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$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

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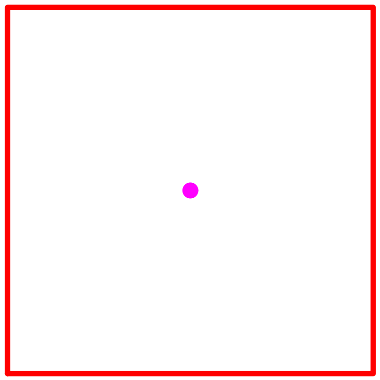

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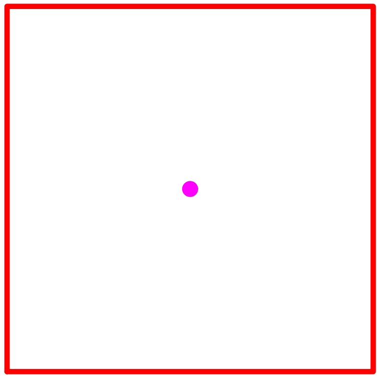

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(1) f *is continuous on R;*

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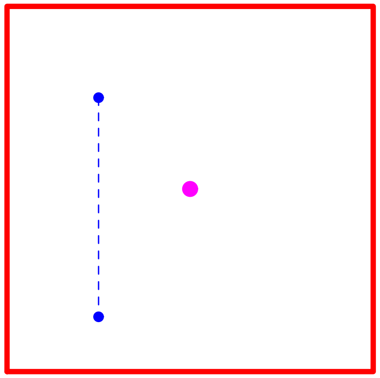

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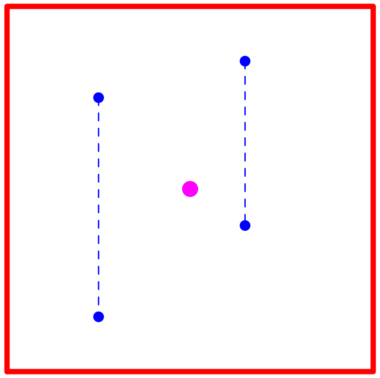

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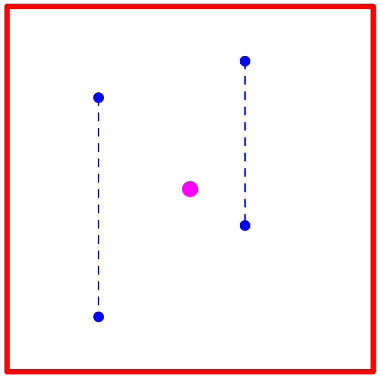

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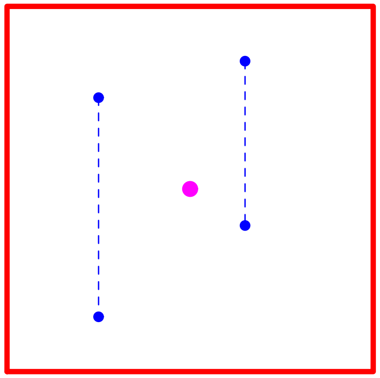

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- If (2) holds, say that $f(t, y)$ satisfies a *Lipschitz condition* with respect to the *2nd argument*.

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- If f and f_y are cont. in a nbd of (t_0, y_0), then $\exists\ R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$ on which f and f_y are continuous.

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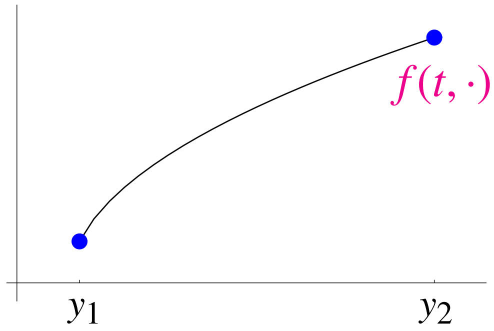

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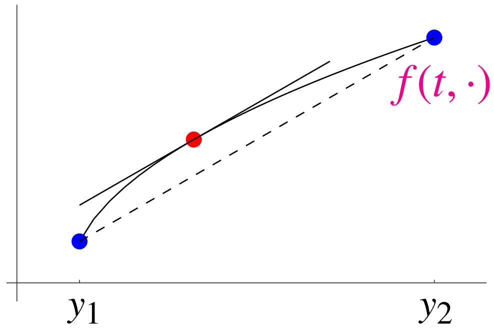

