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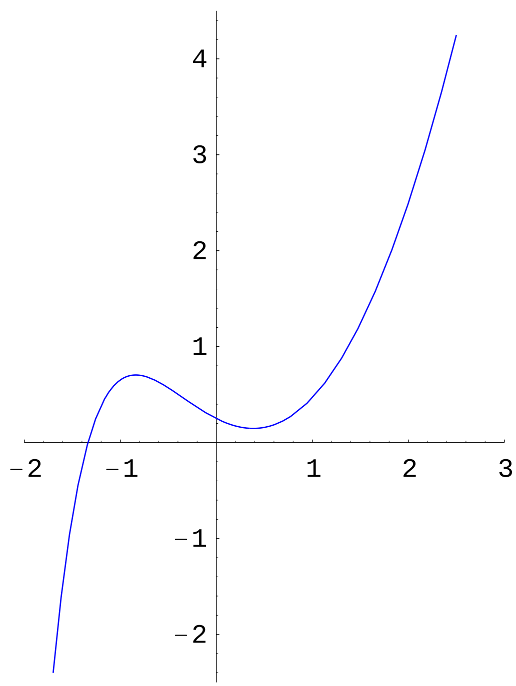

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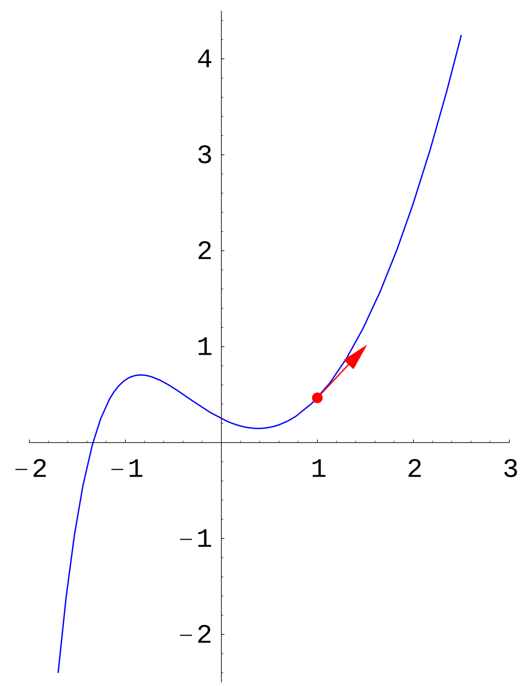

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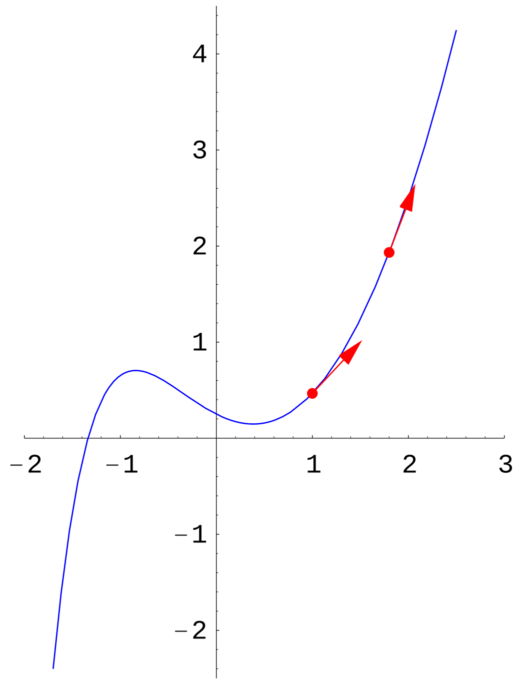

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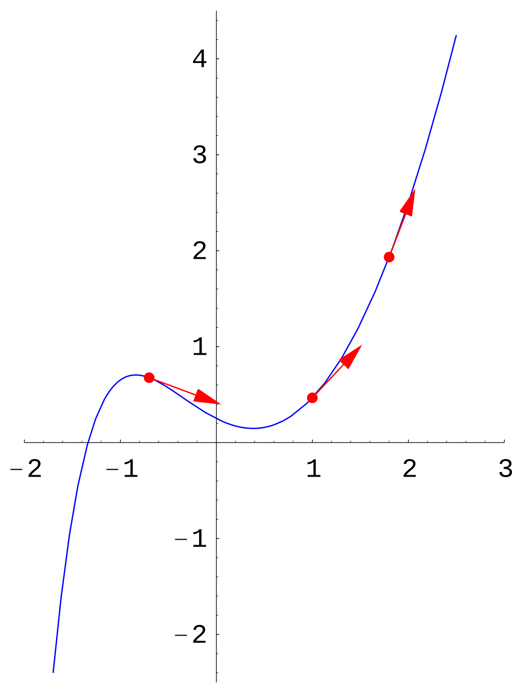

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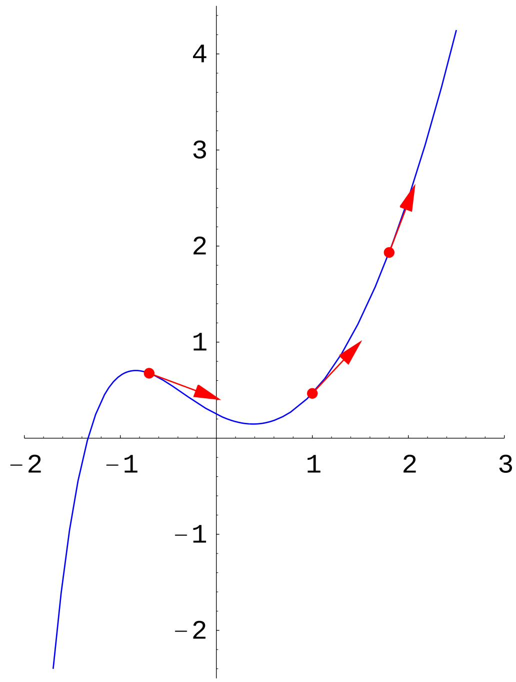

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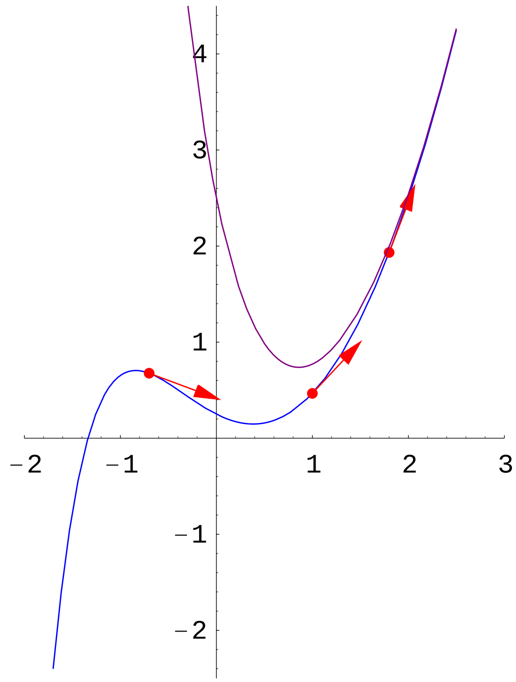

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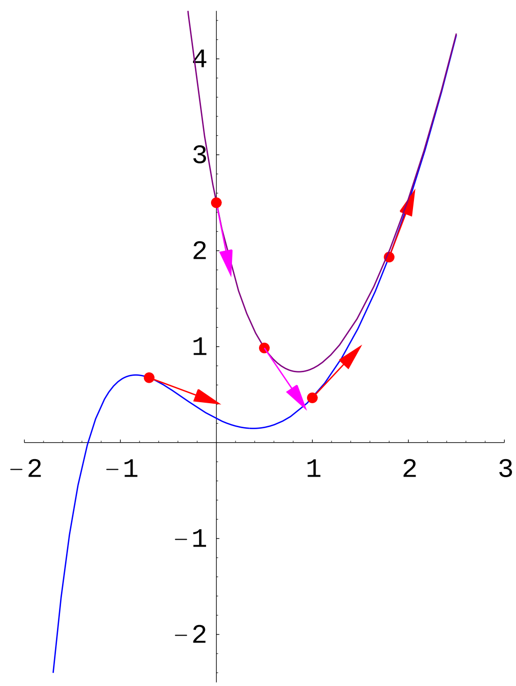

