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Since y_1, y_2 are solutions to the DE:

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Second order linear nonhomogeneous DE

Theorem *The general solution to the nonhomogeneous DE*

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Proof Because L is linear. Use the following

Lemma *Let* X, Y *be vector spaces. Let* $T : X \longrightarrow Y$ *be a linear transformation. Let* $b \in Y$. *Suppose* a *is an element in* X *satisfying* $Ta = b$. *Then*

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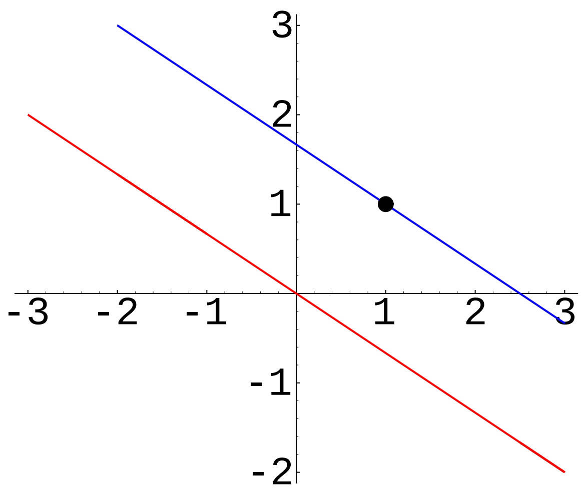

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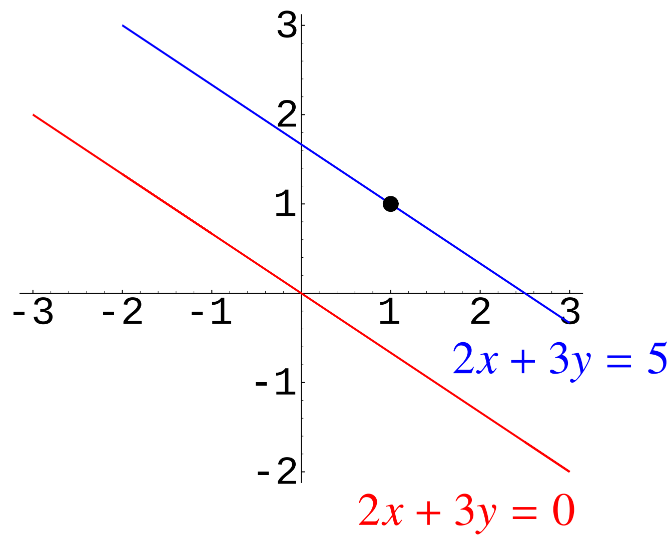

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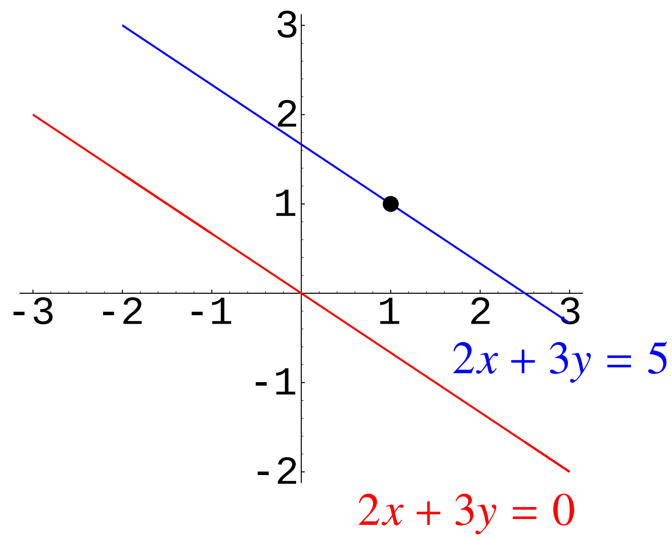

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}, \qquad T(x, y) = 2x + 3y$$

Method of undetermined coefficients

Consider a 2nd order linear *non-homogeneous* DE with *constant coefficients*

$$ay'' + by' + cy = g(t)$$

To solve the DE, just need to find a particular solution.

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- For certain *"familiar"* $g(t)$, there is a particular solution in special form.

Theorem *Consider the DE*

$$ay'' + by' + cy = P_n(t)\mathrm{e}^{\alpha t}$$

where $\alpha, a, b, c \in \mathbb{R}$*,* $a \neq 0$ *and* $P_n(t)$ *is a polynomial of degree* n*.*

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Proof Operator notation

$$a(D - \lambda_1)(D - \lambda_2)y = \mathrm{e}^{\alpha t} P_n(t) \tag{1}$$

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Proof Integration by parts and M.I.

Case 1 α is not a characteristic root *($\alpha \neq \lambda_1$ and $\alpha \neq \lambda_2$)*

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y &= e^{\alpha t} B_n(t) + \frac{C}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + D\, e^{\lambda_2 t}
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w &= e^{\alpha t} A_n(t) + C\, e^{\lambda_1 t} \\
e^{-\lambda_2 t}\left(y' - \lambda_2 y\right) &= \left(e^{\alpha t} A_n(t) + C\, e^{\lambda_1 t}\right) e^{-\lambda_2 t} \\
\left(e^{-\lambda_2 t} y\right)' &= e^{(\alpha-\lambda_2)t} A_n(t) + C\, e^{(\lambda_1-\lambda_2)t} \\
e^{-\lambda_2 t} y &= e^{(\alpha-\lambda_2)t} B_n(t) + \frac{C}{\lambda_1-\lambda_2} e^{(\lambda_1-\lambda_2)t} + D && \text{by Lemma} \\
y &= e^{\alpha t} B_n(t) + \frac{C}{\lambda_1-\lambda_2} e^{\lambda_1 t} + D\, e^{\lambda_2 t}
\end{aligned}
$$

