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$$x\,\mathrm{d}x + \frac{y}{\sin y}\,\mathrm{d}y = 0$$ (variables separated)

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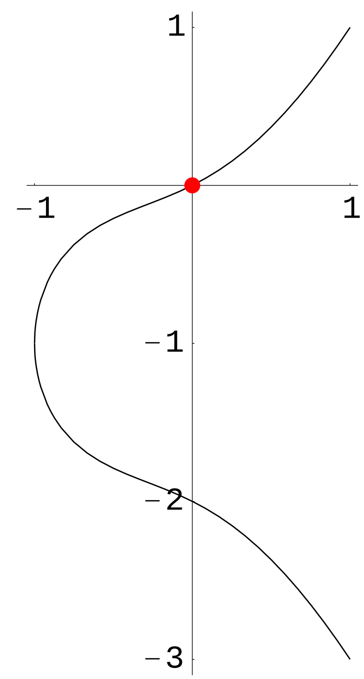

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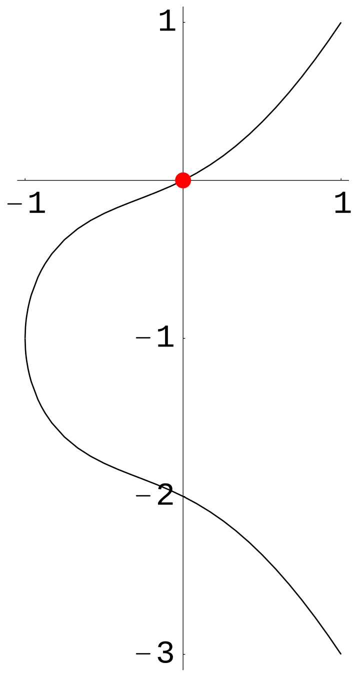

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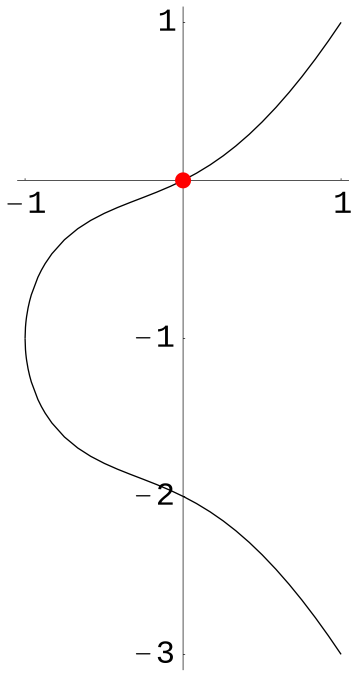

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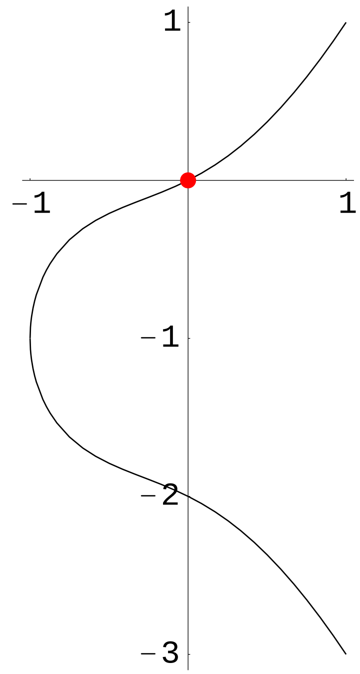

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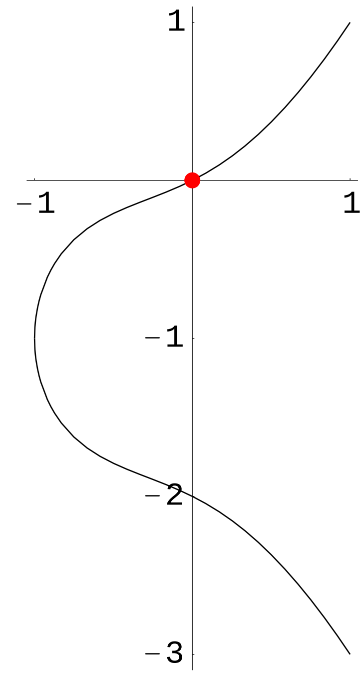
$y \to -1$ as $x \to -1^+$



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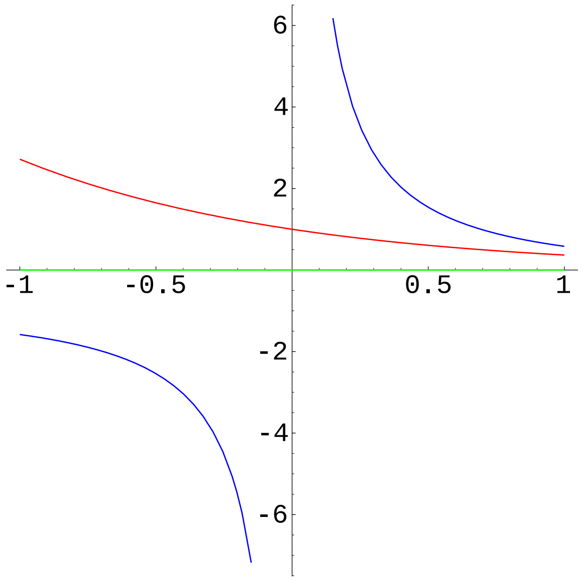

Exact Equations

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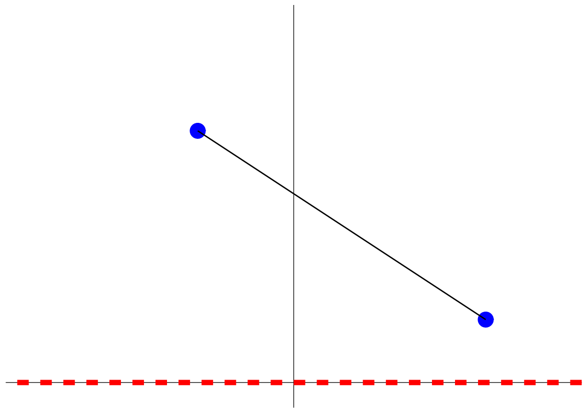

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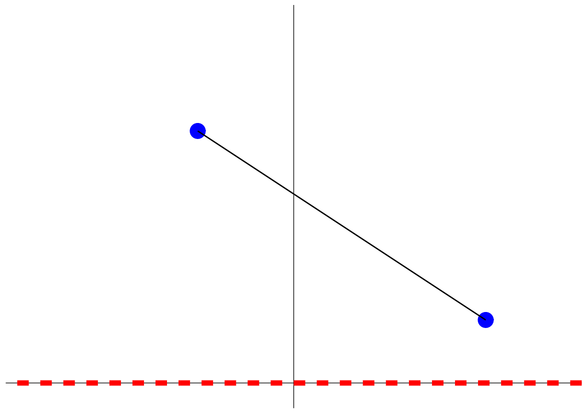