Compare with previous version

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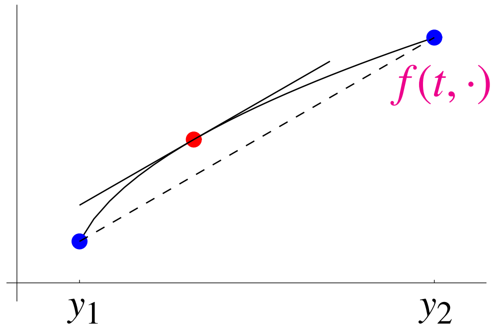

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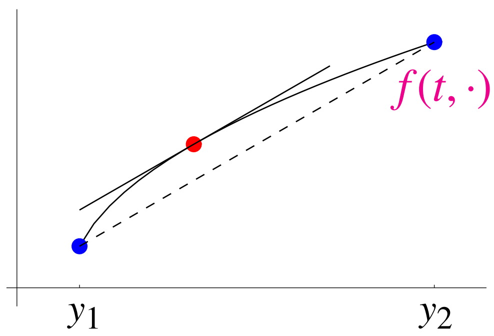

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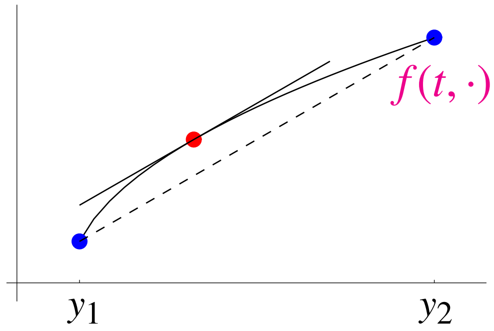

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Idea of Proof of E&U Thm

$y = \varphi(t)$ is solution IVP in neighborhood of $t_0 \Longleftrightarrow$

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Picard iteration method (φ_n are called *Picard iterates*)

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

f and f_y continuous on $R = [t_0 - a, t_0 + a] \times [y_0 - b, y_0 + b]$

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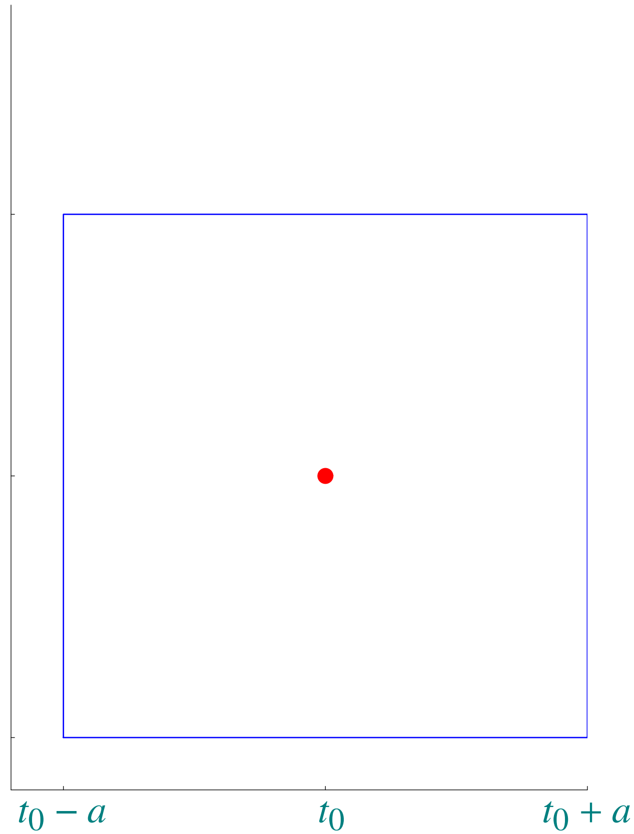