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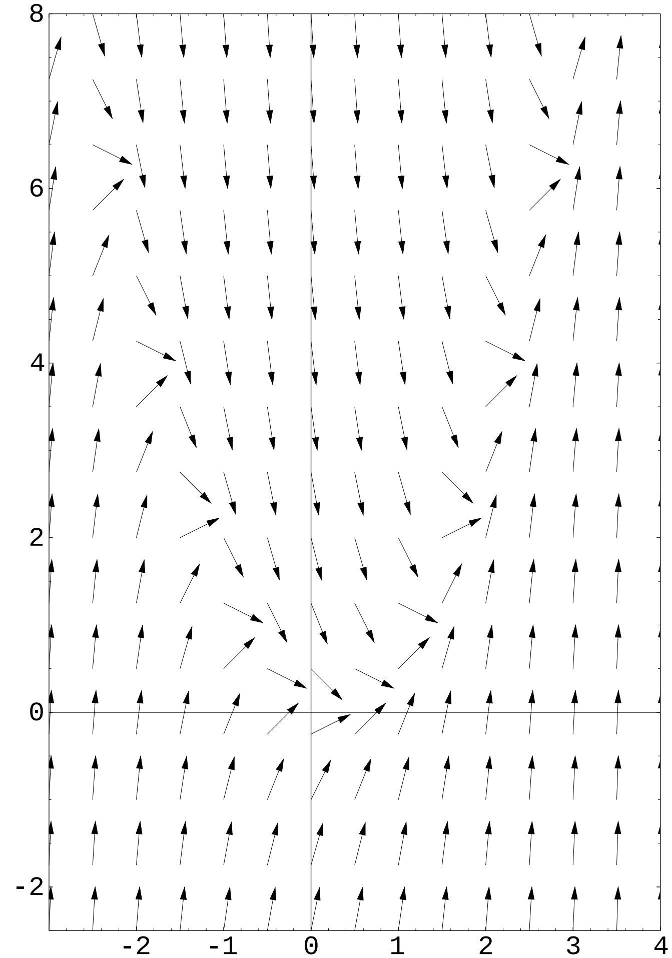

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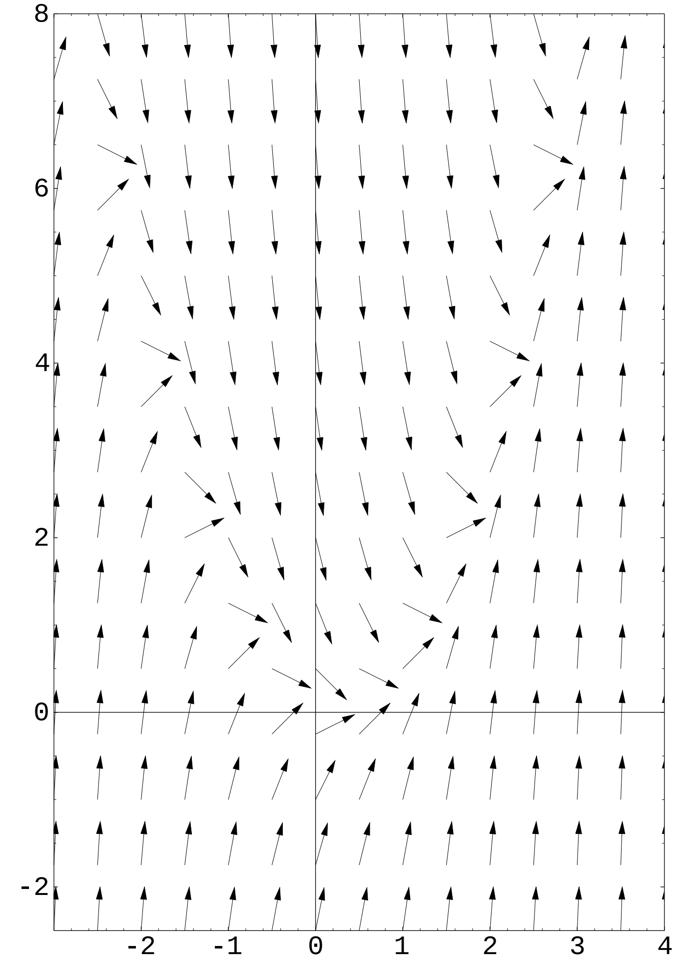

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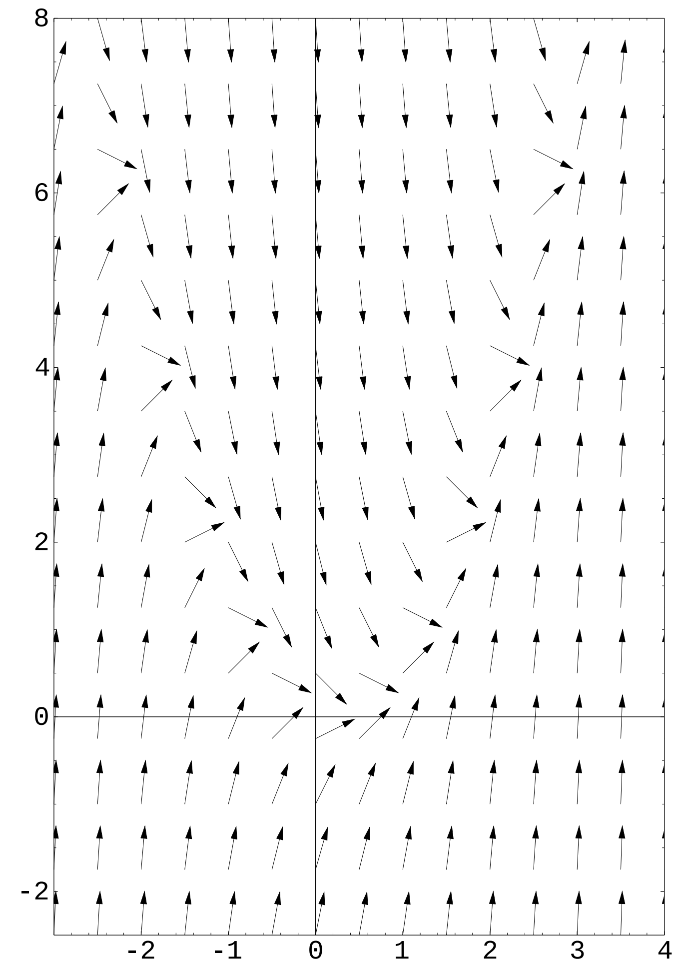

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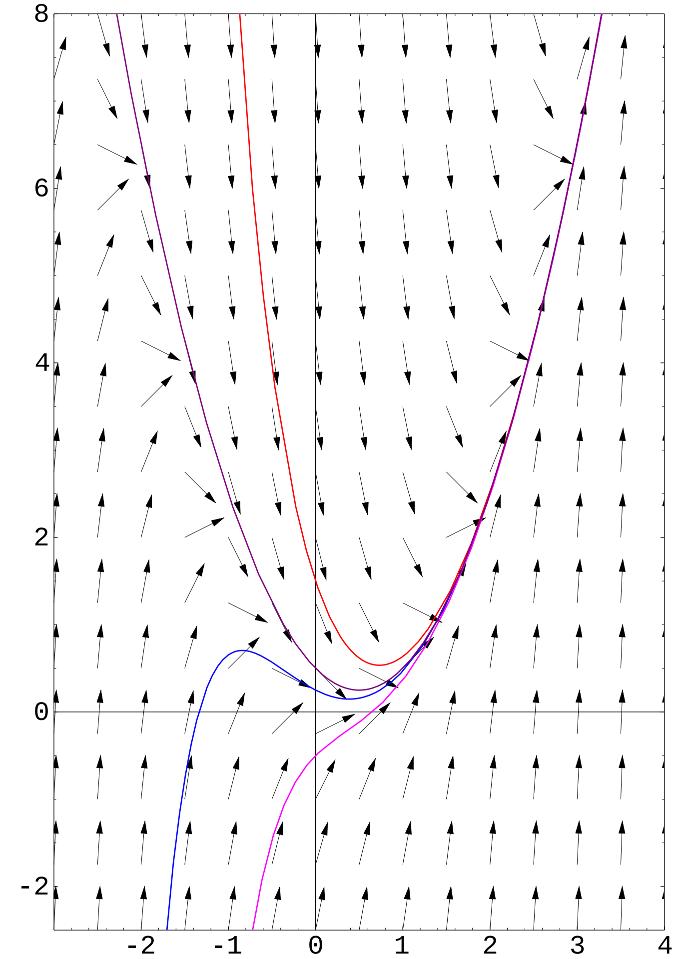