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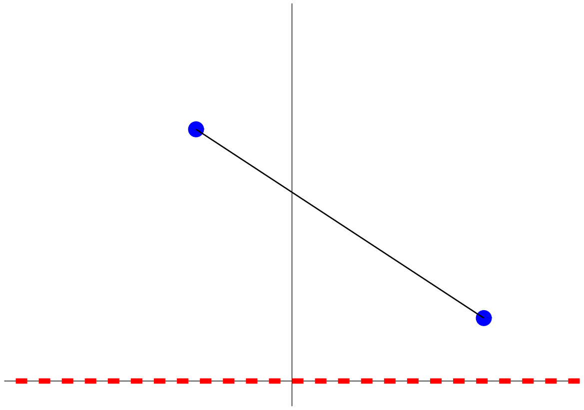

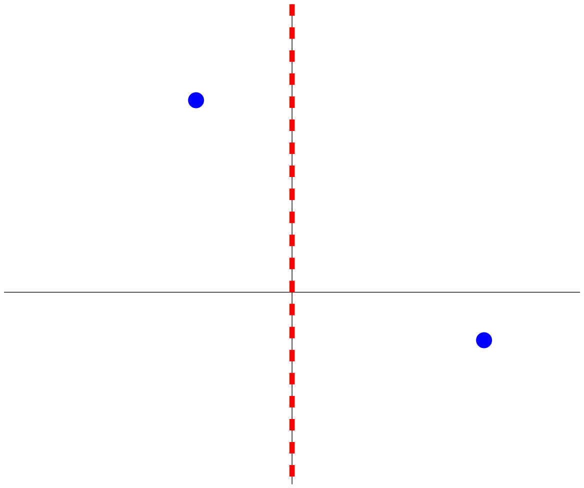

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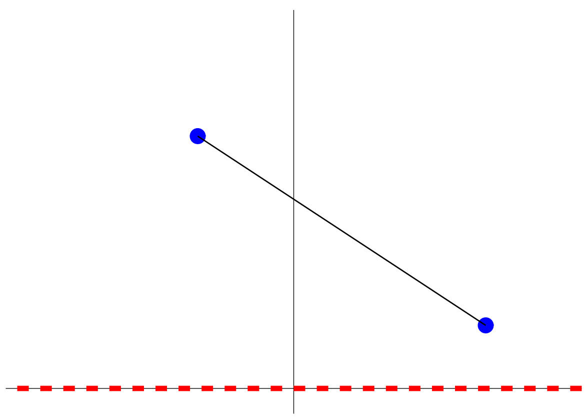

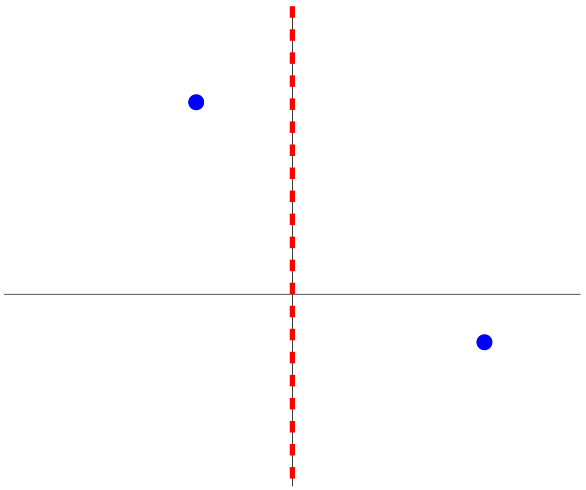

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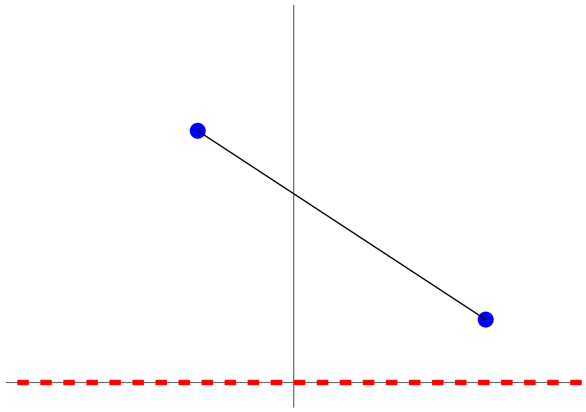

