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since $\mathrm{Si}(t) = \int_0^t \frac{\sin s}{s}\,\mathrm{d}s$ is a primitive for $\frac{\sin t}{t}$

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We can define $\varphi(0)$ such that φ satisfies DE at $t = 0$ also.

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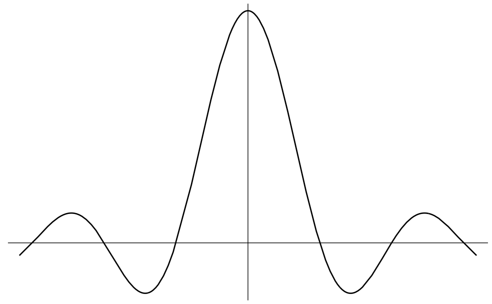

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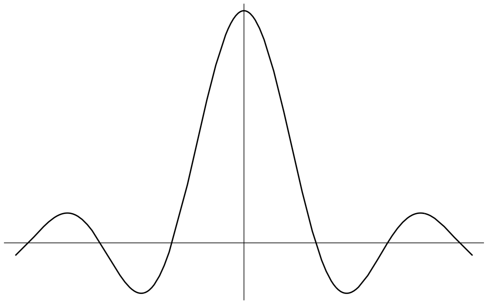

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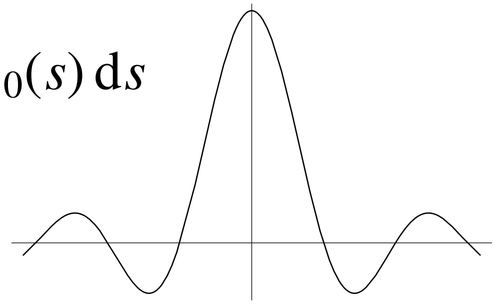

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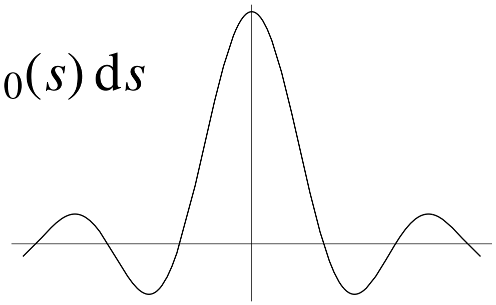

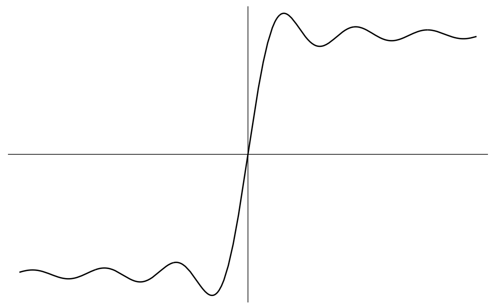

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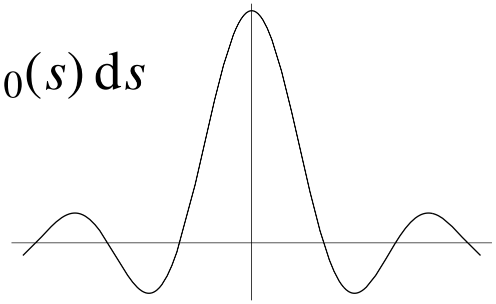

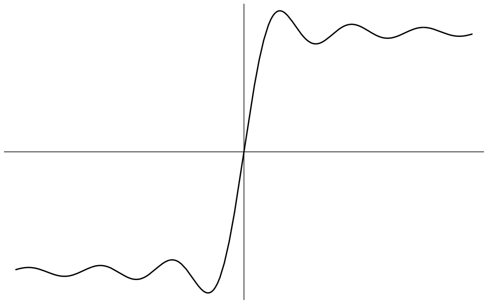

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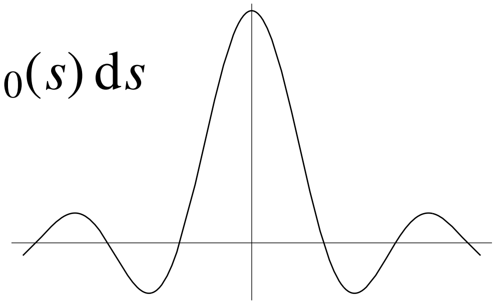

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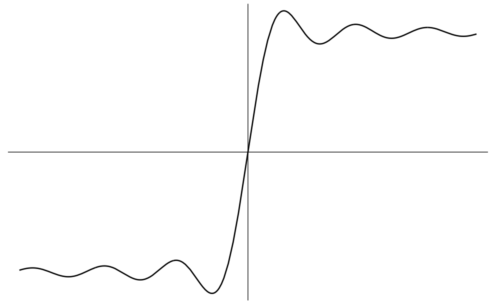