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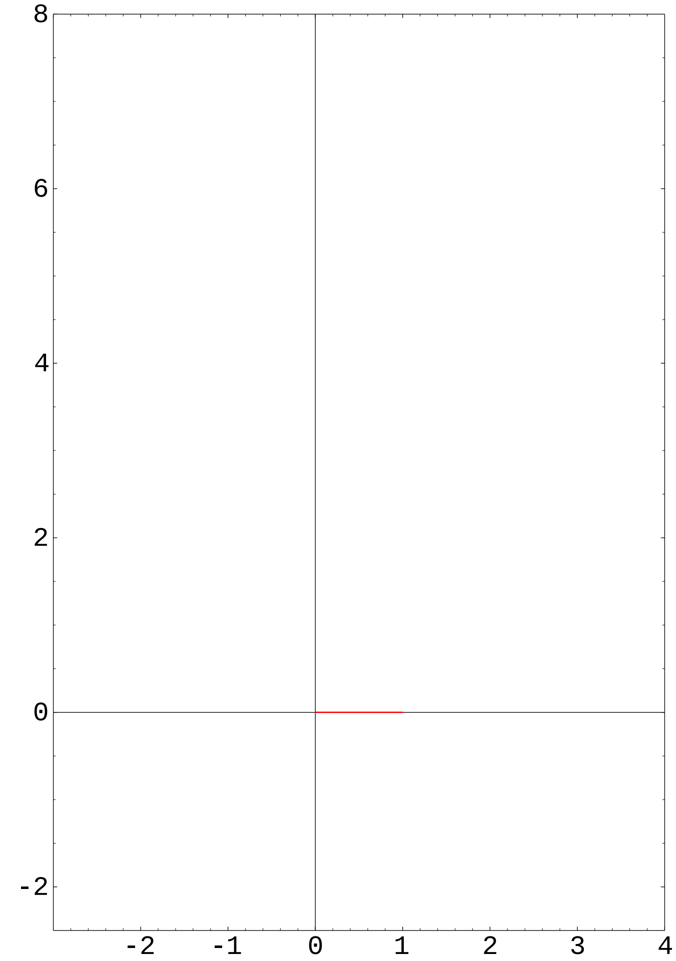

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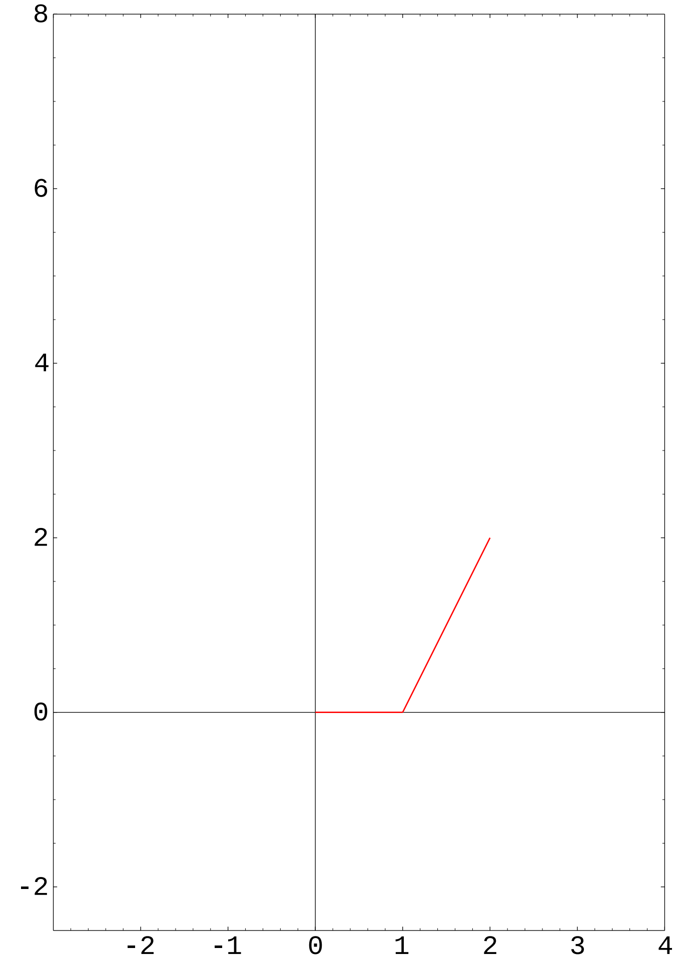

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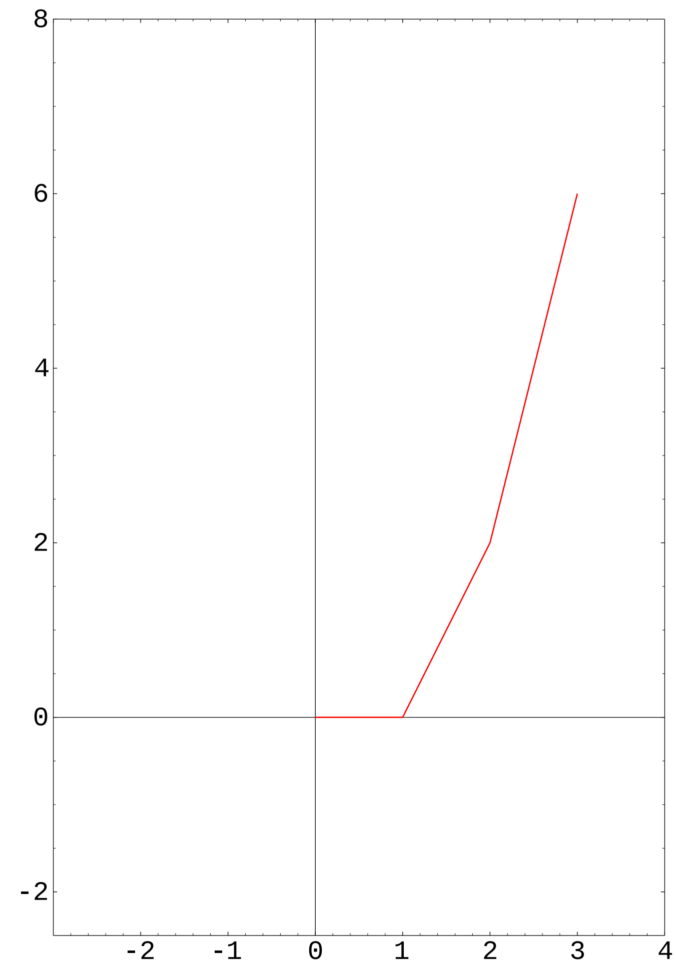

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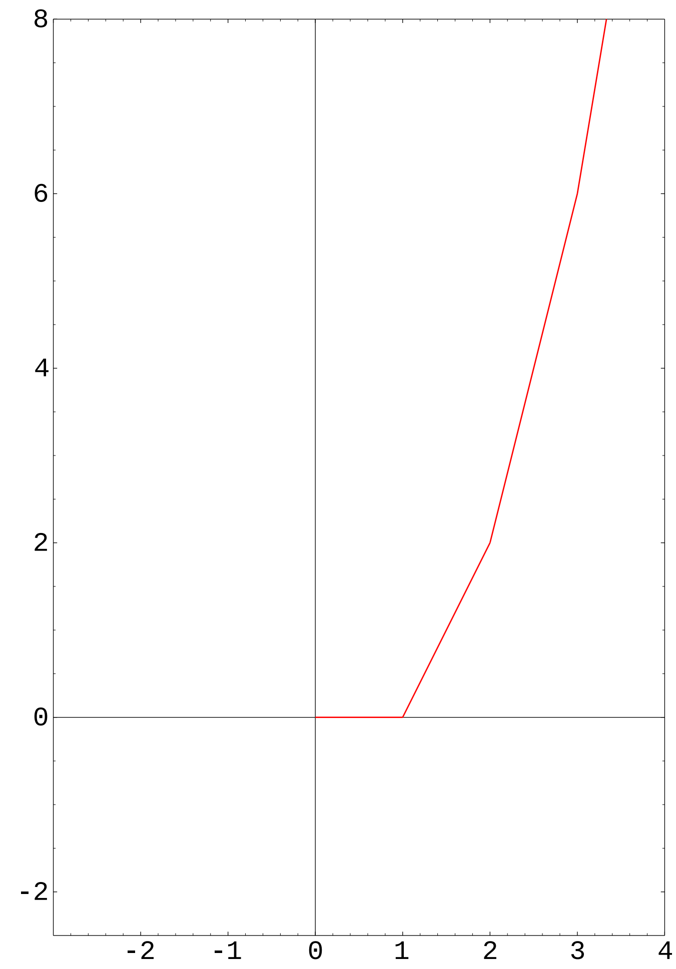

