Example Consider (1st order, non-linear) DE

$$y' + 2ty^2 = 0$$

- The following are solutions:
  - ◇ $\varphi(t) \equiv 0$
  - ◇ $\varphi_c(t) = (t^2 + c)^{-1}$ (where c is a constant)

$$\begin{aligned} \varphi_c'(t) &= -(t^2 + c)^{-2} \cdot 2t \\ &= -2t[\varphi_c(t)]^2 \end{aligned}$$

- No more solution *(by E&U Thm)*.
- Interval of Validity ?
  - ◇ $\varphi \equiv 0$ solution on $\mathbb{R}$
  - ◇ ○ if $c > 0$, φ_c solution on $\mathbb{R}$
    - ○ if $c = 0$, φ_0 solution on $(-\infty, 0)$ and also on $(0, \infty)$
    - ○ if $c = -p < 0$, φ_c solution on $(-\infty, -\sqrt{p})$, $(-\sqrt{p}, \sqrt{p})$ and $(\sqrt{p}, \infty)$.

      $\varphi_c(t) = \dfrac{1}{t^2 - p}$ is undefined at $\pm\sqrt{p}$

Note Above DE $y' + 2ty^2 = 0$

- has infinitely many solutions $\begin{cases} \varphi(t) \equiv 0 \\ \varphi_c(t) = (t^2 + c)^{-1} \quad \text{(where } c \text{ is a constant)} \end{cases}$

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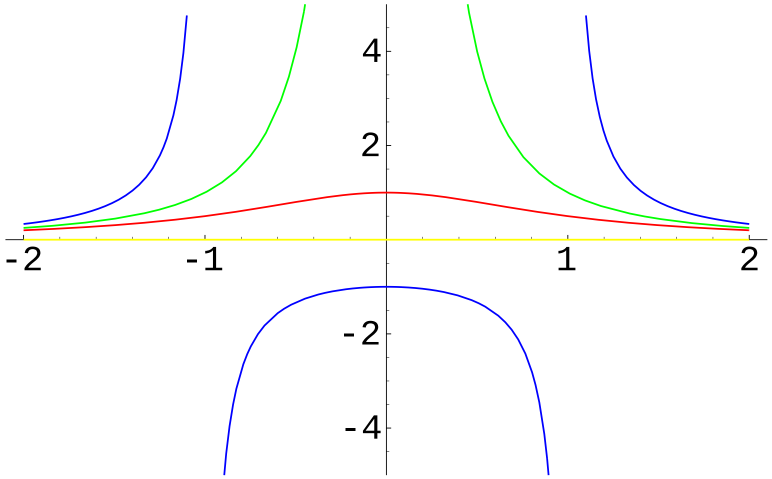

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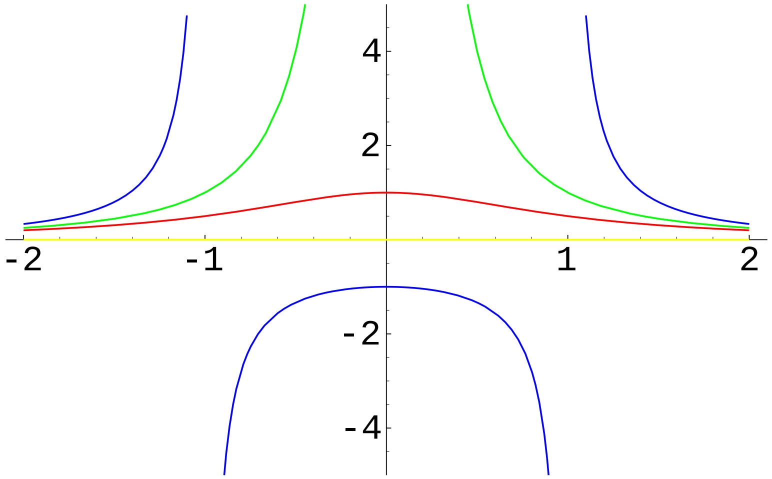

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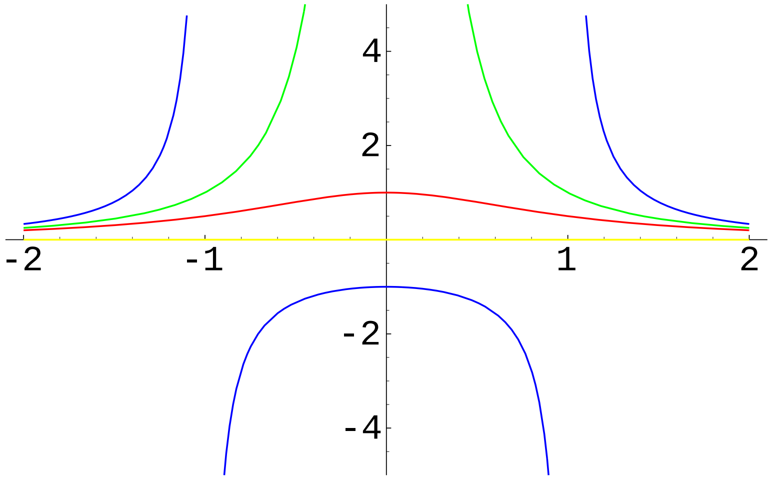