Put $C = D = 0$, get particular solution $y_p(t) = e^{\alpha t} B_n(t)$

Case 2 α is a simple root, may assume $\alpha = \lambda_1, \ \lambda_1 \neq \lambda_2$

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$$\begin{aligned}(e^{-\lambda_1 t} w)' &= e^{(\alpha - \lambda_1)t} P_n(t) \\ &= P_n(t)\end{aligned}$$

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(\mathrm{e}^{-\lambda_1 t} w)' &= \mathrm{e}^{(\alpha-\lambda_1)t} P_n(t) \\
&= P_n(t) \\
\mathrm{e}^{-\lambda_1 t} w &= A_{n+1}(t) + C
\end{aligned}
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w &= \mathrm{e}^{\lambda_1 t} A_{n+1}(t) + C\,\mathrm{e}^{\lambda_1 t}
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\left(e^{-\lambda_2 t} y\right)' &= e^{(\lambda_1-\lambda_2)t} A_{n+1}(t) + C\, e^{(\lambda_1-\lambda_2)t}
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\left(e^{-\lambda_2 t} y\right)' &= e^{(\lambda_1 - \lambda_2)t} A_{n+1}(t) + C\, e^{(\lambda_1 - \lambda_2)t} \\
e^{-\lambda_2 t} y &= e^{(\lambda_1 - \lambda_2)t} B_{n+1}(t) + \frac{C}{\lambda_1 - \lambda_2} e^{(\lambda_1 - \lambda_2)t} + D
\end{aligned}
$$

by Lemma

From above

$$e^{-\lambda_2 t} y = e^{(\lambda_1-\lambda_2)t} B_{n+1}(t) + \frac{C}{\lambda_1 - \lambda_2} e^{(\lambda_1-\lambda_2)t} + D$$

$$y = e^{\lambda_1 t} B_{n+1}(t) + \frac{C}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + D\, e^{\lambda_2 t}$$

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$$
\begin{aligned}
e^{-\lambda_2 t} y &= e^{(\lambda_1-\lambda_2)t} B_{n+1}(t) + \frac{C}{\lambda_1-\lambda_2} e^{(\lambda_1-\lambda_2)t} + D \\
y &= e^{\lambda_1 t} B_{n+1}(t) + \frac{C}{\lambda_1-\lambda_2} e^{\lambda_1 t} + D\, e^{\lambda_2 t} \\
&= e^{\lambda_1 t}\left(b_{n+1} t^{n+1} + \cdots + b_1 t + b_0\right) + \frac{C}{\lambda_1-\lambda_2} e^{\lambda_1 t} + D\, e^{\lambda_2 t}
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From above

$$
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e^{-\lambda_2 t} y &= e^{(\lambda_1-\lambda_2)t} B_{n+1}(t) + \frac{C}{\lambda_1-\lambda_2} e^{(\lambda_1-\lambda_2)t} + D \\
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\end{aligned}
$$

Put $D = 0$ and $C = -b_0(\lambda_1 - \lambda_2)$,

From above

$$\mathrm{e}^{-\lambda_2 t} y = \mathrm{e}^{(\lambda_1-\lambda_2)t} B_{n+1}(t) + \frac{C}{\lambda_1-\lambda_2}\,\mathrm{e}^{(\lambda_1-\lambda_2)t} + D$$

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Particular solution $y_p(t) = \mathrm{e}^{\lambda_1 t}\left(b_{n+1}t^{n+1} + \cdots + b_1 t\right)$

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$$
\begin{aligned}
y_p(t) &= e^{\lambda_1 t}\left(b_{n+1} t^{n+1} + \cdots + b_1 t\right) \\
&= t\, e^{\alpha t} Q_n(t)
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$$

Case 3 α is a double root, $(\alpha = \lambda_1 = \lambda_2)$

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Particular solution
$$\begin{aligned} y_p(t) &= e^{\lambda_2 t}\left(b_{n+2} t^{n+2} + \cdots + b_2 t^2\right) \\ &= t^2\, e^{\alpha t} Q_n(t) \end{aligned}$$

Example Find the general solution to the DE

$$y'' - 3y' - 4y = e^t$$

Solution

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Solution

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  Characteristic roots: $\lambda_1 = -1$ and $\lambda_2 = 4$.

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$$y'' - 3y' - 4y = e^{t} \;=\; 1 \cdot e^{1\,t}$$

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- Substitute into the DE $Ae^t - 3Ae^t - 4Ae^t \;=\; e^t$

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$$-6Ae^t \;=\; e^t$$

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- Substitute into the DE

$$\begin{aligned} Ae^t - 3Ae^t - 4Ae^t &= e^t \\ -6Ae^t &= e^t \\ A &= -\tfrac{1}{6} \end{aligned}$$

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- General solution $y(t) = -\frac{1}{6}e^t + c_1 e^{-t} + c_2 e^{4t}$

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Leibnitz Rule $(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$

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$$y'' - 2y' + y = te^t$$

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- Sub. $(At^3 + Bt^2)e^t + 2(3At^2 + 2Bt)e^t + (6At + 2B)e^t$

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Solution

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- Try particular solution $y_p(t) = t^2(At + B)e^t \quad \because$ 1 is a double root

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- Sub. $(At^3 + Bt^2)e^t + 2(3At^2 + 2Bt)e^t + (6At + 2B)e^t$

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$$y'' - 2y' + y = te^t$$

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Comparing coefficients $\begin{cases} 6A = 1, \\ 2B = 0 \end{cases}$ ....

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Write down the form of a particular solution.

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