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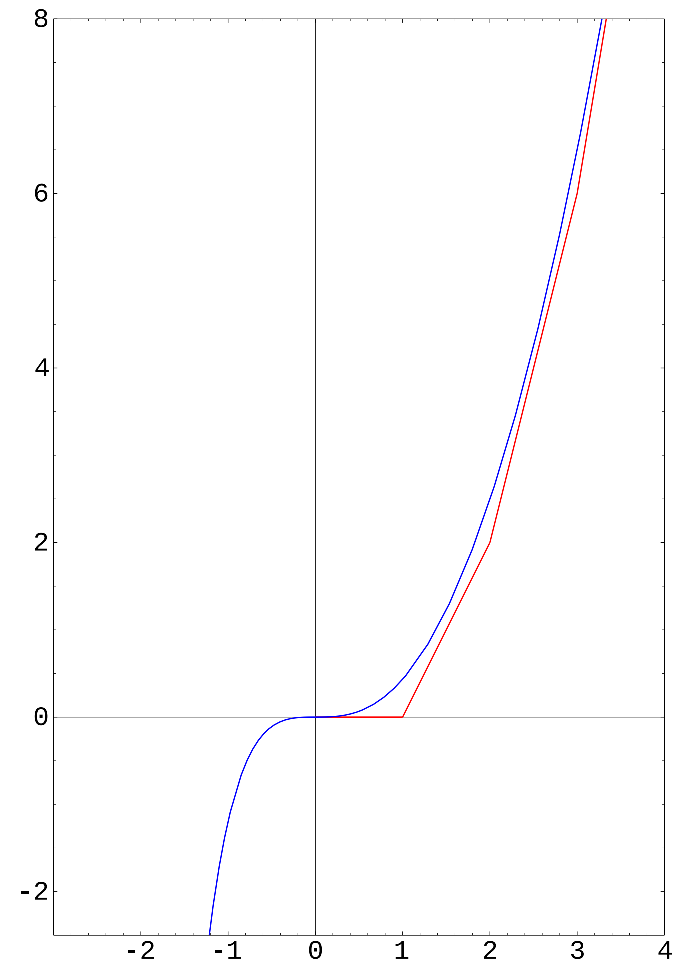

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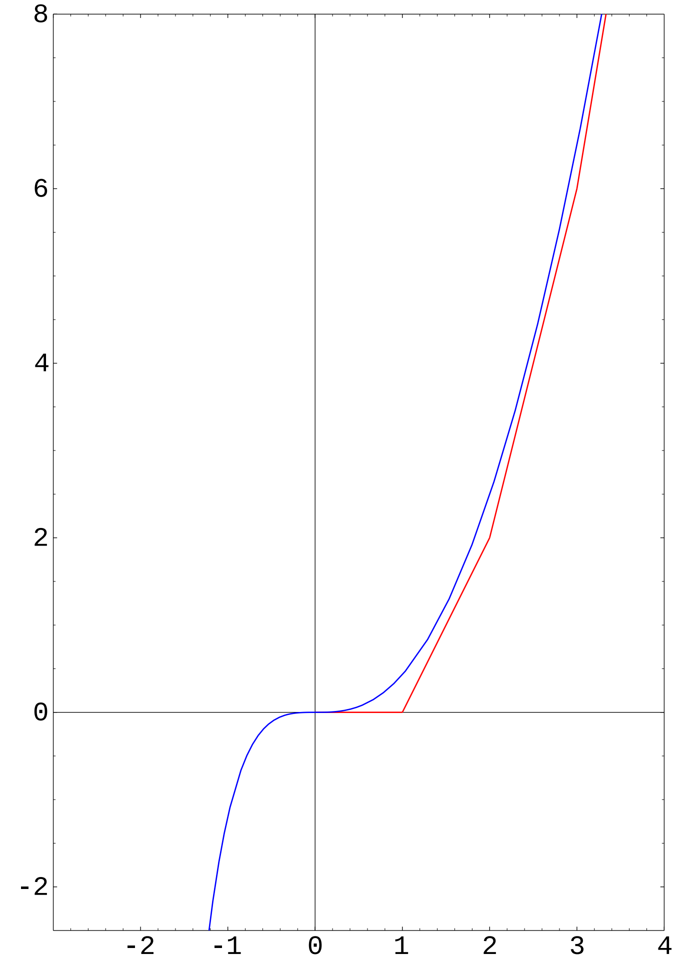

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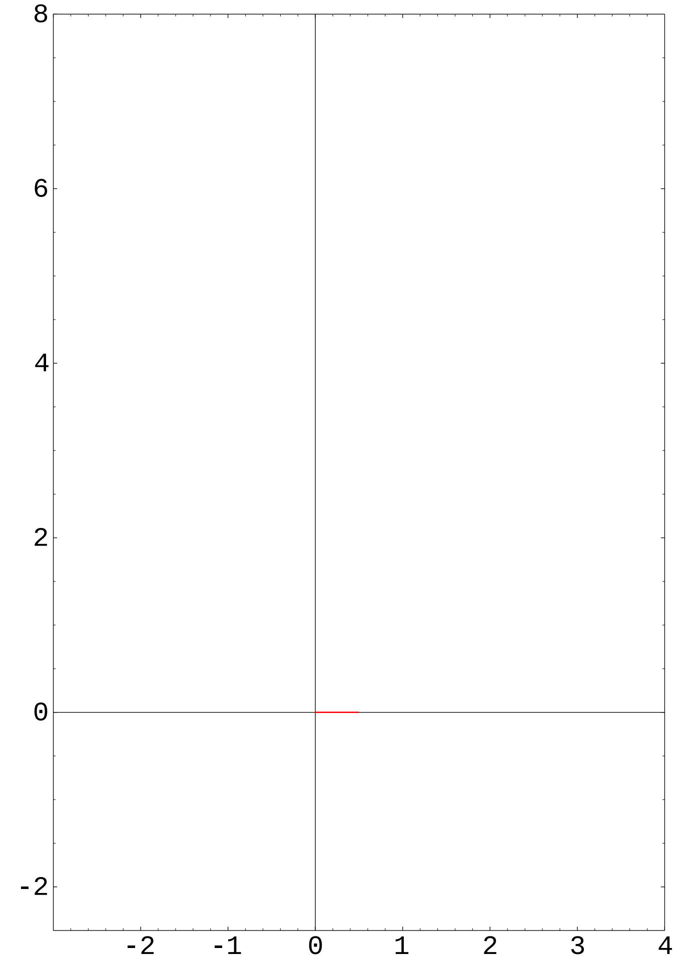

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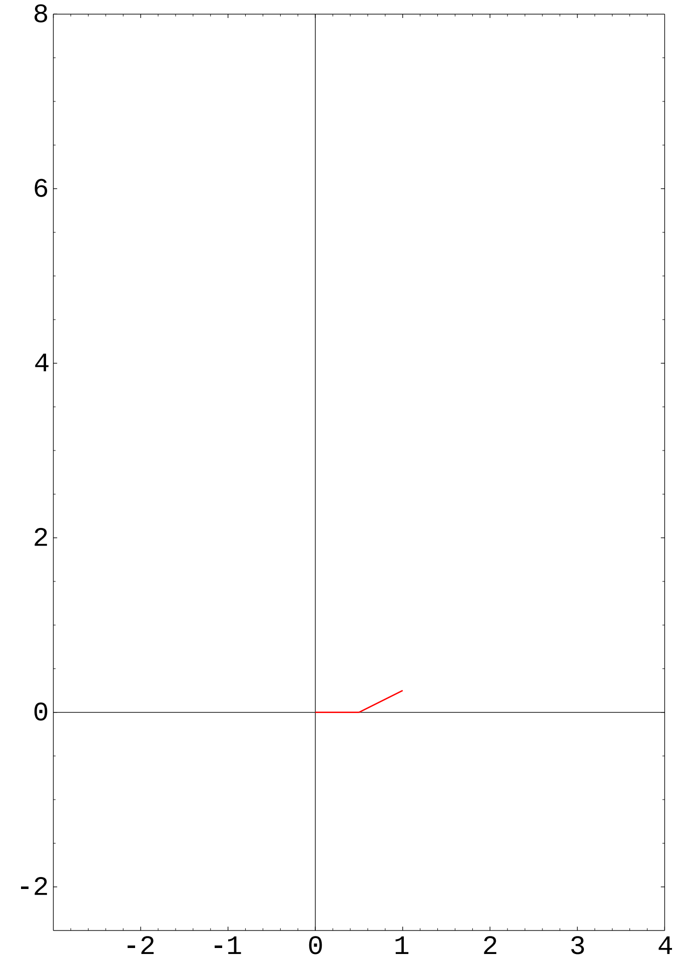

