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nth Order Linear DE/IVP

nth Order Linear DE/IVP 舉 2 反 n

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Chapter 3: First Order Systems

(1) E & U Theorem

(2) Linear Systems

(3) Homogeneous Systems with Constant Coefficients

- Elimination
- Matrix Exponential
- Eigenvalues Method

(4) Non-homogeneous Systems

- Undetermined Coefficients
- Variation of Parameters

First Order System (explicit form, n equations, n unknown functions)

$$\begin{cases} x_1' &= F_1(t, x_1, \ldots, x_n) \\ x_2' &= F_2(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, \ldots, x_n) \end{cases}$$

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Solutions to the system in an open interval (a, b) expressed in the form

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for each $i = 1, 2, \ldots, n$, $\quad \varphi_i'(t) = F_i\Big(t, \varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)\Big) \quad \forall\, t \in (a, b)$

1st order system IVP

- First order system

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$$\begin{cases} x_1(t_0) = x_1^0 \\ x_2(t_0) = x_2^0 \\ \vdots \\ x_n(t_0) = x_n^0 \end{cases}$$

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$\left(t_0, x_1^0, x_2^0, \ldots, x_n^0\right) \in \text{dom}\, F_i$ for all $i = 1, \ldots, n$

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Let $f_i : (a, b) \longrightarrow \mathbb{R}$ $(i = 1, \ldots, n)$ be (real-valued) functions.

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Vector Notation

Let $f_i : (a,b) \longrightarrow \mathbb{R}$ $(i = 1, \ldots, n)$ be (real-valued) functions.

- Define $\mathbf{f} : (a,b) \longrightarrow \mathbb{R}^n$ by

$$\mathbf{f}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix} \qquad a < t < b,$$

  $\mathbf{f}$ is called a *vector-valued function* (of *one variable*)

- Similarly, can define
  - ◇ vector-valued functions of *several variables*.
  - ◇ *matrix-valued* functions of one or several variables.

- ◇ If all the f_i's are continuous, say that $\mathbf{f}$ is continuous.
  - ◇ If all the f_i's are differentiable, say that $\mathbf{f}$ is differentiable.

    Define derivative of $\mathbf{f}$ $\quad \mathbf{f}' : (a,b) \longrightarrow \mathbb{R}^n \qquad \mathbf{f}'(t) = \begin{bmatrix} f_1'(t) \\ \vdots \\ f_n'(t) \end{bmatrix}$

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Vector notation for 1st order system (IVP)

System
$$\begin{cases} x_1' &= F_1(t, x_1, \ldots, x_n) \\ x_2' &= F_2(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, \ldots, x_n), \end{cases}$$

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Existence and Uniqueness Theorem for 1st order system IVP

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Corollary *Consider the nth order IVP*

$$\begin{cases} y^{(n)} = F(t, y, y', \dots, y^{(n-1)}), \\ y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \dots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}$$

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Change to 1st order sys
$$\begin{cases} x'_1 = x_2 \\ x'_2 = x_3 \\ \vdots \\ x'_{n-1} = x_n \\ x'_n = F(t, x_1, x_2, \ldots, x_n) \end{cases}$$

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$$\begin{cases} y^{(n)} = F(t, y, y', \ldots, y^{(n-1)}), \\ y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \ldots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}$$

Suppose that $F, \ \dfrac{\partial F}{\partial y}, \ \dfrac{\partial F}{\partial y'}, \ \ldots, \dfrac{\partial F}{\partial y^{(n-1)}}$ *are continuous on a neighborhood of* $\left(t_0, y_0, y'_0, \ldots, y_0^{(n-1)}\right)$.

Then in some neighborhood of t_0, the IVP has a unique solution.

Proof Put $x_1 = y$

Change to 1st order sys

$$\begin{cases} x'_1 = x_2 & = F_1(t, x_1, x_2, \ldots, x_n) \\ x'_2 = x_3 & = F_2(t, x_1, x_2, \ldots, x_n) \\ \quad \vdots & \\ x'_{n-1} = x_n & = F_{n-1}(t, x_1, x_2, \ldots, x_n) \\ x'_n = F(t, x_1, x_2, \ldots, x_n) & \end{cases}$$

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Iteration formula

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$$\rightarrow \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

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$$\to \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \text{as } n \to \infty \quad \text{soln to system IVP} \quad \begin{cases} x_1 = \cos t \\ x_2 = -\sin t \end{cases}$$

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Solution to original IVP $y(t) = \cos t$

First order Linear systems

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each F_i *"linear"* in $x_1, x_2, \ldots, x_n$

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$$\begin{cases} x_1' &= p_{11}(t)x_1 + \cdots + p_{1n}(t)x_n + g_1(t), \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + \cdots + p_{nn}(t)x_n + g_n(t), \end{cases} \qquad a < t < b$$

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Put $\mathbf{P}(t) = \left[p_{ij}(t)\right]_{n\times n}$

First order Linear systems

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Consider the following system IVP

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- In what follows, $\mathbf{P}$ and $\mathbf{g}$ are assumed to be *continuous*.

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Question $\dim\big($solution space$\big) = ?$