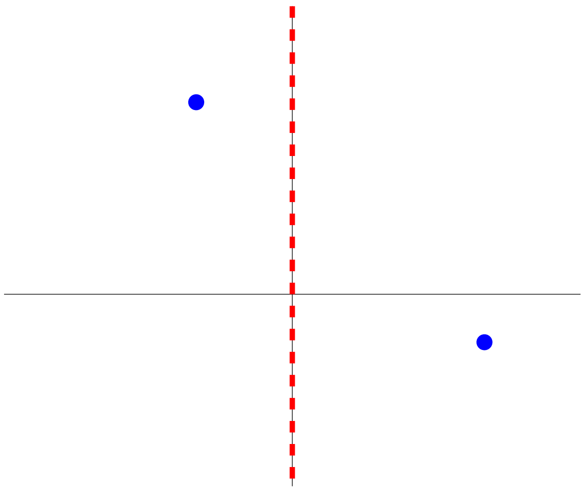

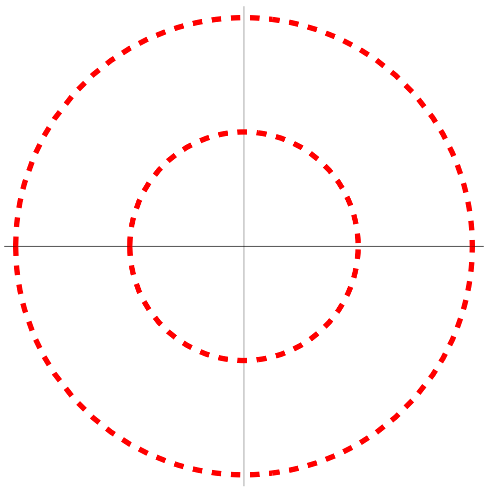

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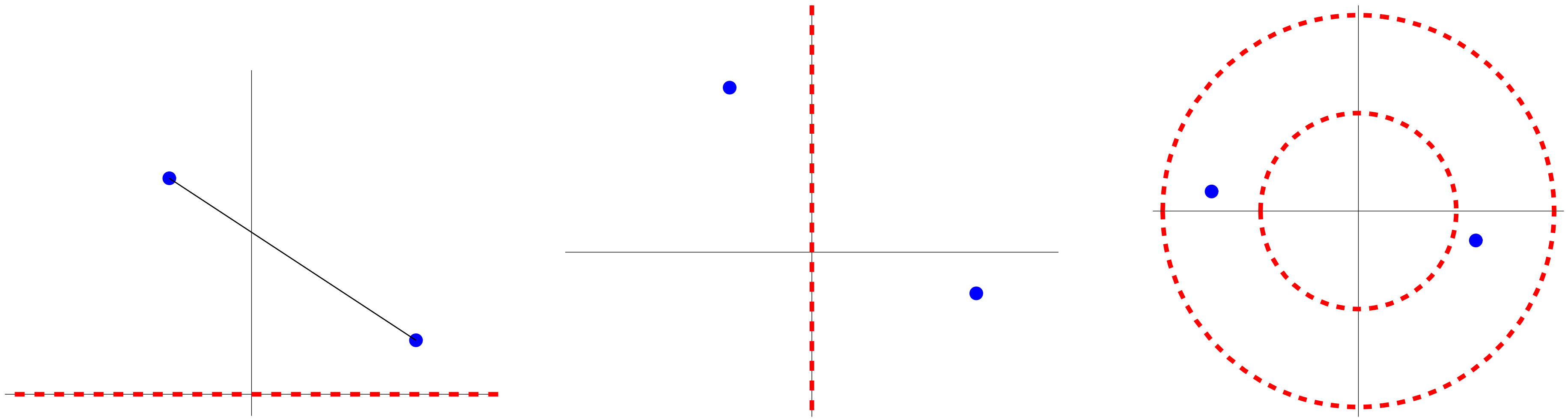

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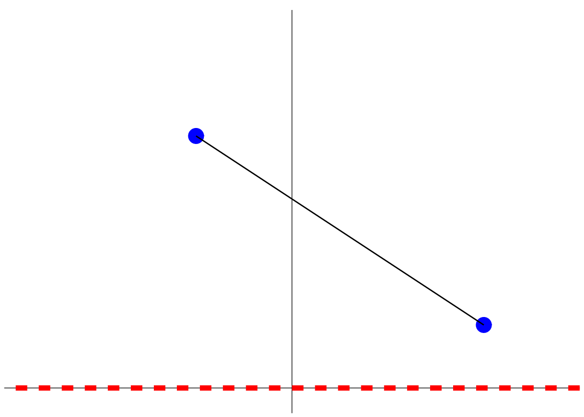
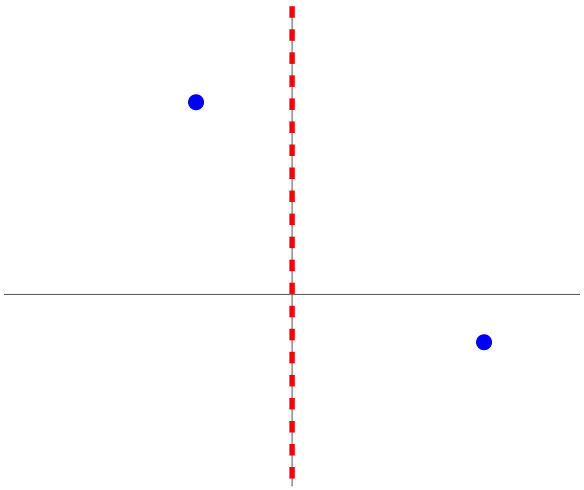
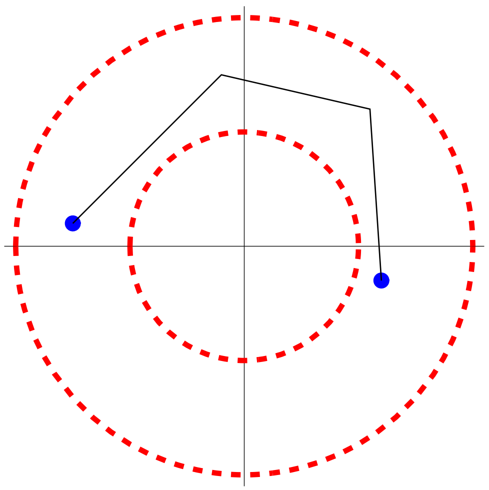

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Can write DE in explicit form $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-M(x,y)}{N(x,y)}$

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For all $(x, y) \in \mathbb{R}^2$,
$$\begin{aligned} F_x(x, y) &= 2xy \\ F_y(x, y) &= x^2 \end{aligned}$$

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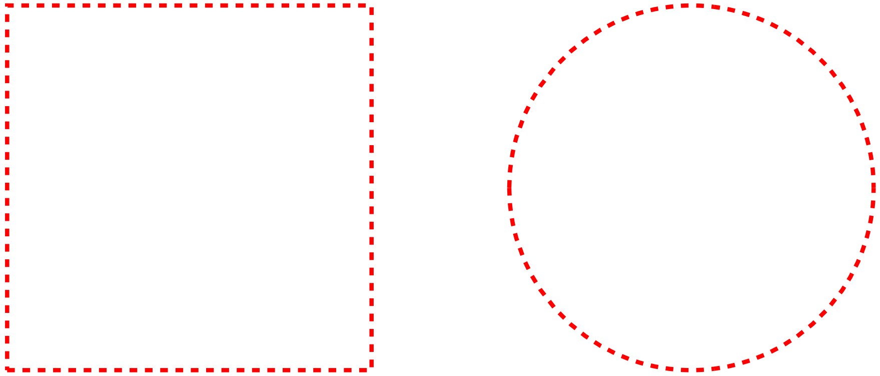

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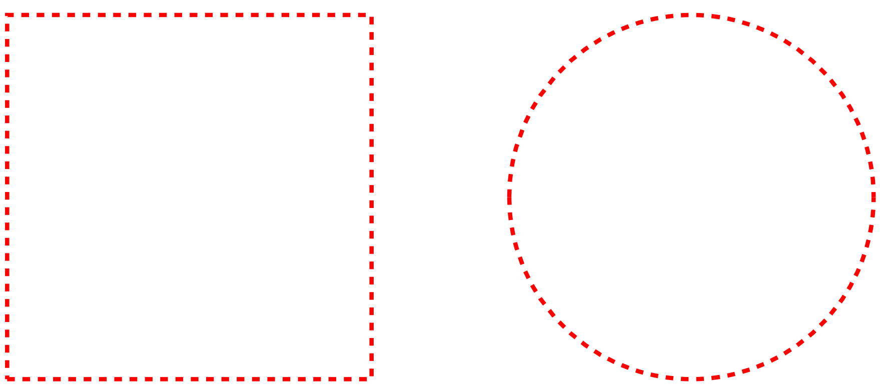