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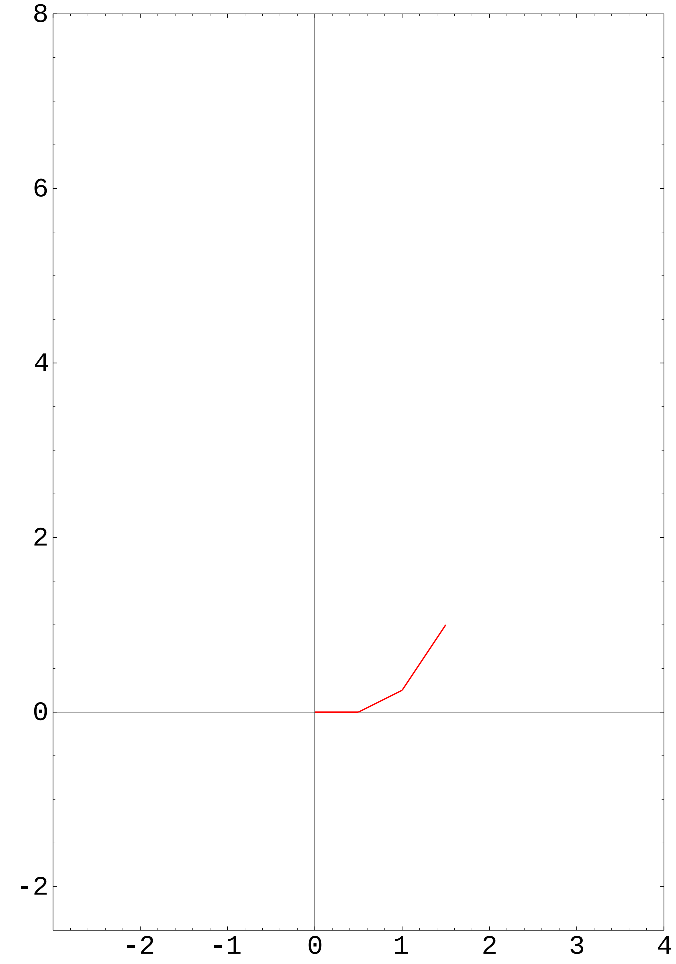

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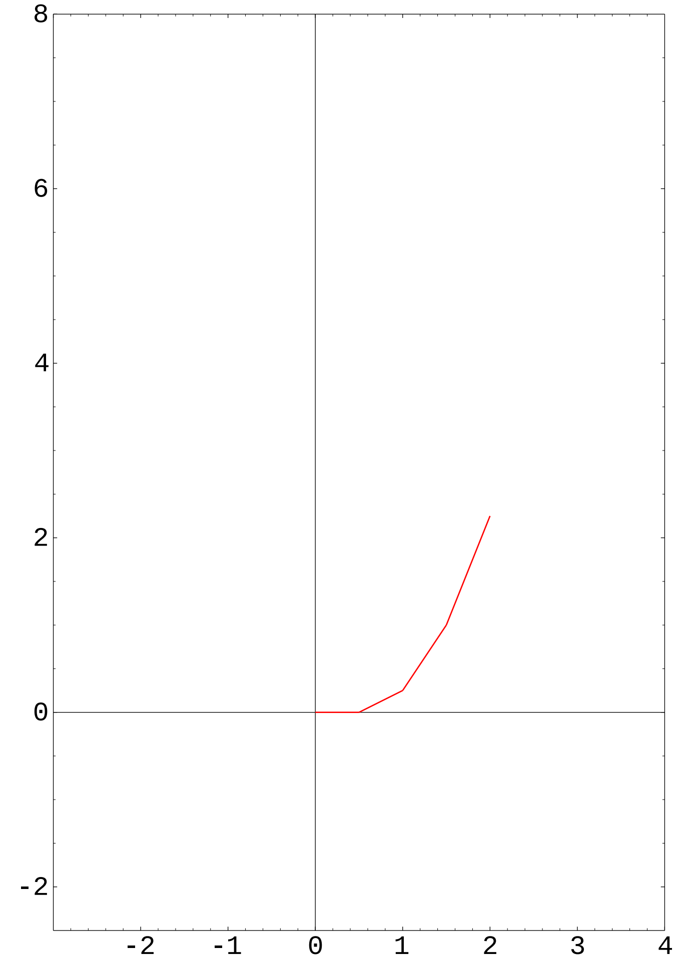

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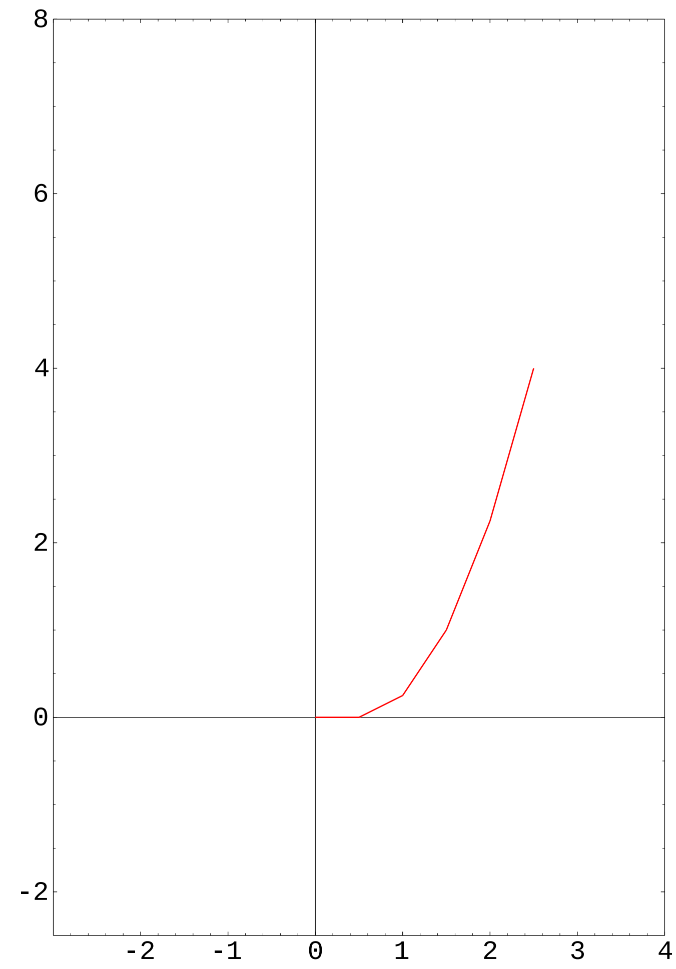

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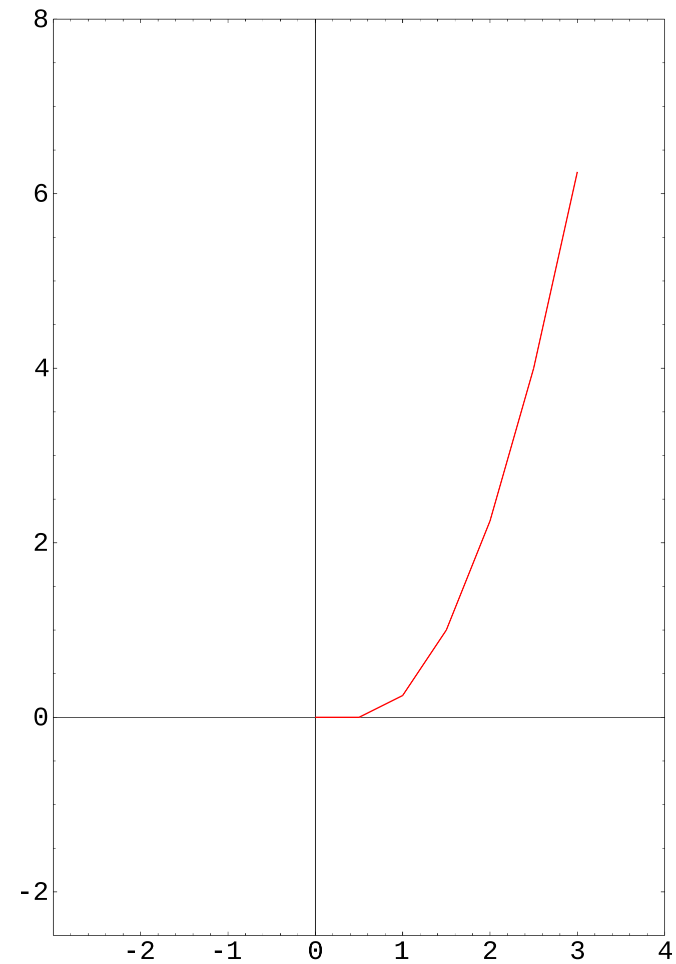