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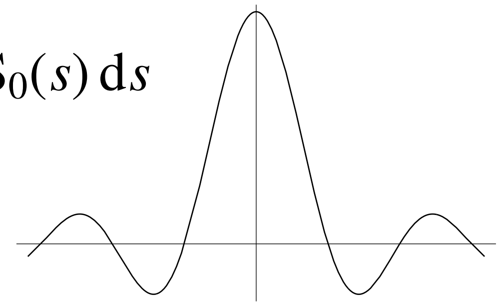

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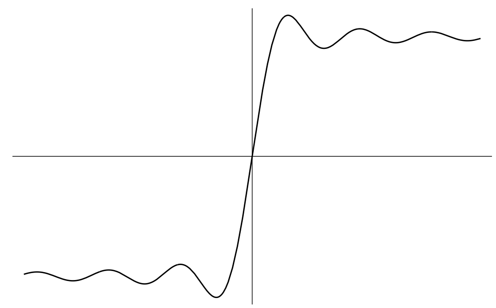

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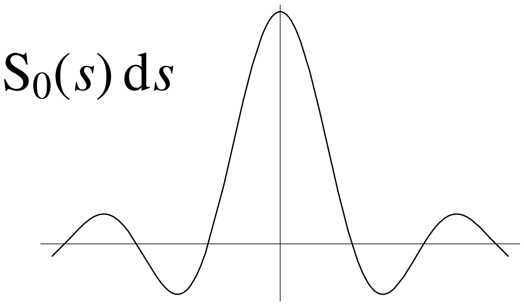

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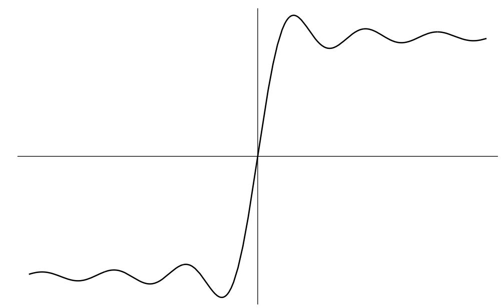

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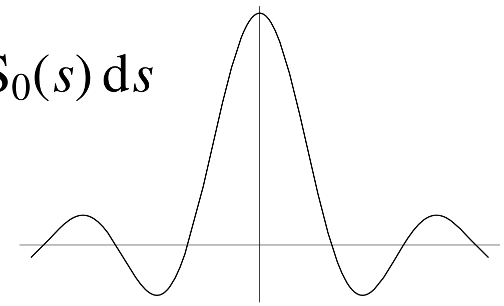

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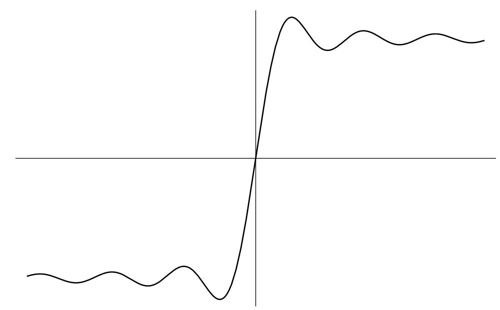

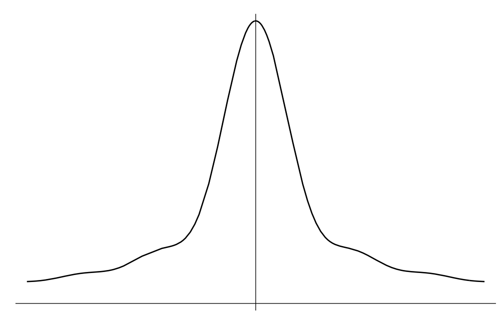

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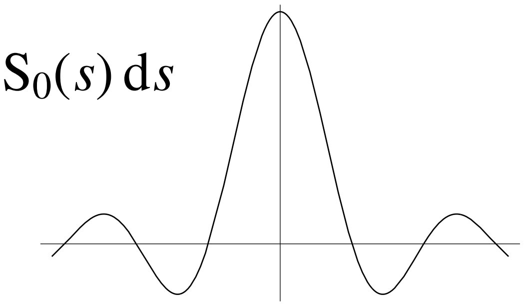

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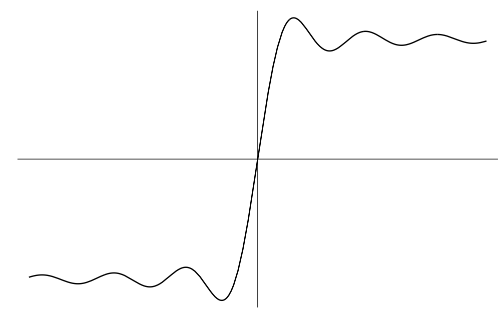