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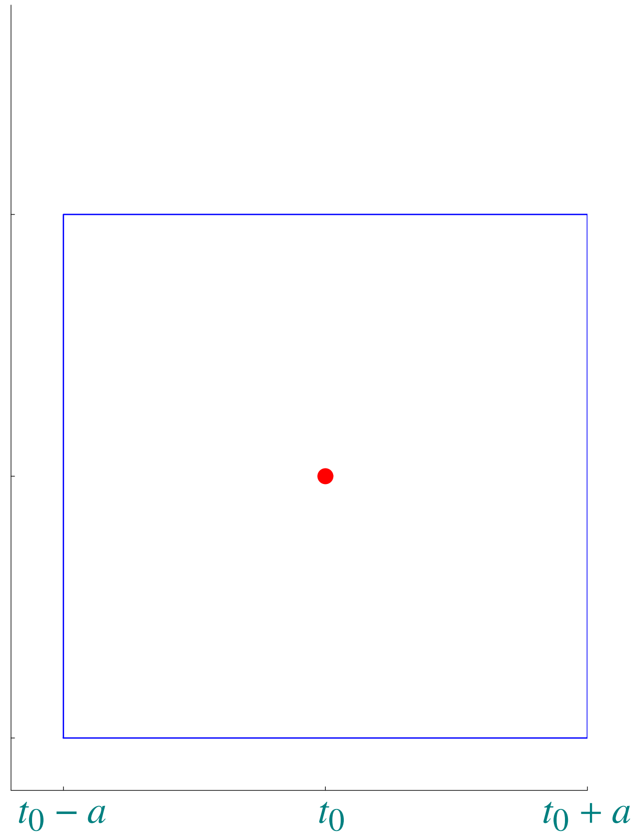

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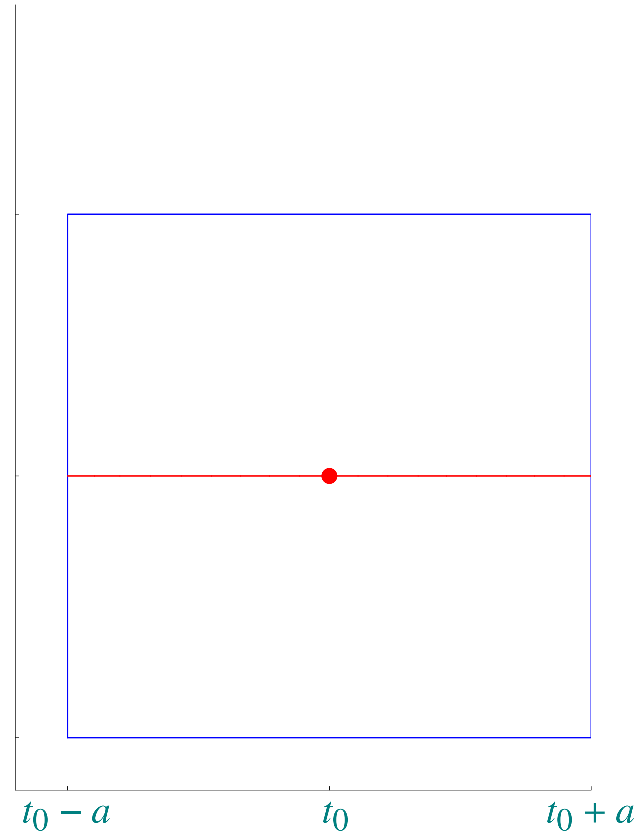

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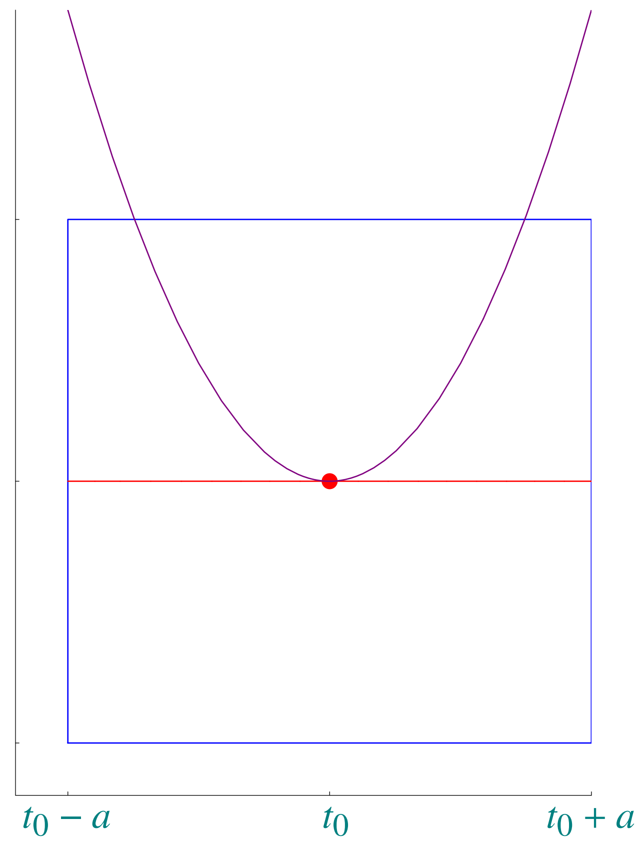