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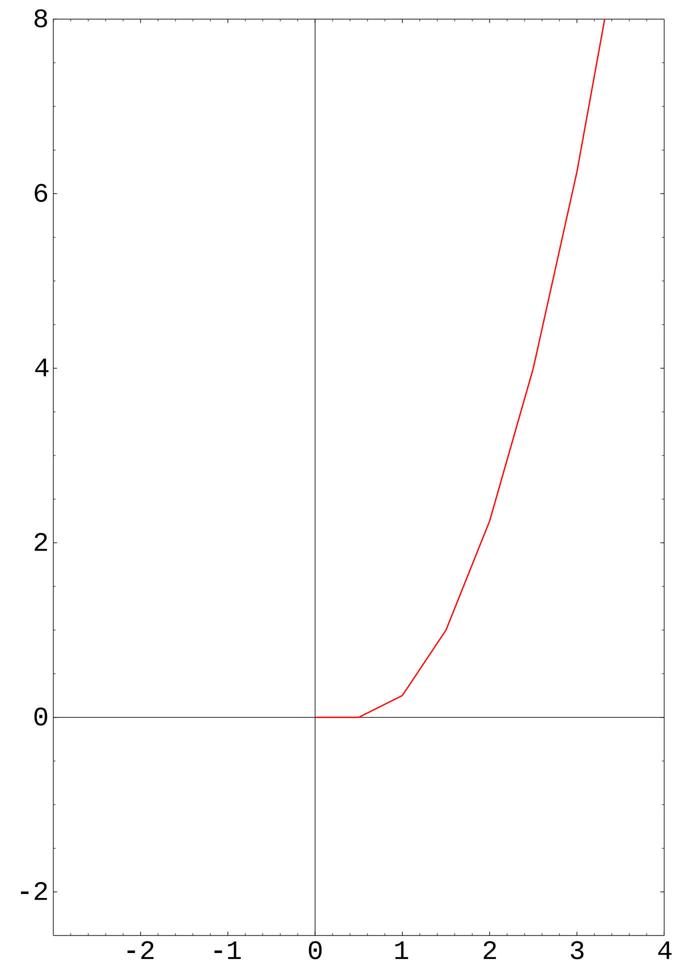

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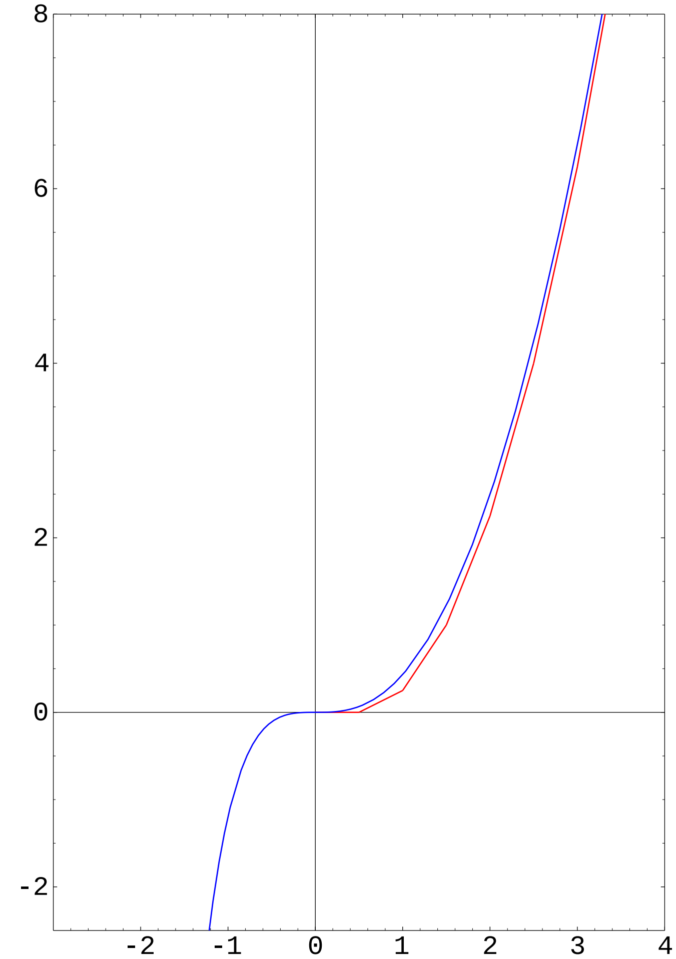

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F is called a *potential function* for $\mathbf{f}$.

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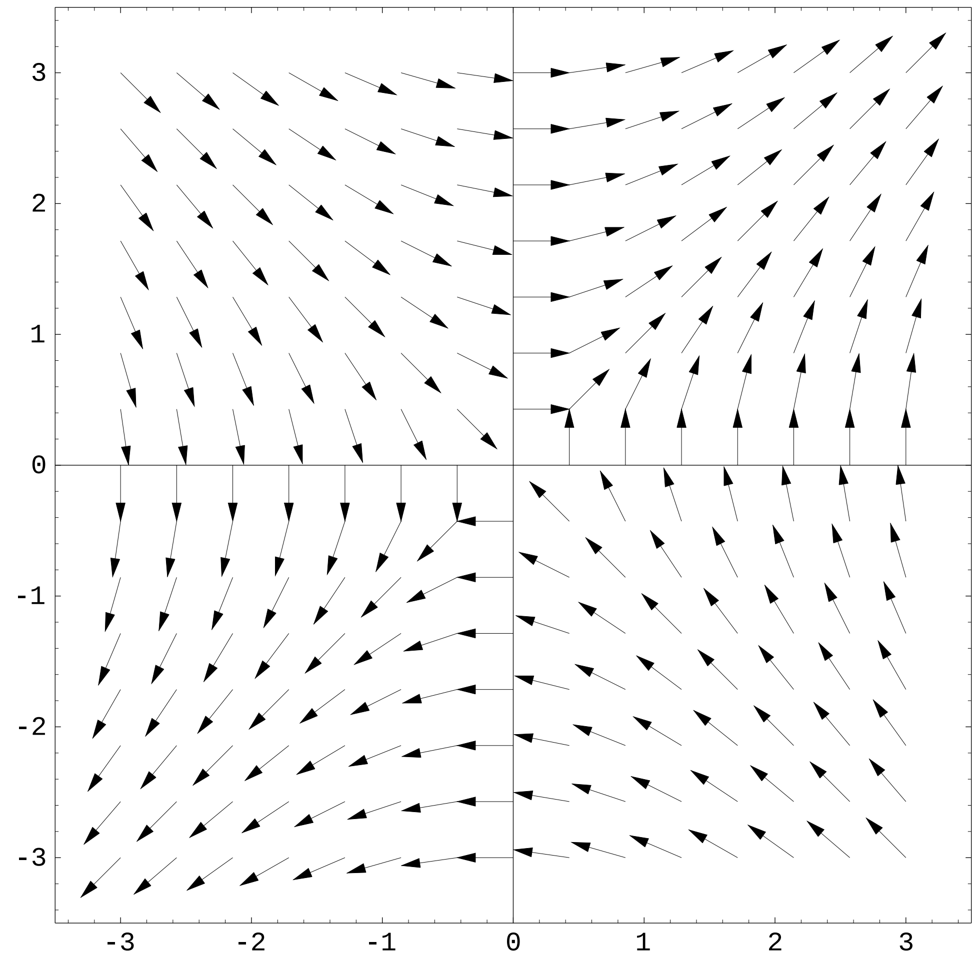