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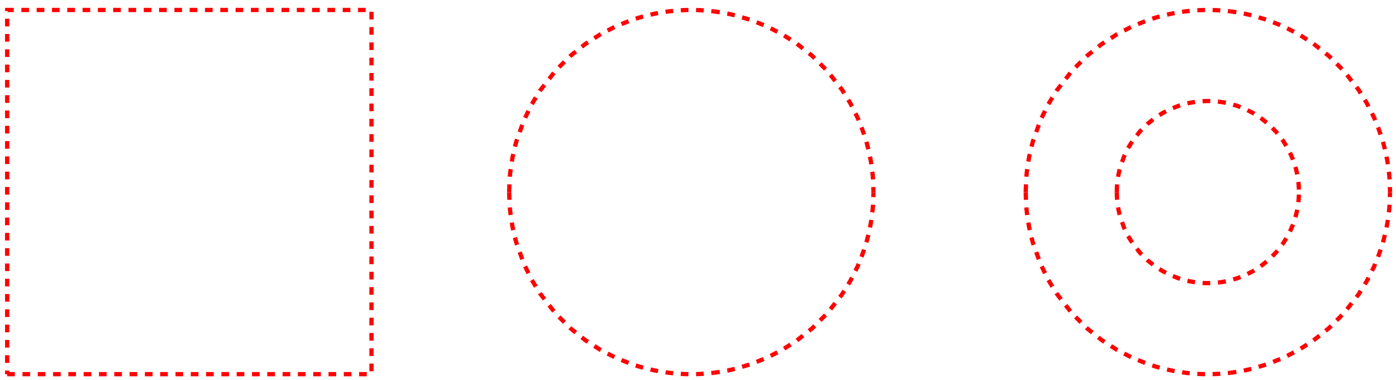

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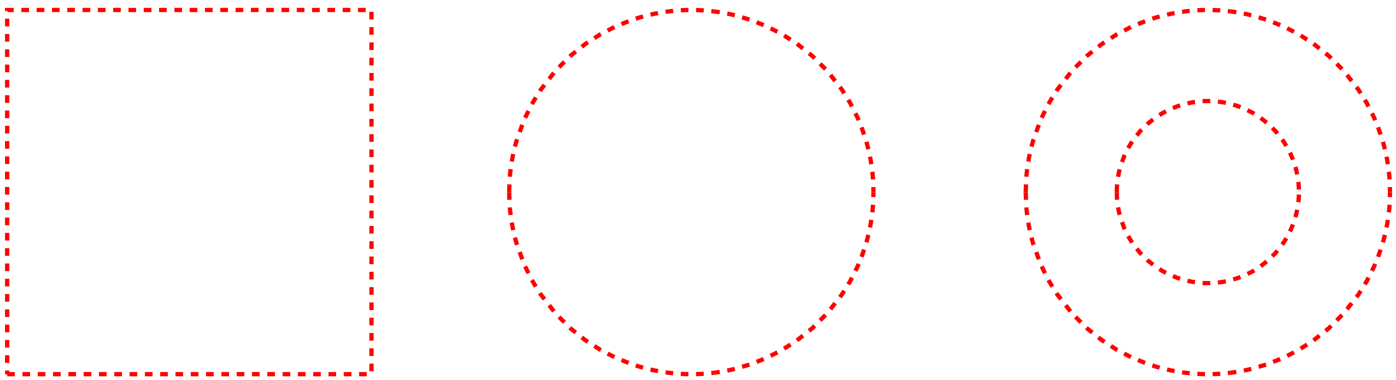

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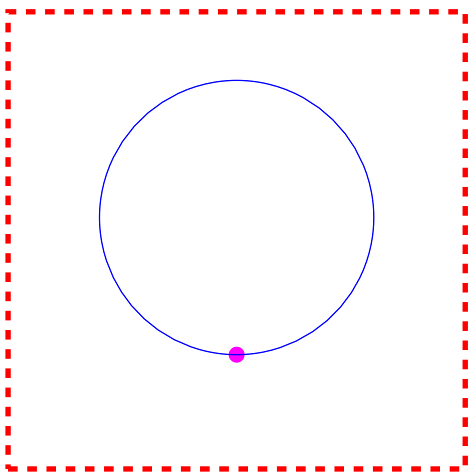

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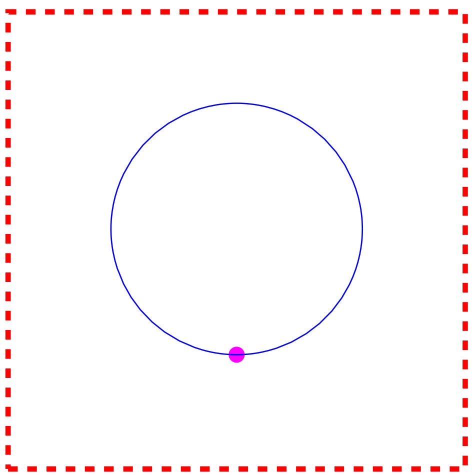

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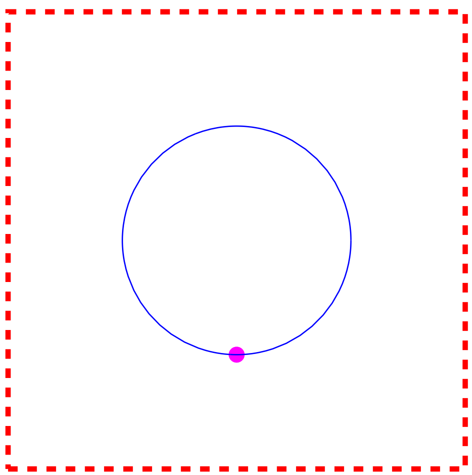

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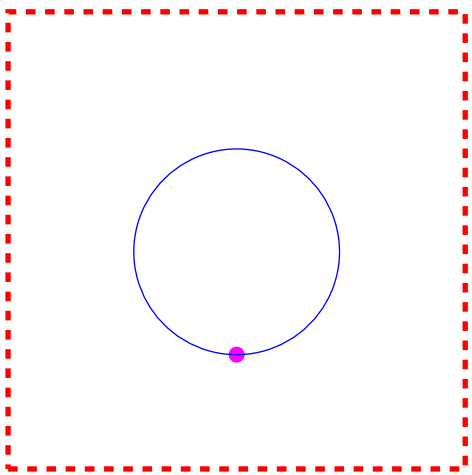

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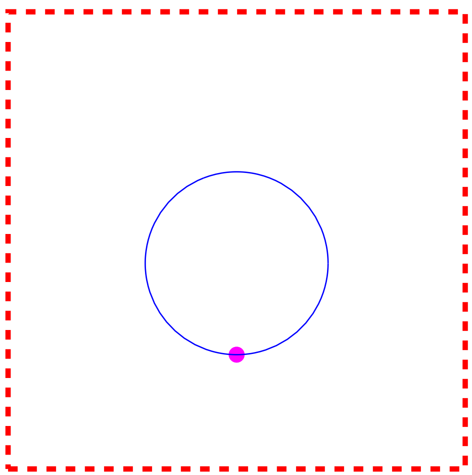