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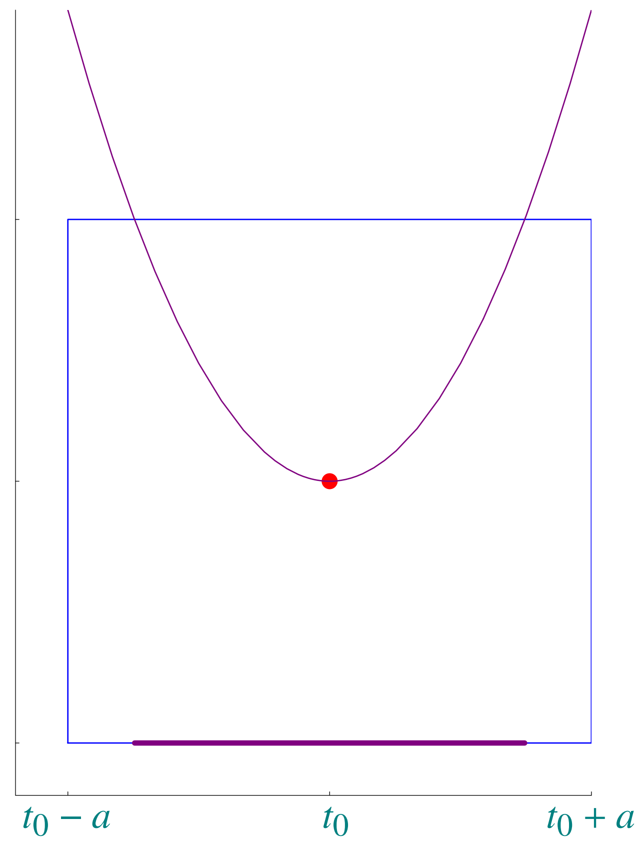

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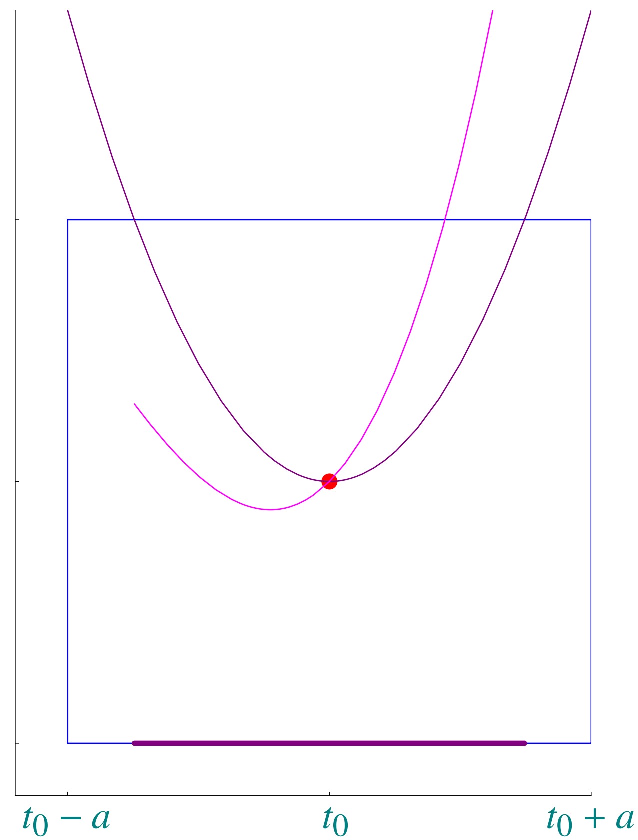