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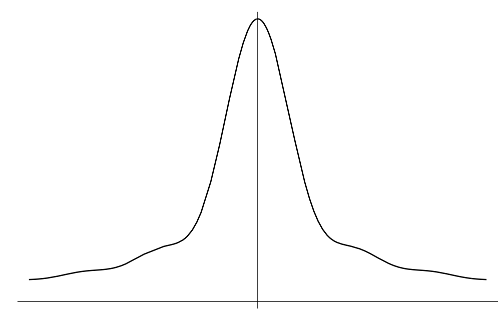

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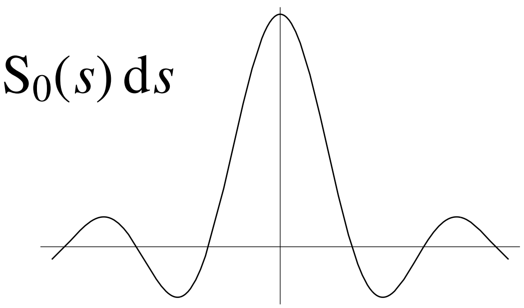

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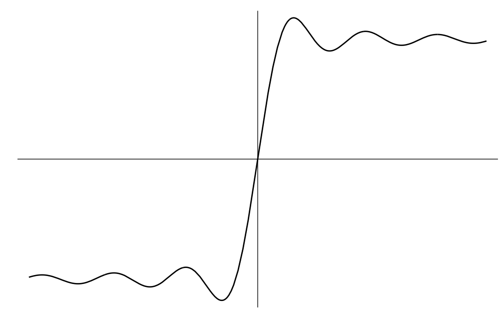

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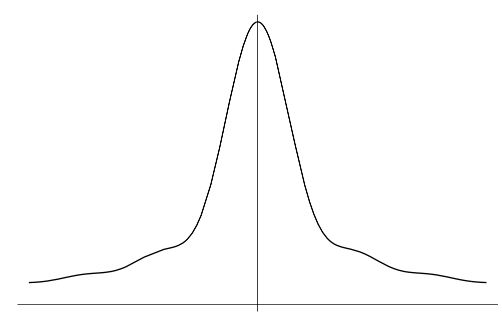

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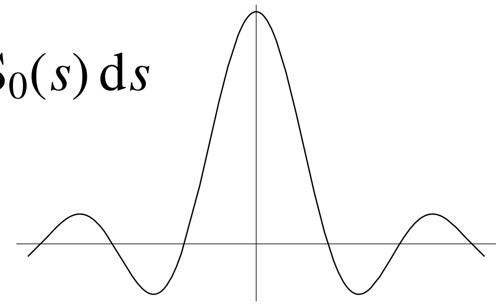

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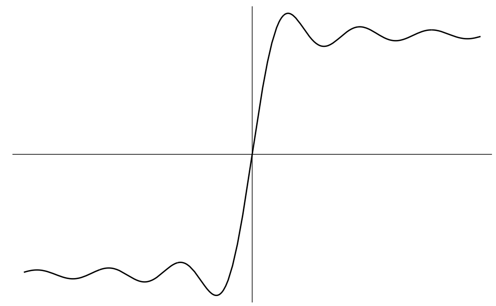

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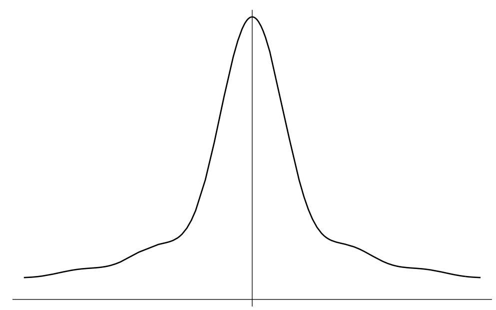
Can show that $\varphi'(0)$ exists.

DE is satisfied when $t = 0$.



Existence and Uniqueness Theorem

Consider the following IVP

$$\begin{cases} y'(t) = f(t, y) \\ y(t_0) = y_0 \end{cases}$$

Suppose f *and* f_y *are continuous on a neighborhood of* (t_0, y_0)*.*

Then in some neighborhood of t_0*, the IVP has a unique solution.*

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- A subset S of $\mathbb{R}$ is said to be a neighborhood of a point $t_0 \in \mathbb{R}$ if there exists $r > 0$ such that $(t_0 - r, t_0 + r) \subset S$

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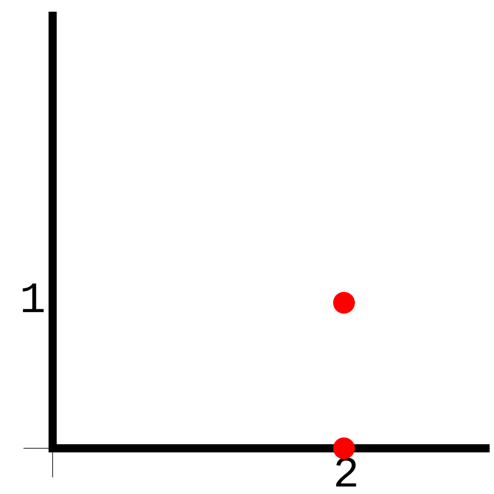