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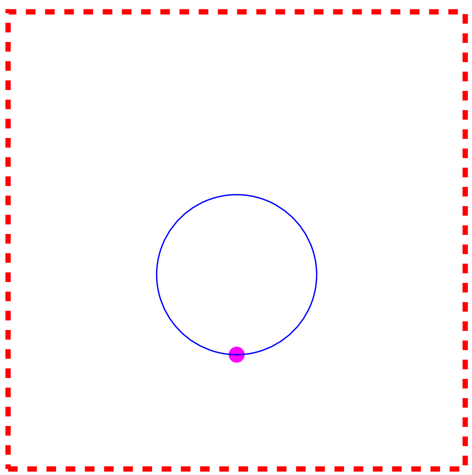

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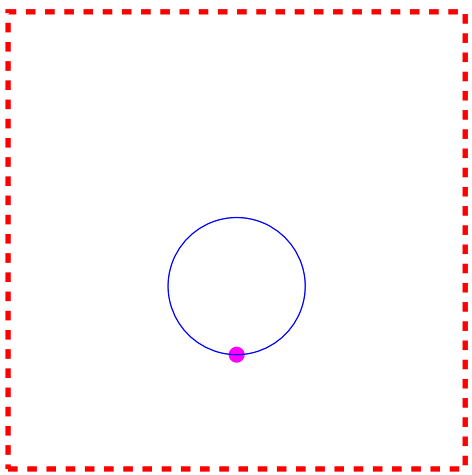

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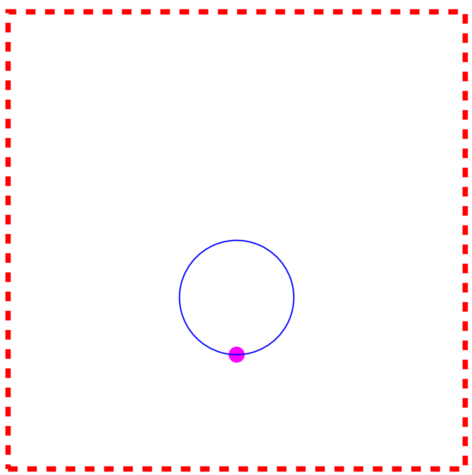

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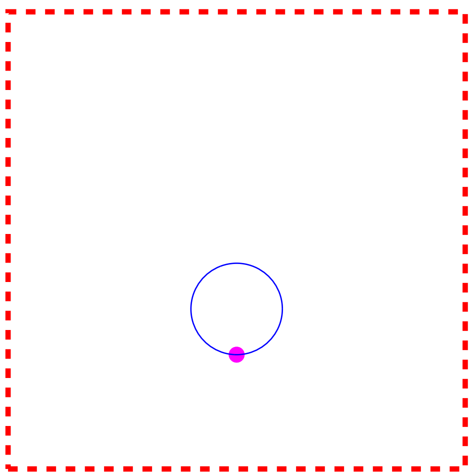

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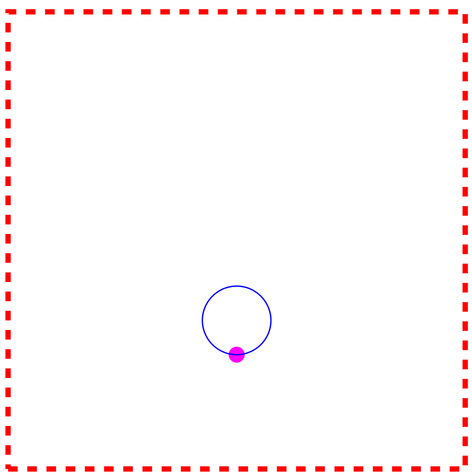

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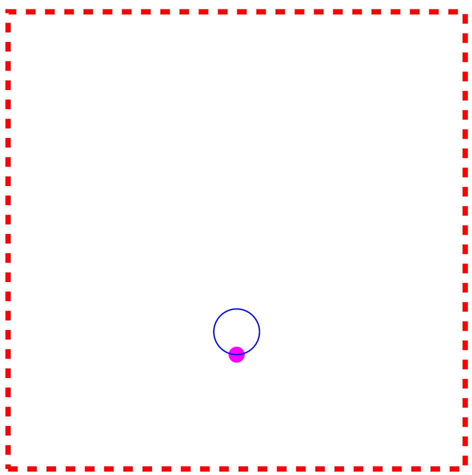

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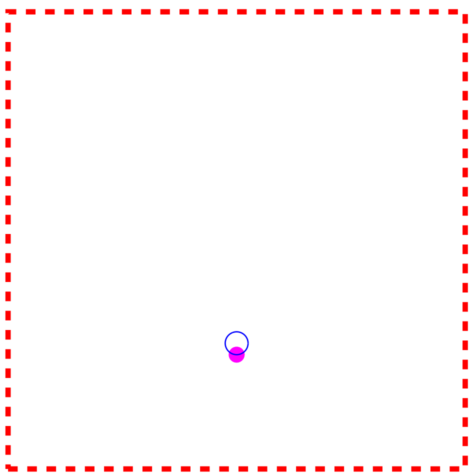

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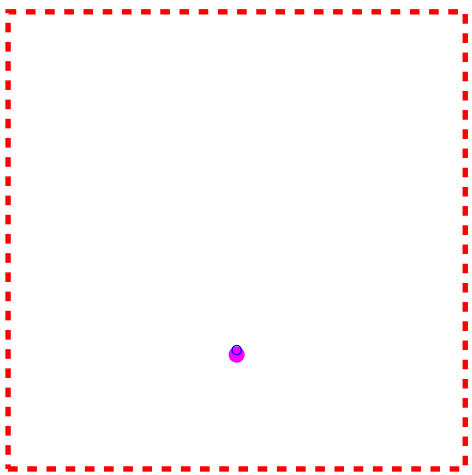

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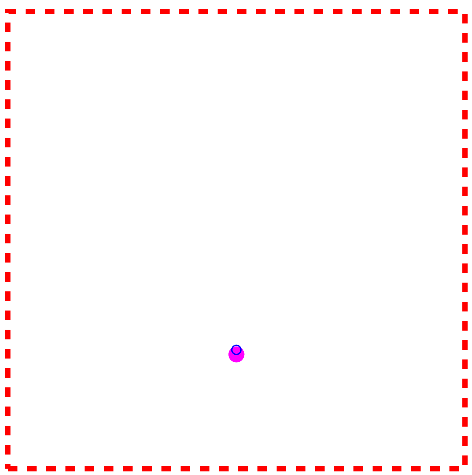