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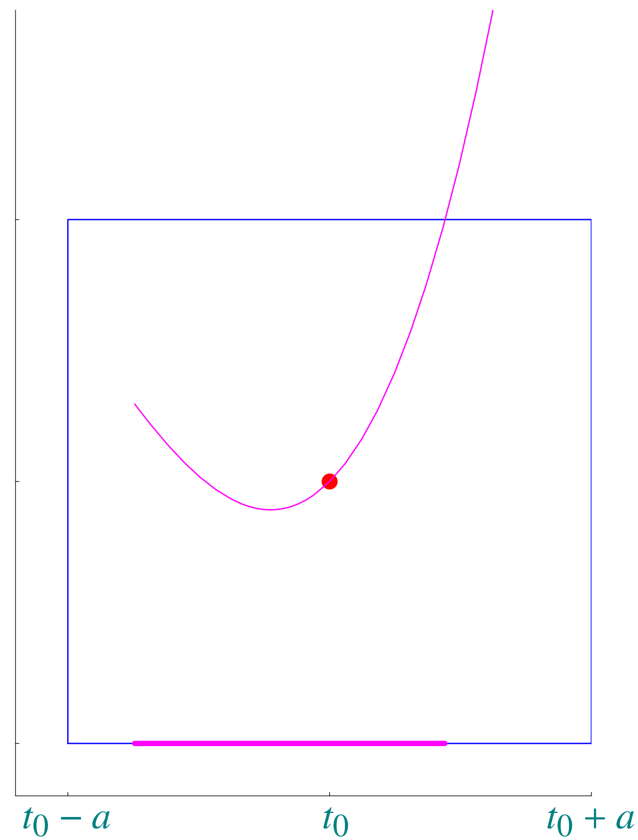

Example Use the Picard iteration method to solve $\begin{cases} y' = y + 1 - t \\ y(0) = 0. \end{cases}$

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E&U Thm for First Order Linear IVP

$$\begin{cases} y' + p(t)y = g(t), & a < t < b \\ y(t_0) = y_0. \end{cases}$$

where $t_0 \in (a, b)$ *and* $y_0 \in \mathbb{R}$. *Suppose* $p, g \in C(a, b)$. *Then the IVP has a unique solution in* (a, b).

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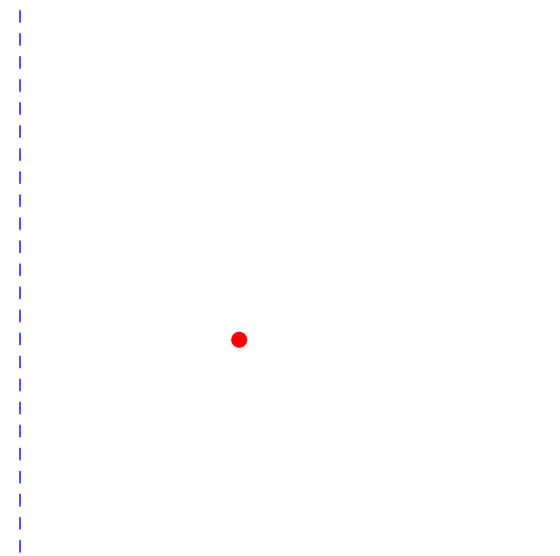

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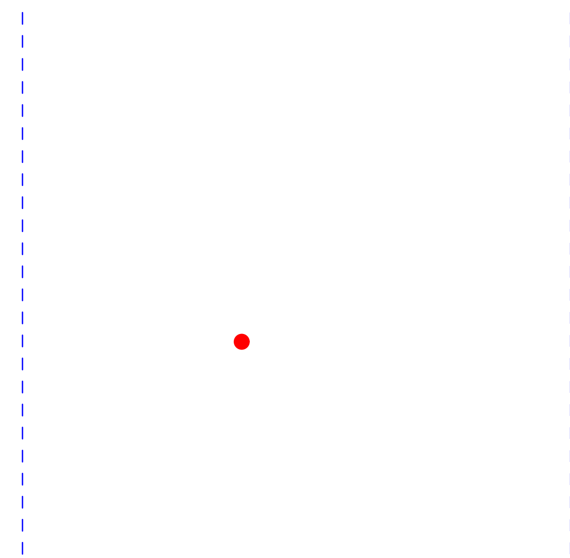

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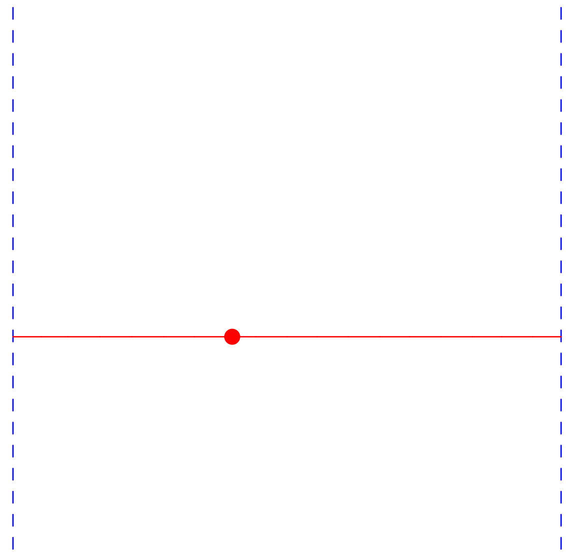

