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$$(D - \lambda_2)[y] = Ae^{\lambda_1 t}$$

Case 1 $b^2 - 4ac > 0$

Case 2 $b^2 - 4ac = 0$

Case 3 $b^2 - 4ac < 0$

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Case 1 $b^2 - 4ac > 0$ (λ_1 *and* λ_2 *are distinct and real*)

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Solving $y' - \lambda_2 y = Ae^{\lambda_1 t}$ integrating factor $e^{-\lambda_2 t}$

$$e^{-\lambda_2 t}y' - \lambda_2 e^{-\lambda_2 t}y = Ae^{\lambda_1 t}e^{-\lambda_2 t}$$

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General solution $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ *where* c_1 *and* c_2 *are arbitrary constants*

Case 2 $b^2 - 4ac = 0$ $(\lambda_1 = \lambda_2)$

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Case 2 $b^2 - 4ac = 0$ $(\lambda_1 = \lambda_2)$

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Case 2 $b^2 - 4ac = 0$ $(\lambda_1 = \lambda_2)$

Solving

$$
\begin{aligned}
y' - \lambda_1 y &= A e^{\lambda_1 t} \qquad \text{integrating factor } e^{-\lambda_1 t} \\
e^{-\lambda_1 t} y' - \lambda_1 e^{-\lambda_1 t} y &= A \\
(e^{-\lambda_1 t} y)' &= A \\
e^{-\lambda_1 t} y &= At + B
\end{aligned}
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Complex-valued functions

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Differentiation of complex-valued functions

Let $\varphi : (a, b) \longrightarrow \mathbb{C}$ be a complex-valued function.

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- If $z(t) = \mathrm{e}^{\lambda t}$, then $z'(t) = \lambda z(t)$ and $z''(t) = \lambda^2 z(t)$

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- If $z(t) = \mathrm{e}^{\lambda t}$, then $z'(t) = \lambda z(t)$ and $z''(t) = \lambda^2 z(t)$
- Hence $z_1(t)$ and $z_2(t)$ satisfy $ay'' + by' + cy = 0$ $(*)$

- Moreover, all complex-valued function satisfying $(*)$ can be written in the form $c_1 z_1(t) + c_2 z_2(t)$ where $c_1, c_2 \in \mathbb{C}$.

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Proposition *Suppose φ is a complex-valued function satisfying a linear homogeneous DE*

$$a_n(t)y^{(n)} + \cdots + a_1(t)y' + a_0(t)y = 0$$

all $a_i(t)$ are real-valued functions. Then $\mathrm{Re}\,\varphi$ and $\mathrm{Im}\,\varphi$ are solutions to the DE.

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Solutions to $(*)$

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Solutions to $(*)$

- $y_1(t) = e^{\alpha t}\cos\beta t, \qquad y_2(t) = e^{\alpha t}\sin\beta t$
- Linear combinations over $\mathbb{R}$ of y_1 and y_2 are also solutions.

Claim All (real) solutions to $(*)$ can be written as $c_1 y_1(t) + c_2 y_2(t)$ where $c_1, c_2 \in \mathbb{R}$

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$$\varphi(t) = Az_1(t) + Bz_2(t), \qquad -\infty < t < \infty$$

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$t = \pi/2\beta$ $\quad \varphi(\pi/2\beta) = Ae^{\frac{\alpha\pi}{2\beta}}(\cos\frac{\pi}{2} + \mathrm{i}\ \sin\frac{\pi}{2}) + Be^{\frac{\alpha\pi}{2\beta}}(\cos\frac{\pi}{2} - \mathrm{i}\ \sin\frac{\pi}{2}) =$

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Conclusion General solution to DE $\quad y(t) = c_1e^{\alpha t}\cos\beta t + c_2e^{\alpha t}\sin\beta t$

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Solution Space and Wronskian

Consider the DE

$$y'' + p(t)y' + q(t)y = 0, \qquad a < t < b \tag{1}$$

where p and q are continuous functions on (a, b).

- The solutions to (1) form a vector subspace of $C(a, b)$.

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Theorem *The general solution for*

$$y' + p(t)y = 0, \qquad a < t < b,$$

where $p \in C(a, b)$, is

$$y(t) = C\mathrm{e}^{-H(t)}, \qquad a < t < b,$$

where H is a primitive for p on (a, b).

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For (2), need condition(s) on y_1 and y_2 so that

- they are linearly independent,
- they span the whole solution space.

Existence and Uniqueness Theorem (for 2nd order linear IVP)

Consider the IVP

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- Matrix form $\dfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix}\begin{bmatrix} y(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} 0 \\ g(t) \end{bmatrix}, \qquad \begin{bmatrix} y(t_0) \\ z(t_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$

  That is, $\mathbf{x}'(t) = \mathbf{P}(t)\mathbf{x}(t) + \mathbf{g}(t), \qquad \mathbf{x}(t_0) = \mathbf{x}_0$

- Apply result for 1st order linear system IVP.

Corollary *Suppose φ_1 and φ_2 are solutions to a DE*

$$y'' + p(t)y' + q(t)y = g(t), \quad a < t < b$$

where $p, q, g \in C(a, b)$ and $\exists\, t_0 \in (a, b)$ such that

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