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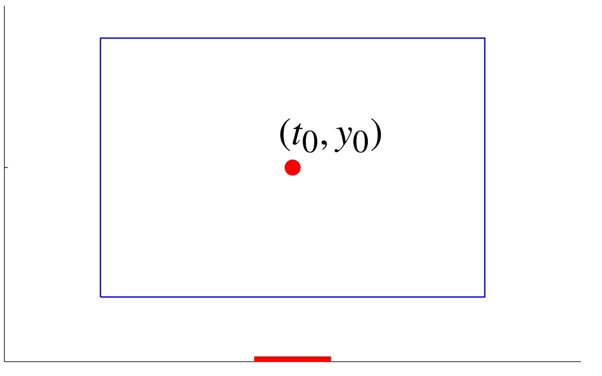

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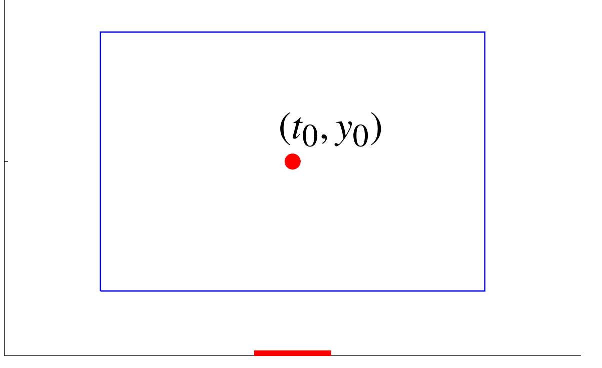

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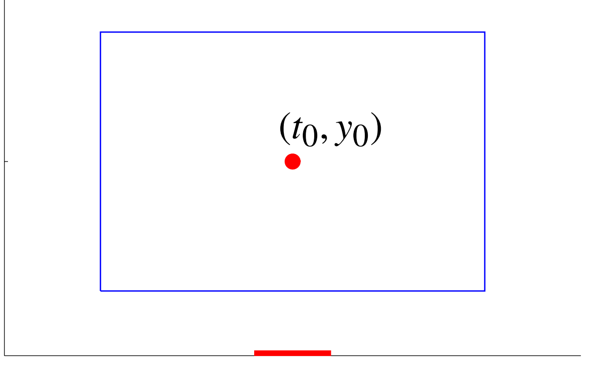

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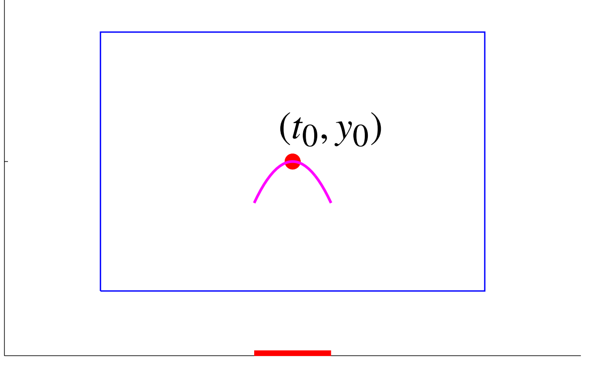

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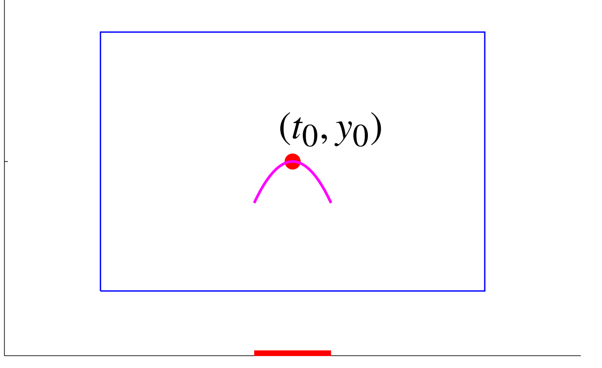

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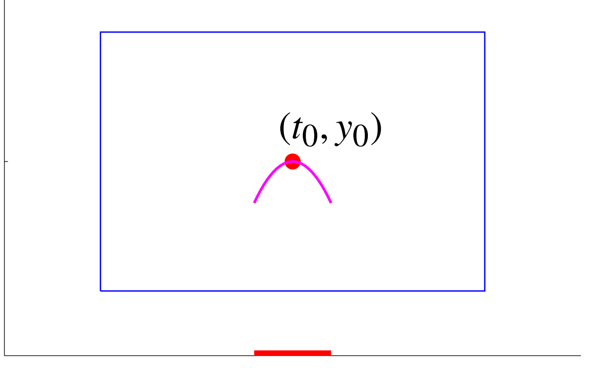