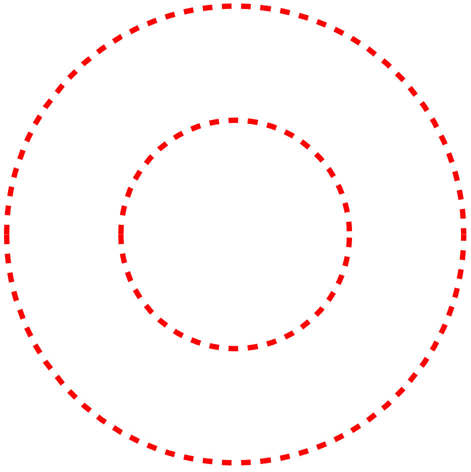

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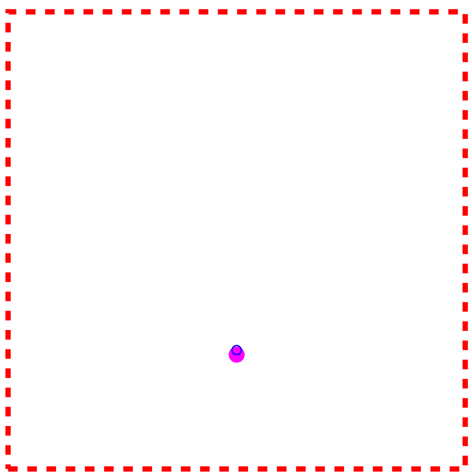

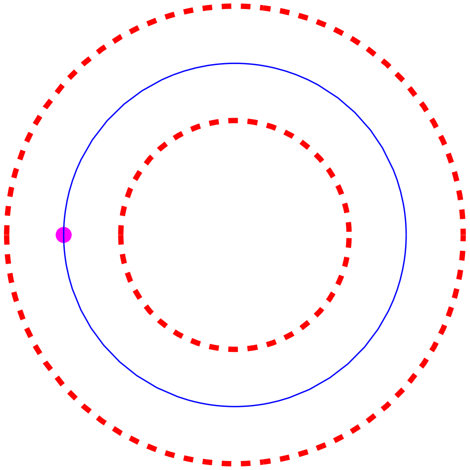

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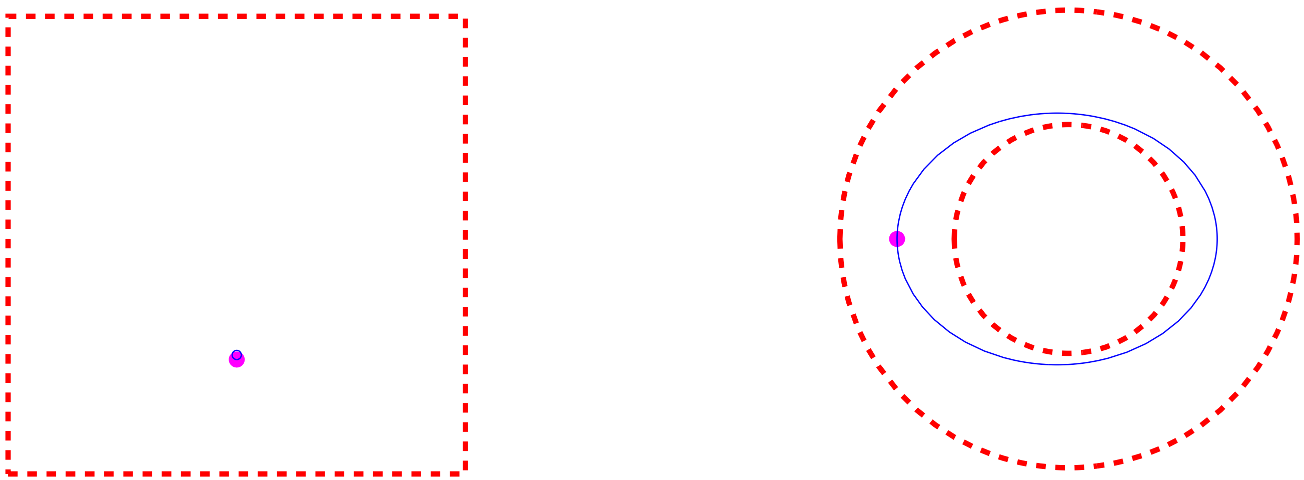

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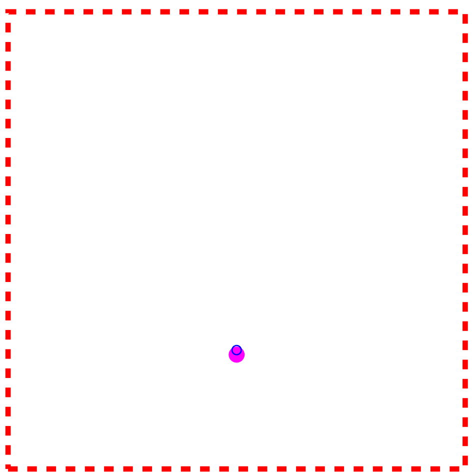

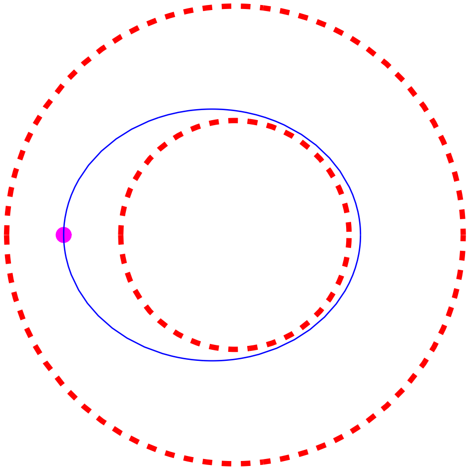

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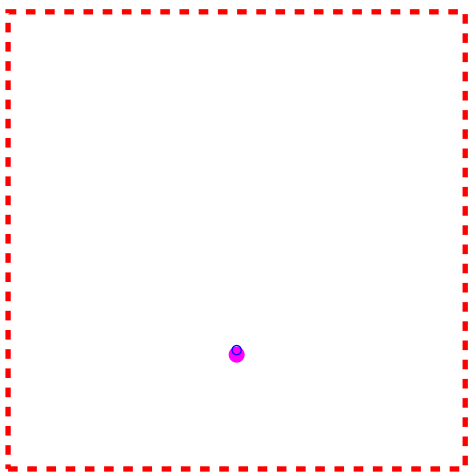

