x}_1^{(1)}(t) & \mathbf{x}_1^{(2)}(t) & \cdots & \mathbf{x}_1^{(n)}(t) \\ \mathbf{x}_2^{(1)}(t) & \mathbf{x}_2^{(2)}(t) & \cdots & \mathbf{x}_2^{(n)}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{x}_n^{(1)}(t) & \mathbf{x}_n^{(2)}(t) & \cdots & \mathbf{x}_n^{(n)}(t) \end{bmatrix}$$

the square matrix whose i th column is $\mathbf{x}^{(i)}(t)$

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*
- n solutions to 1st order linear homogeneous *system* with n equations

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*
- n solutions to 1st order linear homogeneous *system* with n equations

Question Are the two definitions the same ?

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*
- n solutions to 1st order linear homogeneous *system* with n equations

Question Are the two definitions the same ?

- Suppose y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*
- n solutions to 1st order linear homogeneous *system* with n equations

Question Are the two definitions the same ?

- Suppose y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*
- n solutions to 1st order linear homogeneous *system* with n equations

Question Are the two definitions the same ?

- Suppose y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- Rewrite DE as *1st order system*.

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*
- n solutions to 1st order linear homogeneous *system* with n equations

Question Are the two definitions the same ?

- Suppose y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$

  $$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- Rewrite DE as *1st order system*.

  Each y_i gives a solution $\mathbf{x}^{(i)}$ to the system.

Remark Wronskian is defined for

- two solutions to 2nd order linear homogeneous *DE*
- n solutions to 1st order linear homogeneous *system* with n equations

Question Are the two definitions the same ?

- Suppose y_1, y_2 are solutions to $y'' + p(t)y' + q(t)y = 0$

  $$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

- Rewrite DE as *1st order system*.

  Each y_i gives a solution $\mathbf{x}^{(i)}$ to the system.

  Can consider $W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}]$

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b)*,*

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b)*,*

according to whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly dependent on* (a, b) *or not.*

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b)*,*

according to whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly dependent on* (a, b) *or not.*

Proof Two cases

- the solutions are dependent on (a, b)

- the solutions are independent

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b)*,*

according to whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly dependent on* (a, b) *or not.*

Proof Two cases

- the solutions are dependent on (a, b)

  By Lemma, for all $t \in (a, b)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are dependent $\in \mathbb{R}^n$

- the solutions are independent

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b),

according to whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly dependent on* (a, b) *or not.*

Proof Two cases

- the solutions are dependent on (a, b)

  By Lemma, for all $t \in (a, b)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are dependent $\in \mathbb{R}^n$

  Hence for all $t \in (a, b)$, $W(t) = 0$

- the solutions are independent

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b)*,*

according to whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly dependent on* (a, b) *or not.*

Proof Two cases

- the solutions are dependent on (a, b)

  By Lemma, for all $t \in (a, b)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are dependent $\in \mathbb{R}^n$

  Hence for all $t \in (a, b)$, $W(t) = 0$ $\quad$ det $= 0$ iff columns are dependent

- the solutions are independent

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b),

according to whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly dependent on* (a, b) *or not.*

Proof Two cases

- the solutions are dependent on (a, b)

  By Lemma, for all $t \in (a, b)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are dependent $\in \mathbb{R}^n$

  Hence for all $t \in (a, b)$, $W(t) = 0$

  det $= 0$ iff columns are dependent

- the solutions are independent

  By Lemma, for all $t \in (a, b)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are not dependent $\in \mathbb{R}^n$

Corollary *Let* $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *be solutions to a homogeneous 1st order system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \qquad (n = \text{size of system})$$

Then the Wronskian $W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}]$ *is either*

- *identically zero in* (a, b) *or*
- *never zero in* (a, b)*,*

according to whether $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ *are linearly dependent on* (a, b) *or not.*

Proof Two cases

- the solutions are dependent on (a, b)

  By Lemma, for all $t \in (a, b)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are dependent $\in \mathbb{R}^n$

  Hence for all $t \in (a, b)$, $W(t) = 0$

  det $= 0$ iff columns are dependent

- the solutions are independent

  By Lemma, for all $t \in (a, b)$, $\mathbf{x}^{(1)}(t), \ldots, \mathbf{x}^{(n)}(t)$ are not dependent $\in \mathbb{R}^n$