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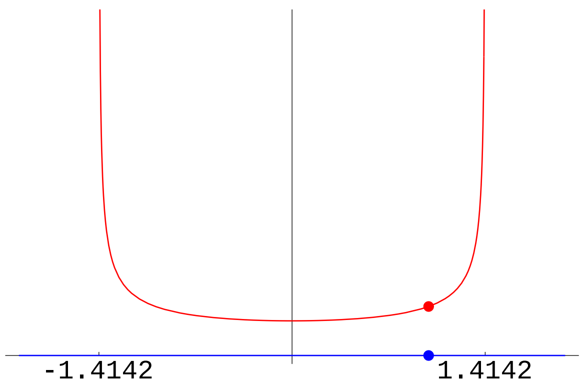

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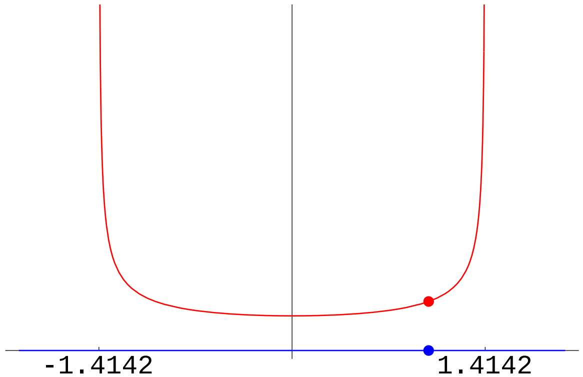

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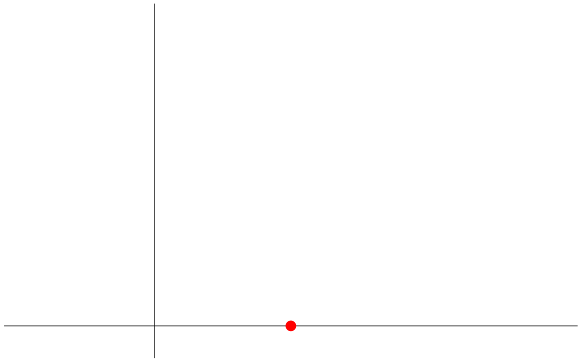

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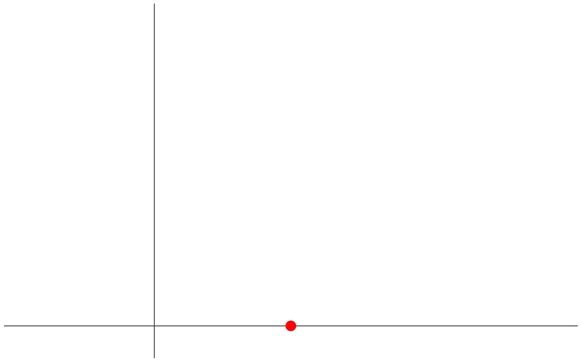

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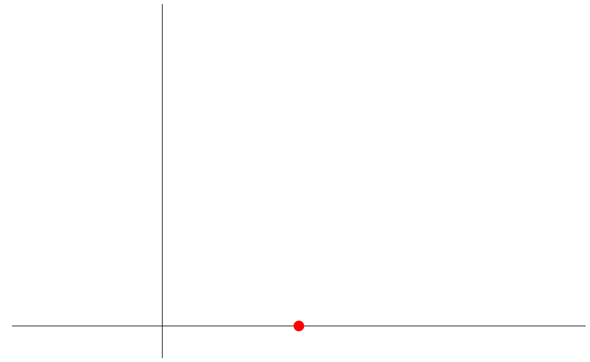

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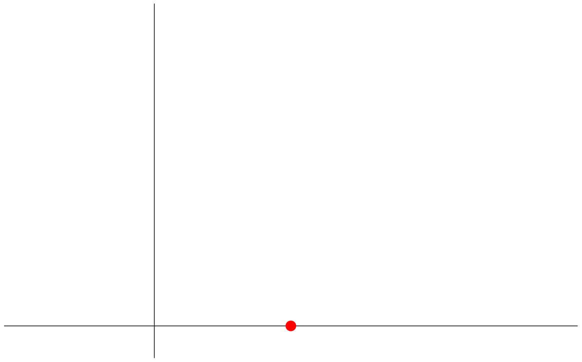