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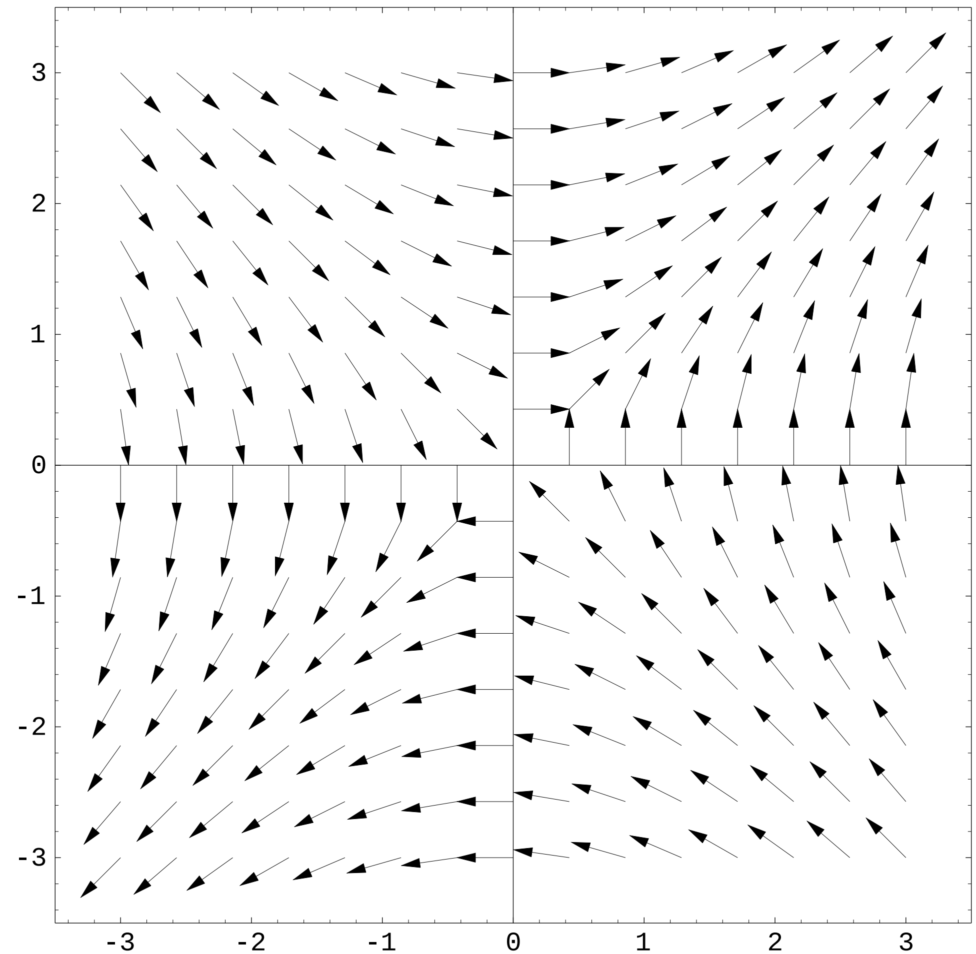

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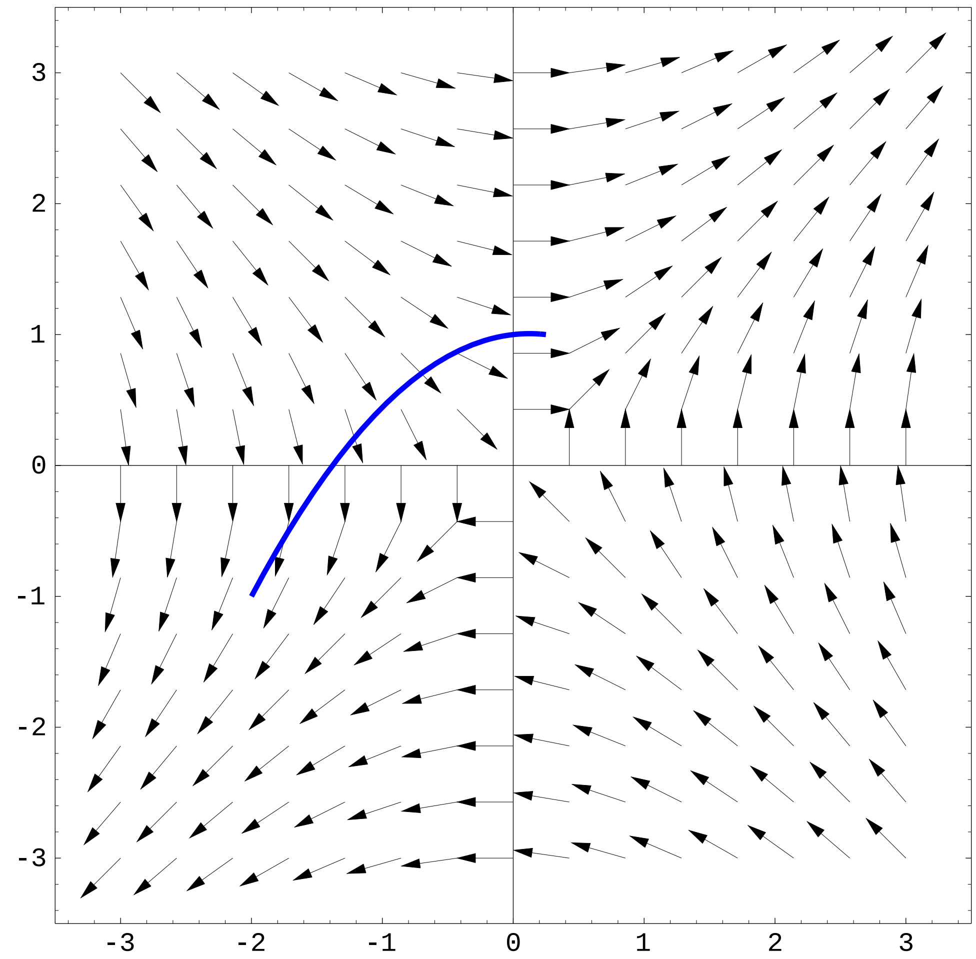

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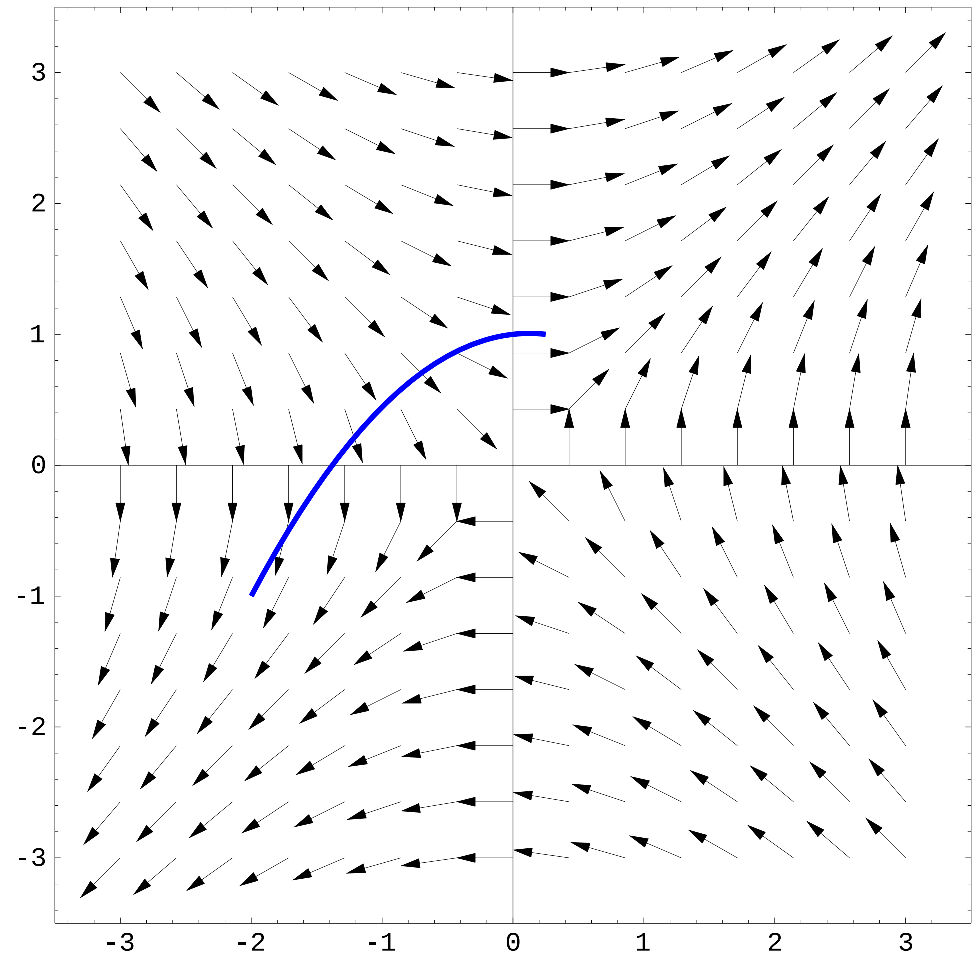