  Hence for all $t \in (a, b)$, $W(t) \neq 0$

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b)*. Then*

$$\dim(\textit{solution space}) = n$$

where $n =$ *size of system.*

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b)*. Then*

$$\dim(\textit{solution space}) = n$$

where n = *size of system.*

Proof (1) *"Find"* n solutions

(2) Show that the solutions form a basis for solution space

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b)*. Then*

$$\dim(\textit{solution space}) = n$$

where $n =$ *size of system.*

Proof (1) *"Find"* n solutions

(2) Show that the solutions form a basis for solution space

(1) Fix $t_0 \in (a, b)$

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b)*. Then*

$$\dim(\textit{solution space}) = n$$

where $n =$ *size of system.*

Proof (1) *"Find"* n solutions

(2) Show that the solutions form a basis for solution space

(1) Fix $t_0 \in (a, b)$

For each $i = 1, \ldots, n$, consider IVP

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b). *Then*

$$\dim(\textit{solution space}) = n$$

where $n =$ *size of system.*

Proof (1) *"Find"* n solutions

(2) Show that the solutions form a basis for solution space

(1) Fix $t_0 \in (a, b)$

For each $i = 1, \ldots, n,$ consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \end{cases}$

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b). *Then*

$$\dim(\textit{solution space}) = n$$

where $n =$ *size of system.*

Proof (1) *"Find"* n solutions

(2) Show that the solutions form a basis for solution space

(1) Fix $t_0 \in (a, b)$

For each $i = 1, \ldots, n$, consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b, \\ \mathbf{x}(t_0) = \mathbf{e}^{(i)} \end{cases}$ where

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b)*. Then*

$$\dim(\textit{solution space}) = n$$

where $n =$ *size of system.*

Proof (1) *"Find"* n solutions

(2) Show that the solutions form a basis for solution space

(1) Fix $t_0 \in (a, b)$

For each $i = 1, \ldots, n$, consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b, \\ \mathbf{x}(t_0) = \mathbf{e}^{(i)} \end{cases}$ where

$$\mathbf{e}^{(i)} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T \leftarrow i \text{ th entry} = 1$$

Theorem *Consider the system*

$$\mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \qquad a < t < b$$

Suppose $\mathbf{P}$ *is continuous on* (a, b)*. Then*

$$\dim(\textit{solution space}) = n$$

where $n =$ *size of system.*

Proof (1) *"Find"* n solutions

(2) Show that the solutions form a basis for solution space

(1) Fix $t_0 \in (a, b)$

For each $i = 1, \ldots, n$, consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b, \\ \mathbf{x}(t_0) = \mathbf{e}^{(i)} \end{cases}$ where

$$\mathbf{e}^{(i)} = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T \leftarrow i \text{ th entry} = 1$$

By E&U Theorem, each IVP has unique solution, denoted by $\mathbf{x}^{(i)}$.

(2) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a basis for solution space

(2) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a basis for solution space

(a) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ linearly independent $\in C\big((a,b), \mathbb{R}^n\big)$

(b) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ span solution space

(2) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a basis for solution space

(a) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ linearly independent $\in C\big((a,b), \mathbb{R}^n\big)$

(b) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ span solution space

Show (a) Since

$$W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t_0)$$

(2) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a basis for solution space

(a) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ linearly independent $\in C\big((a, b), \mathbb{R}^n\big)$

(b) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ span solution space

Show (a) Since

$$W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t_0) \quad = \quad \det\left[\mathbf{e}^{(1)} \quad \mathbf{e}^{(2)} \quad \cdots \quad \mathbf{e}^{(n)}\right]$$

(2) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a basis for solution space

(a) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ linearly independent $\in C\big((a,b), \mathbb{R}^n\big)$

(b) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ span solution space

Show (a) Since

$$\begin{aligned} W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t_0) &= \det\begin{bmatrix}\mathbf{e}^{(1)} & \mathbf{e}^{(2)} & \cdots & \mathbf{e}^{(n)}\end{bmatrix} \\ &= \det\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \end{aligned}$$

(2) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a basis for solution space

(a) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ linearly independent $\in C\big((a,b), \mathbb{R}^n\big)$