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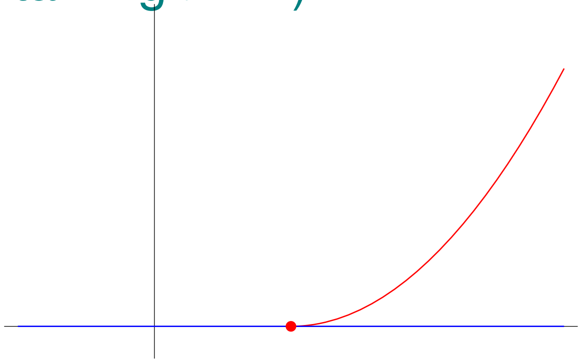

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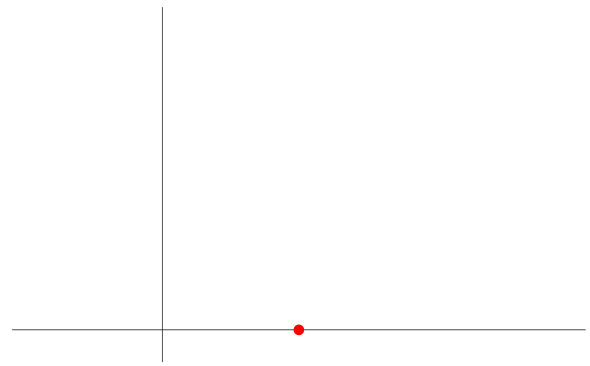

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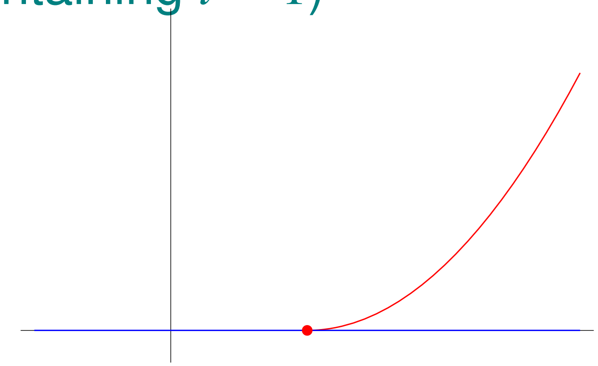

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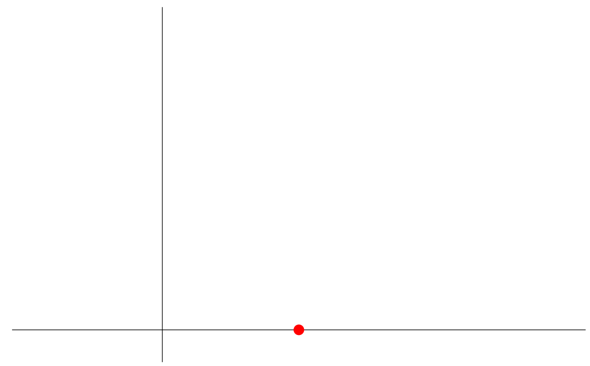

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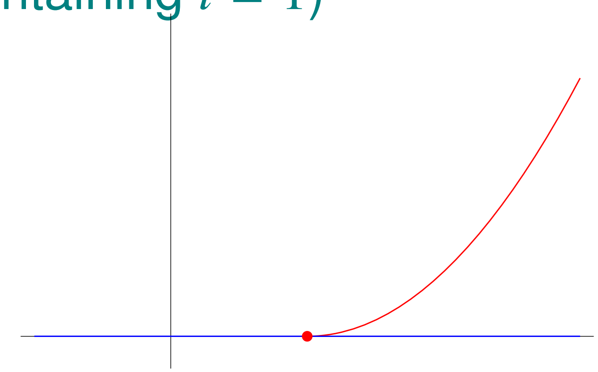

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$$y' = f(t, y)$$

where f *and* f_y *are continuous on a rectangular region* $(a, b) \times (c, d)$*.*

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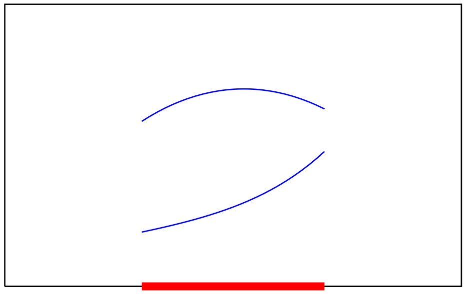

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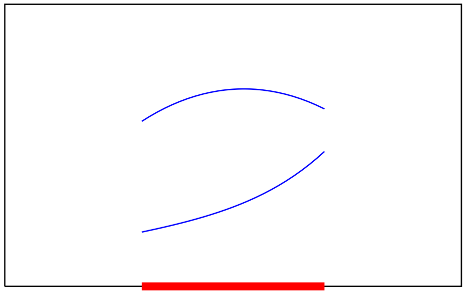

