Existence and Uniqueness Theorem (for 2nd order linear IVP)

Consider the IVP

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t), & a < t < b, \\ y(t_0) = y_0, \quad y'(t_0) = y_1 \end{cases}$$

where $t_0 \in (a, b)$ *and* $y_0, y_1 \in \mathbb{R}$. *Suppose* p, q *and* g *are continuous on* (a, b). *Then the IVP has a unique solution in* (a, b).

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Corollary *Suppose* φ_1 *and* φ_2 *are solutions to a DE*

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b$$

where $p, q, g \in C(a, b)$ *and* $\exists\, t_0 \in (a, b)$ *such that*

$$\varphi_1(t_0) = \varphi_2(t_0) \quad \text{and} \quad \varphi_1'(t_0) = \varphi_2'(t_0).$$

Then $\varphi_1 \equiv \varphi_2$ *on* (a, b).

Theorem *The solution space for the second order linear homogeneous DE*

$$y'' + p(t)y' + q(t)y = 0, \qquad a < t < b$$

where p and q are continuous on (a, b), is 2-dimensional.

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Proof

Step 1 *"Find"* two solutions y_1, y_2

- Fix $t_0 \in (a, b)$.
- Consider the following IVPs

$$\begin{cases} y'' + p(t)y' + q(t)y = 0, \\ y(t_0) = 1, \quad y'(t_0) = 0, \end{cases} \qquad \begin{cases} y'' + p(t)y' + q(t)y = 0, \\ y(t_0) = 0, \quad y'(t_0) = 1. \end{cases}$$

- By E&U Theorem, above IVP's have unique solutions y_1 and y_2 respectively.

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Step 2 Show y_1, y_2 form a basis for solution space.

- Apply next thm $\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

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Need to show

- y_1 and y_2 are linearly independent *elements in* $C^2(a, b) \subset C(a, b)$
- y_1 and y_2 span whole solution space

Definition A *finite subset* $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ (where $\mathbf{x}_i \neq \mathbf{x}_j$ for $i \neq j$) of a linear space E is said to be

- *linearly dependent* if there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, *not all zero*, such that

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- S is *linearly independent* means that *non-trivial linear combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are never $\mathbf{0}$*.

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$$\dim E = \text{number of elements in } B$$

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- Since $\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} \neq 0$, it follows that $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

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Definition Let $y_1, y_2 \in C^1(a,b)$. The *Wronskian* of y_1, y_2 is the function $W(y_1, y_2) : (a,b) \longrightarrow \mathbb{R}$ given by

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Will show/consider

- If the Wronskian is *non-zero somewhere*, it is *non-zero everywhere*.
- The relation between Wronskian and linear dependency.

Terminology We say that two functions y_1 and y_2 are linearly dependent *on an interval* (a,b) if $\{y_1, y_2\}$ is a linearly *dependent subset* of the **vector space of all functions on** (a,b)

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- $\Longleftarrow$ *(in Remark 1)* not true in general.
- But *true if* y_1 *and* y_2 *are solutions to* $y'' + p(t)y' + q(t)y = 0$, $a < t < b$, *where* $p, q \in C(a, b)$

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Consider $\begin{cases} \alpha y_1(t_0) + \beta y_2(t_0) = 0 \\ \alpha y_1'(t_0) + \beta y_2'(t_0) = 0 \end{cases}$ system of linear equations

Determinant of coefficient matrix is

$$\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = W(y_1, y_2)(t_0) = 0 \qquad \text{by assumption}$$

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By E&U Theorem, $c_1 y_1 + c_2 y_2 \equiv 0$ on (a, b).

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- Can use Wronskian to check linear dependency.

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Remark

- Can use Wronskian to check linear dependency.
- Wronskian is either always 0 or never 0.

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Suffice to prove the formula for the wronskian.