Method of Variation of Parameters

Consider 2nd order linear non-homogeneous DE

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- Reduces to $$c_1'(t)y_1'(t) + c_2'(t)y_2'(t) = g(t) \qquad (**)$$

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- Integrate to get $c_1(t)$ and $c_2(t)$ with two arbitrary constants.

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$$y'' + y = \csc t, \qquad 0 < t < \pi \tag{1}$$

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- Thus $y'(t) = c_1(t) \cos t - c_2(t) \sin t$

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- Differentiate $y'(t) = c_1(t) \cos t + c_1'(t) \sin t - c_2(t) \sin t + c_2'(t) \cos t$
- Impose condition

$$c_1'(t) \sin t + c_2'(t) \cos t = 0 \tag{2}$$

- Thus

$$\begin{aligned} y'(t) &= c_1(t) \cos t - c_2(t) \sin t \\ y''(t) &= c_1'(t) \cos t - c_1(t) \sin t - c_2'(t) \sin t - c_2(t) \cos t \end{aligned}$$

- Substitute and simplify...

$$c_1'(t) \cos t - c_2'(t) \sin t = \csc t \tag{3}$$

- Solve

$$\begin{cases} (2) \qquad c_1'(t)\sin t + c_2'(t)\cos t &= 0 \\ (3) \qquad c_1'(t)\cos t - c_2'(t)\sin t &= \csc t \end{cases}$$

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Reduction of order Consider

$$y'' + p(t)y' + q(t)y = g(t), \qquad a < t < b, \tag{4}$$

where $p, q, g \in C(a, b)$.

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Solve (6), get w (with an arbitrary constant)

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Example Find two linearly independent solutions to the DE

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Theorem *Let* φ *be a non-trivial solution to the homogeneous DE*

$$y'' + p(t)y' + q(t)y = 0$$

where $p, q \in C(a, b)$*. Then zeros of* φ *are isolated.*

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By Uniqueness Theorem, $\varphi \equiv 0$ in (a, b).

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$$\lim_{t \to t_0-} \frac{y_l(t) - a}{t - t_0} = \beta \lim_{t \to t_0+} \frac{y_r(t) - a}{t - t_0} + \gamma y_1'(t_0)$$