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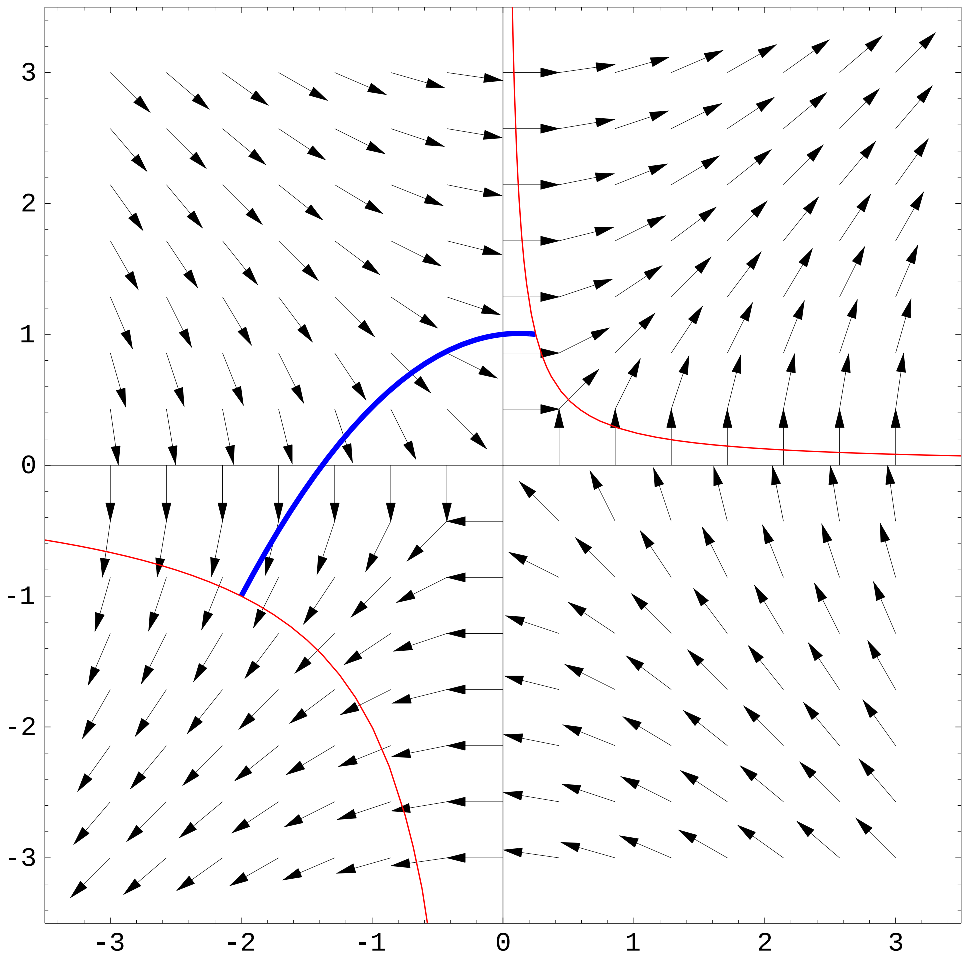

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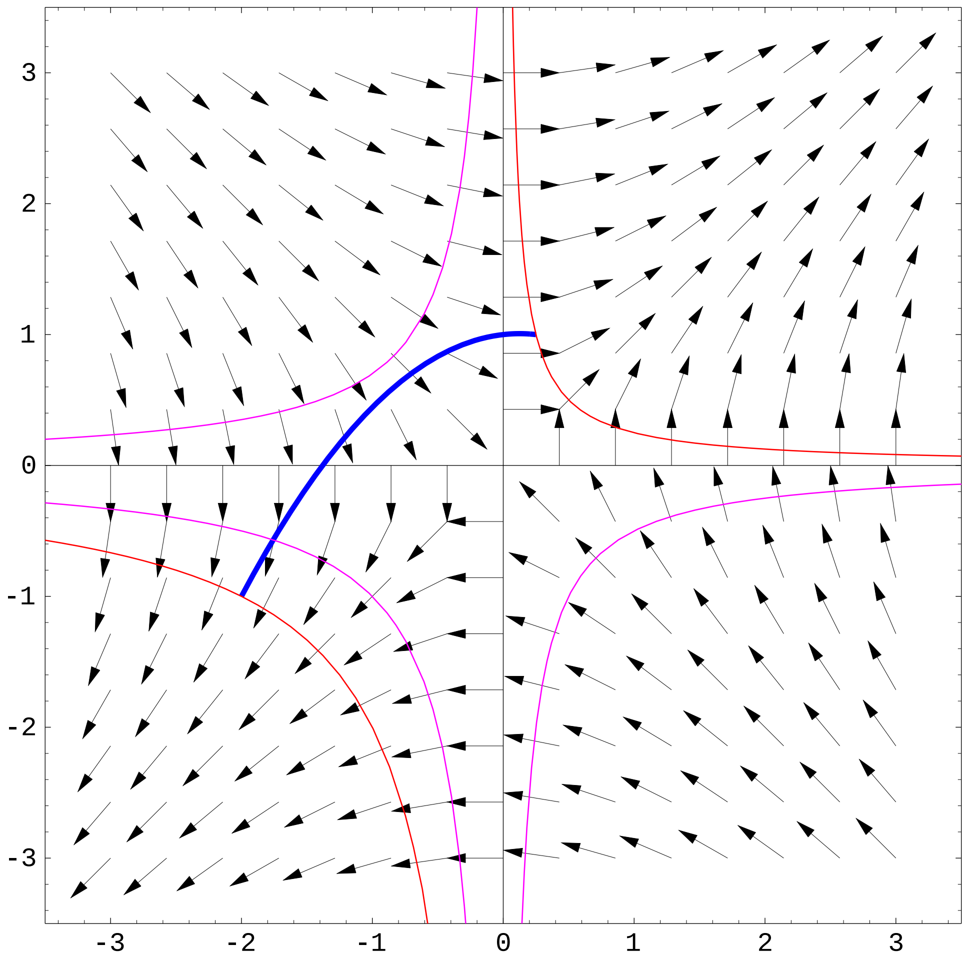

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- in explicit form $\quad y' = \dfrac{-M}{N}, \qquad$ direction field $\quad (1, \frac{-M}{N})$

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- How about double root, for example $y'' + 2y' + y = 0$?

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- $\cdots \overset{\text{vector subspace}}{\subset} C^2(a,b) \overset{\text{vector subspace}}{\subset} C^1(a,b) \overset{\text{vector subspace}}{\subset} C(a,b)$

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$$\begin{aligned}(D^2 + pD + q)[y] &= D^2[y] + pD[y] + q[y] \\ &= y'' + py' + qy\end{aligned}$$

Write $\quad L = D^2 + pD + q : C^2(a, b) \longrightarrow C(a, b), \quad$ DE takes the form

$$L[y] = g$$

Proposition $L : C^2(a,b) \longrightarrow C(a,b)$ *is linear, that is,*

$$L[c_1u_1 + c_2u_2] = c_1L[u_1] + c_2L[u_2]$$

for all $u_1, u_2 \in C^2(a,b)$, $c_1, c_2 \in \mathbb{R}$.

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Proof Exercise.

Proposition $L : C^2(a,b) \longrightarrow C(a,b)$ *is linear, that is,*

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Reason

- $L : C^2(a,b) \longrightarrow C(a,b)$ is linear.
- $\{\text{solution to the DE}\} =$ kernel of L

2nd order linear homogeneous DE with constant coefficients

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- General solution $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$?

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(1) Let α and β be *constants*. Show that

$$(D - \alpha)(D - \beta) = D^2 - (\alpha + \beta)D + \alpha\beta$$

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$$y' - \lambda_2 y = A\mathrm{e}^{\lambda_1 t}$$

integrating factor $\mathrm{e}^{-\lambda_2 t}$

$$\mathrm{e}^{-\lambda_2 t}y' - \lambda_2 \mathrm{e}^{-\lambda_2 t}y = A\mathrm{e}^{\lambda_1 t}\mathrm{e}^{-\lambda_2 t}$$

$$(\mathrm{e}^{-\lambda_2 t}y)' = A\mathrm{e}^{(\lambda_1 - \lambda_2)t}$$

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General solution $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$ *where* c_1 *and* c_2 *are arbitrary constants*

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e^{-\lambda_1 t} y &= At + B
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