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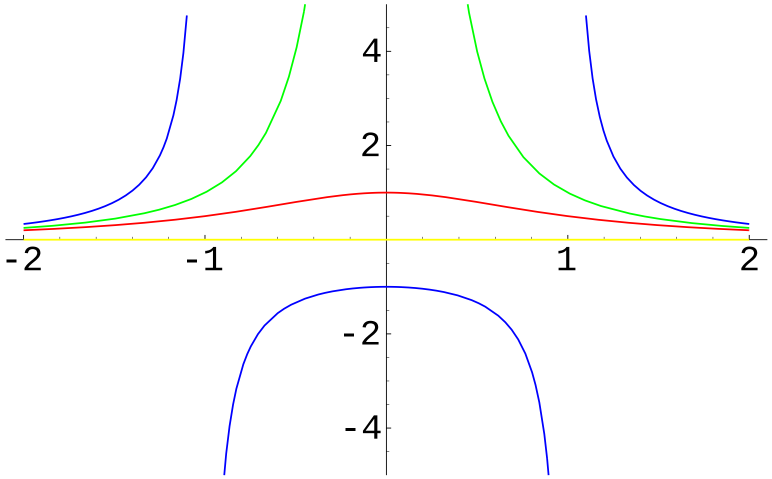
  Interval of validity $(-1, 1)$



Terminology An nth-order *initial value problem* (IVP) is an *nth-order DE together with n initial conditions*

Implicit Form

$$\begin{cases} F(t, y, y', \ldots, y^{(n)}) = 0 \,, \\ y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots \,, \quad y^{(n-1)}(t_0) = y_{n-1} \end{cases}$$

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(2) 2nd order IVP
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Definition A *solution* to IVP $\begin{cases} F(t, y, y', \ldots, y^{(n)}) = 0\ , \\ y(t_0) = y_0, \quad y'(t_0) = y_1, \quad \ldots\ , \quad y^{(n-1)}(t_0) = y_{n-1} \end{cases}$

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Remark Interval of validity of *solution to IVP* is unique.

Example Equation of motion of free falling object (without air resistance) with initial velocity v_0 and initial displacement h_0 governed by the following IVP

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  Unique solution to IVP $y(t) = h_0 + v_0 t - \frac{1}{2}gt^2$

  Interval of validity $(-\infty, \infty)$

Chapter 1: First Order Differential Equations

- Linear DE
- Non-linear DE
  - Separable DE
  - Exact DE
  - Integrating Factors
- Existence and Uniqueness Theorem

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1.1 First order linear DE

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- Special case: constant coefficient.

Example Solve the DE

$$y' = 2y, \qquad -\infty < t < \infty$$

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- By inspection, $y_1(t) = \mathrm{e}^{2t}$ is a solution. *Check* $y_1'(t) = \mathrm{e}^{2t} \cdot 2 = 2y_1$

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- General solution is $y(t) = Ce^{t^2}$

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Remark To solve $y' = 2y$ using separation of variables: $\frac{\mathrm{d}y}{y} = 2\,\mathrm{d}t$
get solution that never vanishes.

To solve

$$y' + p(t)y = g(t), \quad a < t < b \tag{2}$$

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- Since $\mu(t)$ never vanishes in (a, b), solutions to (3) are also solutions to (2) (and vice versa).

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- **Note** Any non-zero multiple of μ is also an integrating factor.

  May write $\mu(t) = e^{\int p(t)\, dt}$

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$$\begin{cases} ty' - 2y = 2t^2 + t, \\ y(1) = 3. \end{cases}$$

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- Unique solution $y(t) = 2t^2 \ln t - t + 4t^2$

  Interval of validity $(0, \infty)$

Theorem *The* initial value problem

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

where $p, g \in C(a, b)$, $t_0 \in (a, b)$ *and* $y_0 \in \mathbb{R}$, *has unique solution in* (a, b).

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- General solution to DE $\quad y(t) \;=\; \mathrm{e}^{-\int_{t_0}^{t} p(s)\,\mathrm{d}s}\left[\displaystyle\int_{t_0}^{t} g(u)\mathrm{e}^{\int_{t_0}^{u} p(s)\,\mathrm{d}s}\,\mathrm{d}u + C\right]$
- By initial condition

$$\begin{aligned} y_0 &= \mathrm{e}^{-\int_{t_0}^{t_0} p(s)\,\mathrm{d}s}\left[\int_{t_0}^{t_0} g(u)\mathrm{e}^{\int_{t_0}^{u} p(s)\,\mathrm{d}s}\,\mathrm{d}u + C\right] \\ &= C \end{aligned}$$

Theorem *The* initial value problem

$$\begin{cases} y' + p(t)y = g(t) \\ y(t_0) = y_0 \end{cases}$$

where $p, g \in C(a,b)$, $t_0 \in (a,b)$ *and* $y_0 \in \mathbb{R}$, *has unique solution in* (a,b).

Proof Take $H(t) = \displaystyle\int_{t_0}^{t} p(s)\,\mathrm{d}s$

- General solution to DE $$y(t) = \mathrm{e}^{-\int_{t_0}^{t} p(s)\,\mathrm{d}s}\left[\int_{t_0}^{t} g(u)\mathrm{e}^{\int_{t_0}^{u} p(s)\,\mathrm{d}s}\,\mathrm{d}u + C\right]$$
- By initial condition $$\begin{aligned} y_0 &= \mathrm{e}^{-\int_{t_0}^{t_0} p(s)\,\mathrm{d}s}\left[\int_{t_0}^{t_0} g(u)\mathrm{e}^{\int_{t_0}^{u} p(s)\,\mathrm{d}s}\,\mathrm{d}u + C\right] \\ &= C \end{aligned}$$
- Unique solution $$y(t) = \mathrm{e}^{-\int_{t_0}^{t} p(s)\,\mathrm{d}s}\left[\int_{t_0}^{t} g(u)\mathrm{e}^{\int_{t_0}^{u} p(s)\,\mathrm{d}s}\,\mathrm{d}u + y_0\right]$$