(b) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ span solution space

Show (a) Since

$$\begin{aligned} W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t_0) &= \det\begin{bmatrix}\mathbf{e}^{(1)} & \mathbf{e}^{(2)} & \cdots & \mathbf{e}^{(n)}\end{bmatrix} \\ &= \det\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \\ &= 1 \neq 0 \end{aligned}$$

(2) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ form a basis for solution space

(a) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ linearly independent $\in C\big((a,b), \mathbb{R}^n\big)$

(b) Show $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ span solution space

Show (a) Since

$$\begin{aligned} W[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}](t_0) &= \det\begin{bmatrix} \mathbf{e}^{(1)} & \mathbf{e}^{(2)} & \cdots & \mathbf{e}^{(n)} \end{bmatrix} \\ &= \det\begin{bmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & 1 \end{bmatrix} \\ &= 1 \neq 0 \end{aligned}$$

by Corollary, $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}$ are linearly independent on (a, b).

Show (b) Let φ be a solution to the system

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \end{cases}$

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*
- $\boldsymbol{\psi}$ also solution to IVP

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*
- $\boldsymbol{\psi}$ also solution to IVP
  - ◇ satisfies system *(homogeneous)*

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*
- $\boldsymbol{\psi}$ also solution to IVP
  - satisfies system *(homogeneous)*
  - satisfies initial condition

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

$$\boldsymbol{\psi}(t_0) = c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0)$$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*
- $\boldsymbol{\psi}$ also solution to IVP
  - satisfies system *(homogeneous)*
  - satisfies initial condition

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\varphi \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

$$\begin{aligned} \boldsymbol{\psi}(t_0) &= c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0) \\ &= c_1\mathbf{e}^{(1)} + \cdots + c_n\mathbf{e}^{(n)} \end{aligned}$$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*
- $\boldsymbol{\psi}$ also solution to IVP
  - satisfies system *(homogeneous)*
  - satisfies initial condition

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

$$\begin{aligned} \boldsymbol{\psi}(t_0) &= c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0) \\ &= c_1\mathbf{e}^{(1)} + \cdots + c_n\mathbf{e}^{(n)} \\ &= c_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \end{aligned}$$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*
- $\boldsymbol{\psi}$ also solution to IVP
  - satisfies system *(homogeneous)*
  - satisfies initial condition

Show (b) Let $\boldsymbol{\varphi}$ be a solution to the system want to show $\boldsymbol{\varphi} \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\boldsymbol{\varphi}(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\boldsymbol{\psi} = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\boldsymbol{\psi} = \boldsymbol{\varphi}$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, & a < t < b, \\ \mathbf{x}(t_0) = \boldsymbol{\varphi}(t_0) \end{cases}$

$$\begin{aligned} \boldsymbol{\psi}(t_0) &= c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0) \\ &= c_1\mathbf{e}^{(1)} + \cdots + c_n\mathbf{e}^{(n)} \\ &= c_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \end{aligned}$$

- $\boldsymbol{\varphi}$ is solution to IVP *(by construction)*
- $\boldsymbol{\psi}$ also solution to IVP
  - ◇ satisfies system *(homogeneous)*
  - ◇ satisfies initial condition

Show (b) Let φ be a solution to the system want to show $\varphi \in \text{span}\,\{\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)}\}$

Write $\varphi(t_0) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Let $\psi = c_1\mathbf{x}^{(1)} + \cdots + c_n\mathbf{x}^{(n)}$

Show $\psi = \varphi$

Consider IVP $\begin{cases} \mathbf{x}' = \mathbf{P}(t)\mathbf{x}, \quad a < t < b, \\ \mathbf{x}(t_0) = \varphi(t_0) \end{cases}$

- φ is solution to IVP *(by construction)*
- ψ also solution to IVP
  - ◇ satisfies system *(homogeneous)*
  - ◇ satisfies initial condition

$$\begin{aligned} \psi(t_0) &= c_1\mathbf{x}^{(1)}(t_0) + \cdots + c_n\mathbf{x}^{(n)}(t_0) \\ &= c_1\mathbf{e}^{(1)} + \cdots + c_n\mathbf{e}^{(n)} \\ &= c_1\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + c_n\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}$$