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Form of part. soln to given DE $y_p(t) = (At^3 + Bt^2 + Ct) + De^{2t}$

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$$ay'' + by' + cy = \begin{cases} P_n(t)\mathrm{e}^{\alpha t}\sin\beta t \\ P_n(t)\mathrm{e}^{\alpha t}\cos\beta t \end{cases}$$

where $a, b, c \in \mathbb{R}$, $a \neq 0$ *and* $P_n(t)$ *is a polynomial of degree* n, $\alpha, \beta \in \mathbb{R}$ ($\beta \neq 0$).

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Particular solution for $(*)$

$$y_p(t) + \overline{y_p(t)}$$

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&= t^m \mathrm{e}^{\alpha t} \text{ Re } \Big[\Big(B_n(t) + \mathrm{i}\, C_n(t) \Big) \Big(\cos \beta t + \mathrm{i} \sin \beta t \Big) \Big]
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deg $B_n(t)$ and deg $C_n(t) \le n$, at least one has degree n.

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Consider a second order linear non-homogeneous DE

$$y'' + p(t)y' + q(t)y = g(t), \qquad a < t < b,$$

where p, q and g are continuous on (a, b).

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(2) *Variation of parameters* (cf. result for 1st order linear DE)

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(1) *Reduction of order* (use Variation of Parameters or Wronskian)

Theorem *The* general solution *to*

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where $p, g \in C(a, b)$*, is*

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$$y' + p(t)y = g(t), \qquad a < t < b, \qquad (*)$$

where $p, g \in C(a,b)$, *is*

$$y(t) = \mathrm{e}^{-H(t)}\left[\int \mathrm{e}^{H(t)} g(t)\,\mathrm{d}t\right], \qquad a < t < b,$$

where H *is a primitive for* p *on* (a,b).

- Complementary solution $y_h(t) = C\mathrm{e}^{-H(t)}$
- General solution to $(*)$ has the form $y(t) = C(t)\mathrm{e}^{-H(t)}$

  To find $C(t)$

  - Substitute into $(*)$ gives $C'(t) =$ something
  - Integrate to get $C(t)$

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