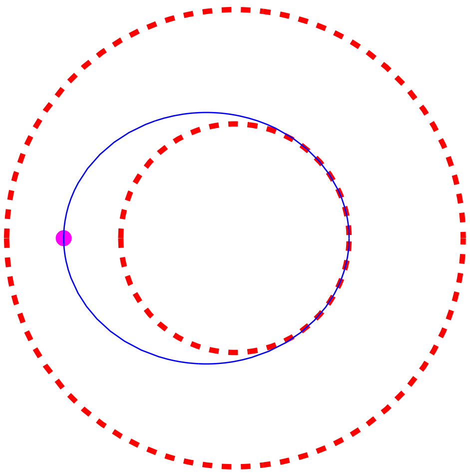

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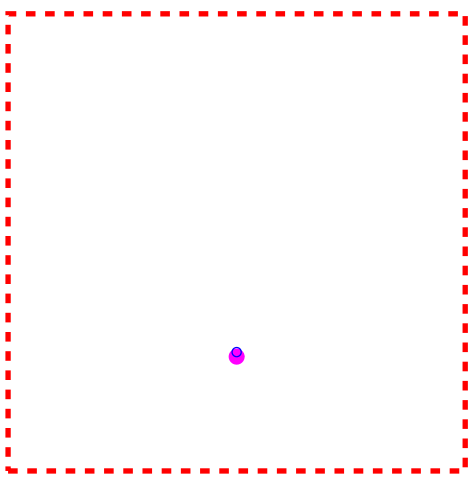
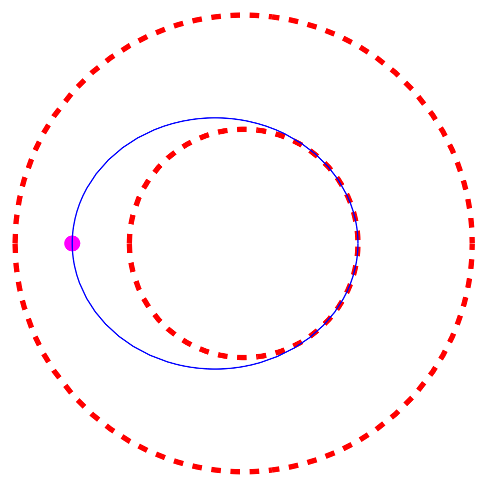
(4) If $\gamma(a) = \gamma(b)$, the path is said to be *closed*.

Explanation A simply connected region is a region that does not have any "*hole*", or equivalently, every closed path in the region can be "*shrunk*" to a point.





Remark In considering IVP, can take a rectangular region containing the initial point (t_0, y_0). Exactness can be tested using above condition.

Proof

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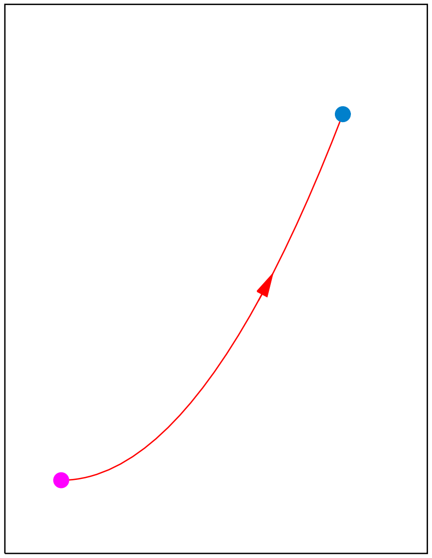

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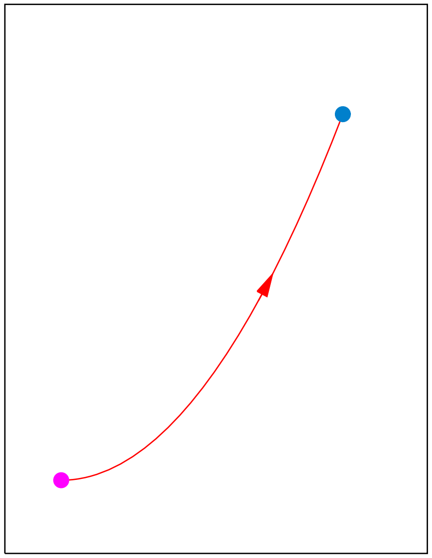

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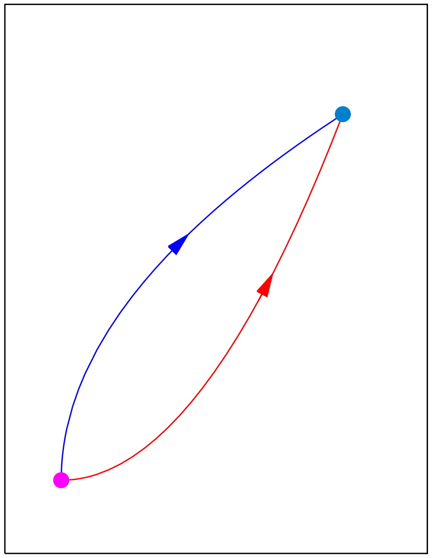

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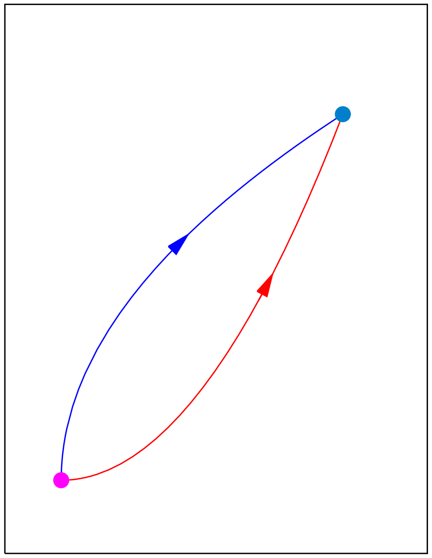

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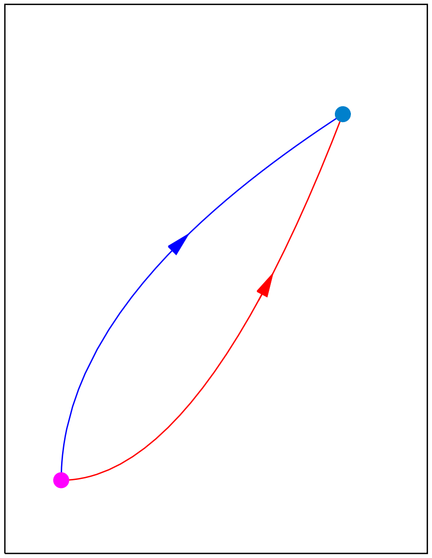

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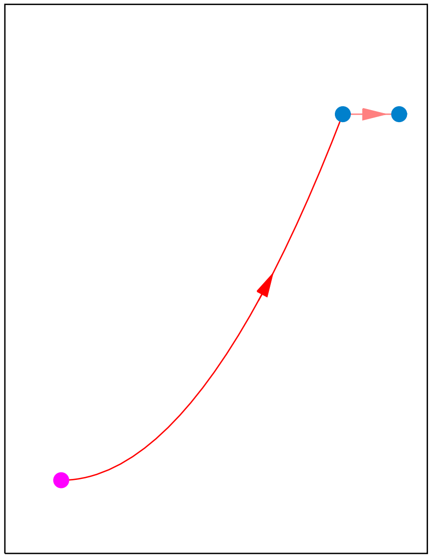