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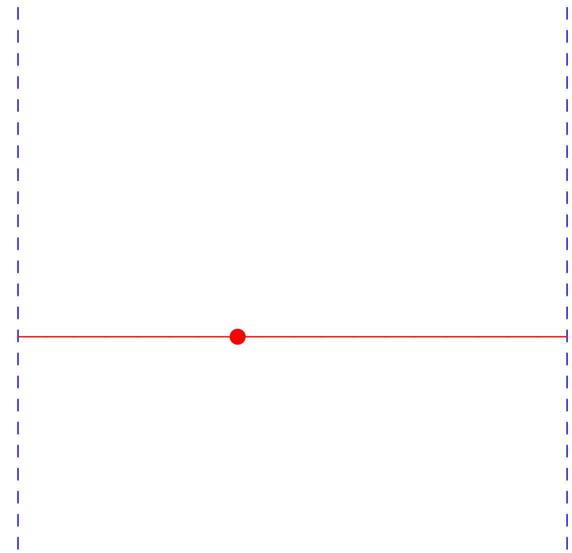

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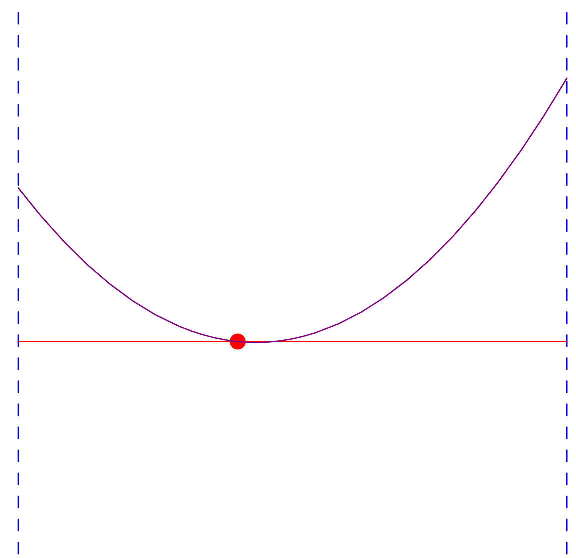

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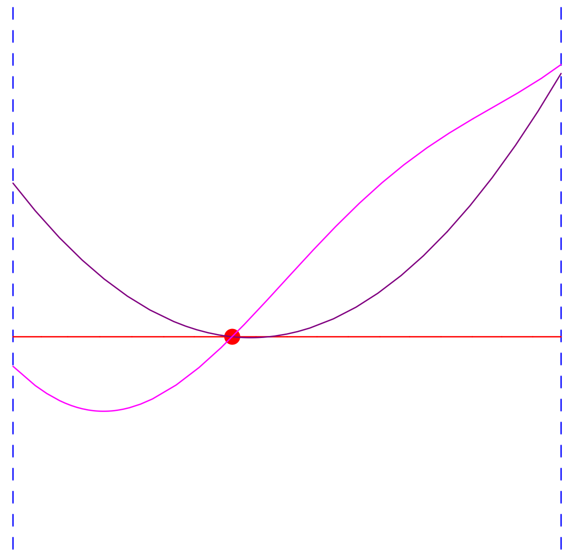

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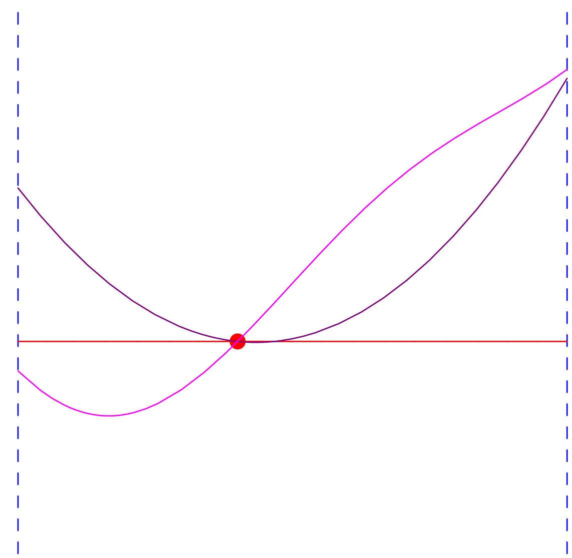

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