- *either* $\varphi_1(t) = \varphi_2(t)$ *for all* $t \in (\alpha, \beta)$
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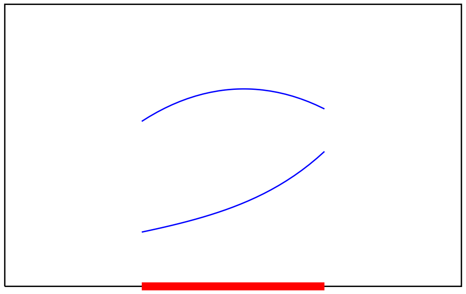

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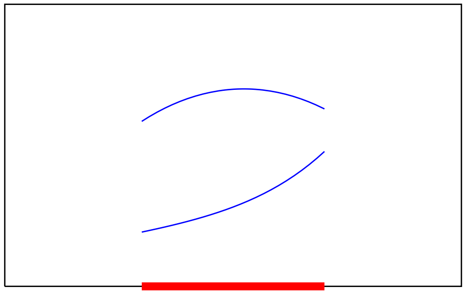

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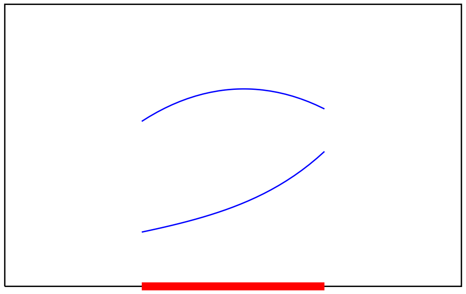

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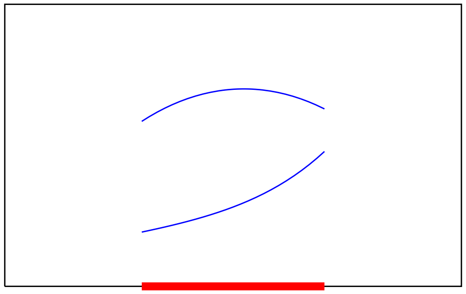

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- If $\alpha_1 = \alpha$ and $\beta_1 = \beta$, done.

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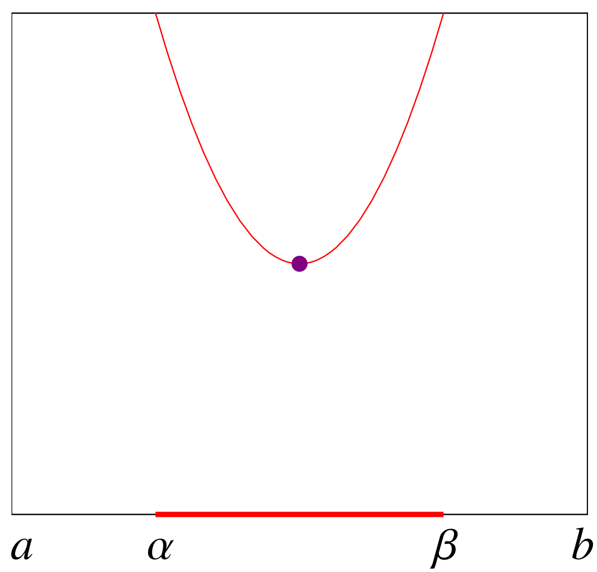

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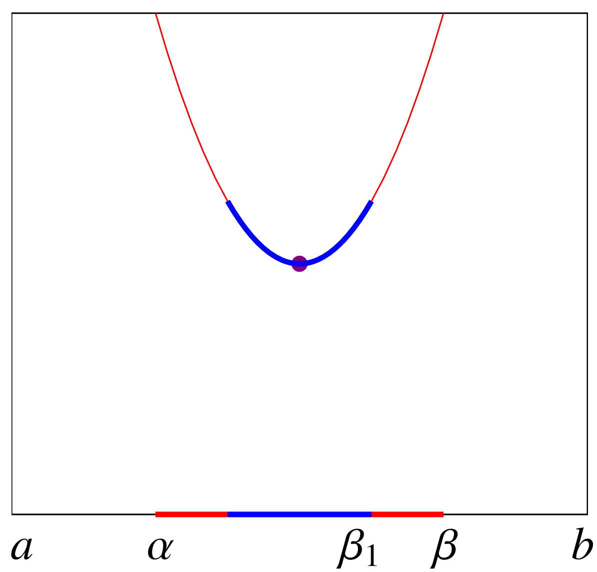