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  $F_x = M$ and $F_y = N$ $\qquad F(x+\Delta x, y) - F(x,y) = \int_{\text{horizontal path}} M\,\mathrm{d}x + N\,\mathrm{d}y$

Example Solve the following DE

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Solution

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$$(2x\,e^y + \cos x + \cos y)\,dx + (x^2 e^y - x\sin y - 1)\,dy = 0$$

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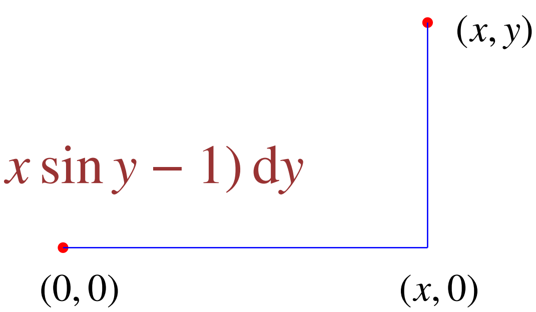

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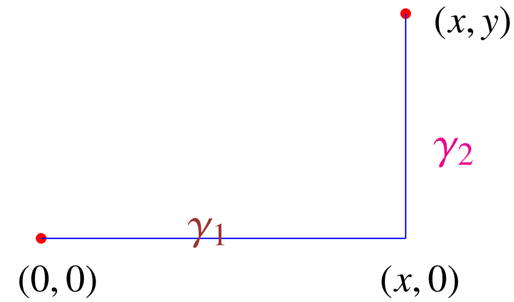

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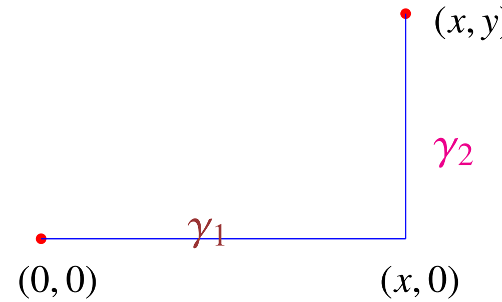

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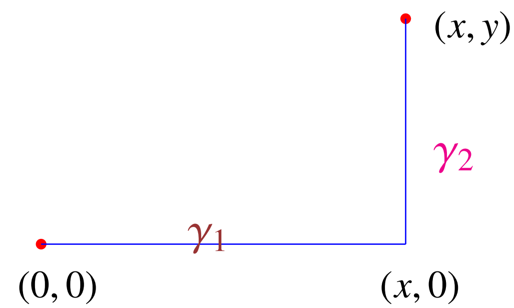

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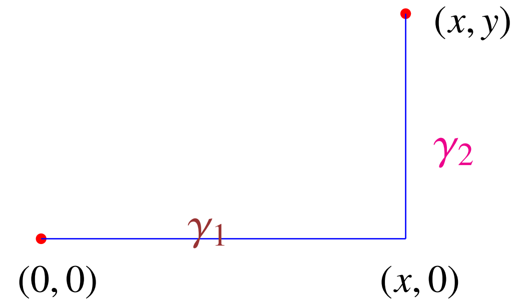

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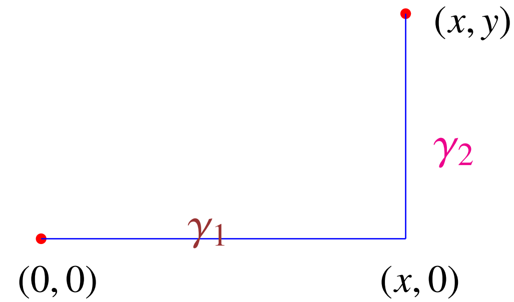

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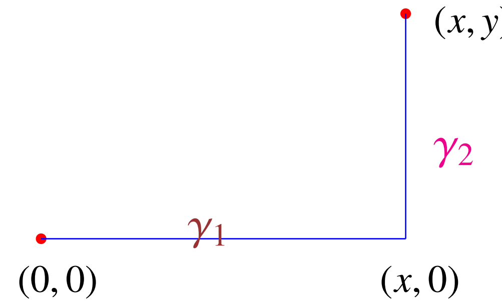

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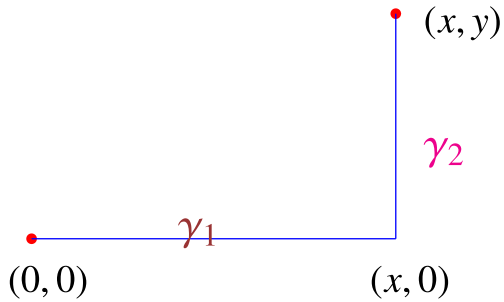

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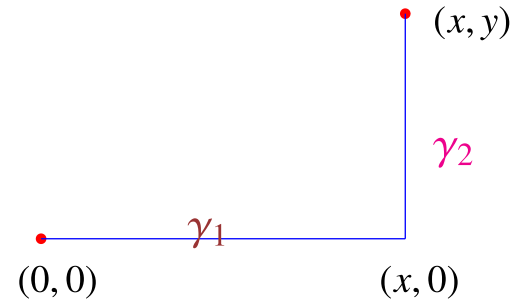

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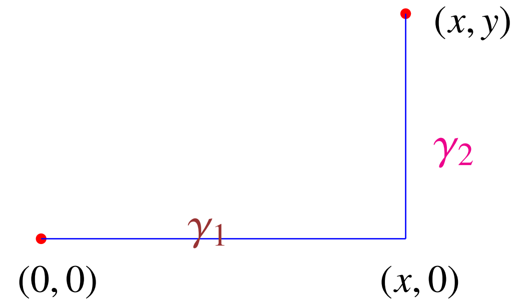

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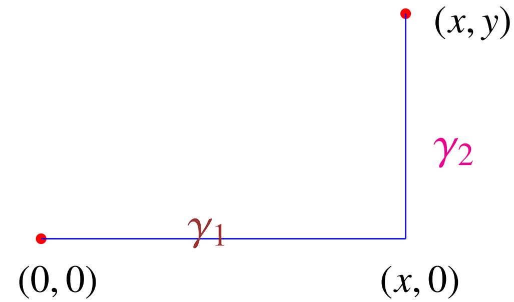

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ethod 2

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ethod 2 From first condition

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Implicit solution to the DE $x^2e^y + \sin x + x\cos y - y = C$.

Integrating Factors

For non-exact DE

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- Such a function μ is called an *integrating factor* for (1) in Ω.

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continue Consider non-exact DE (in a simply connected region Ω)

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{3}$$

Want to find μ such that the following is exact in Ω

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- Discuss the case $\mu = \mu(x)$.

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$$\Big[M(x, y)\mu(x)\Big]_y \quad = \quad \Big[N(x, y)\mu(x)\Big]_x$$

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- μ is a solution to (4), thus integrating factor for the DE.