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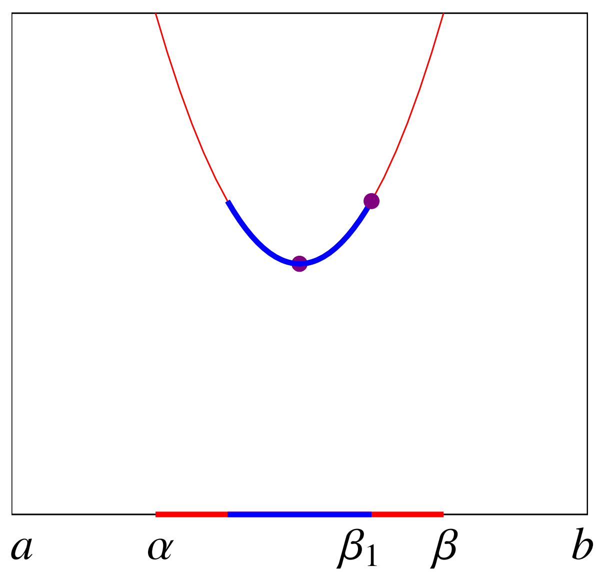

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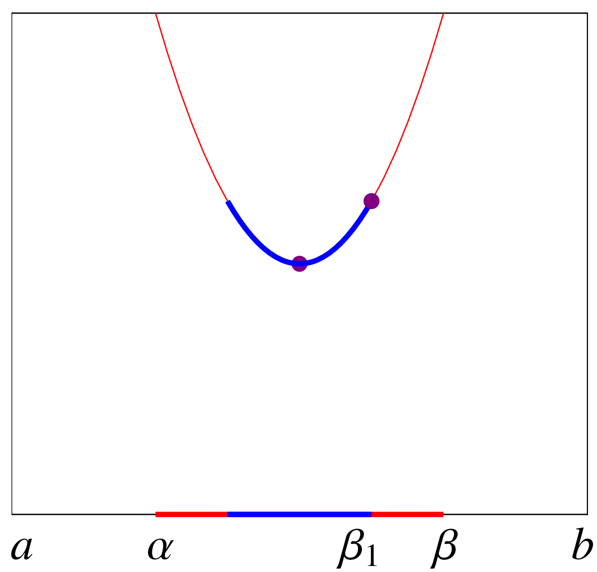

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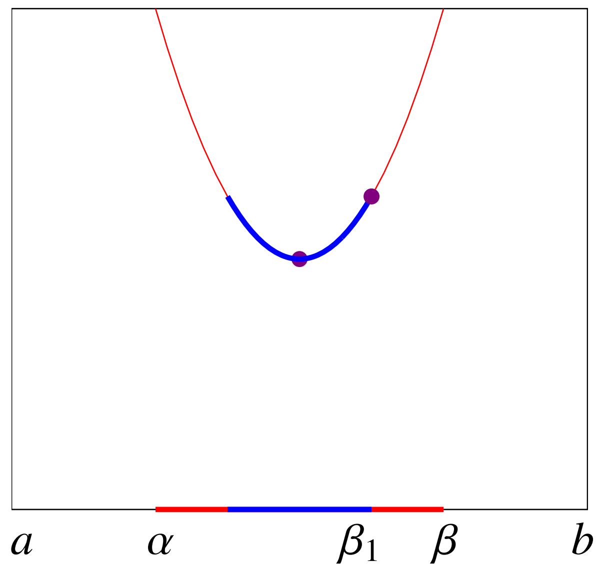

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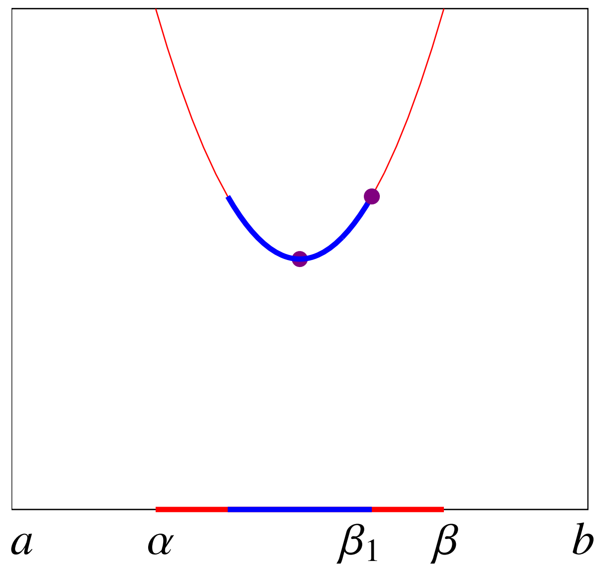
  Get larger interval in which φ_1 and φ_2 are equal.



- **Principle of Continuum Induction**

Consider statements $P(t)$ where $t \in [a, b)$. Suppose

(1) $P(a)$ is true.

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Theorem *Every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum).*

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By Principle of Continuum Induction, required result follows.

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where f and f_y are continuous on a rectangular region $(a, b) \times \mathbb{R}$ and $t_0 \in (a, b)$.

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Remarks to Examples 1 and 2 follow from above corollary.