MATH0802 Contents

(1) One variable Calculus

More differentiation (inverse trigonometric functions, Taylor Theorem, L'Hôpital's rule etc)

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Limits and Continuity

Differentiation (partial derivatives, local extremum points, Lagrange multiplier methods)

Double integrals

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(3) Linear Algebra (elementary)

Matrices

Solving Equations

Eigenvalues, eigenvectors

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Use your imagination

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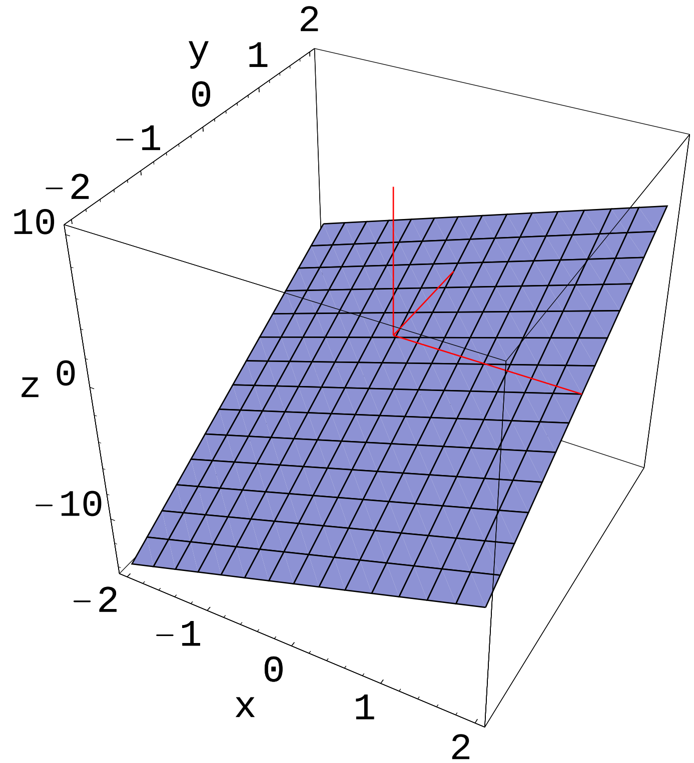

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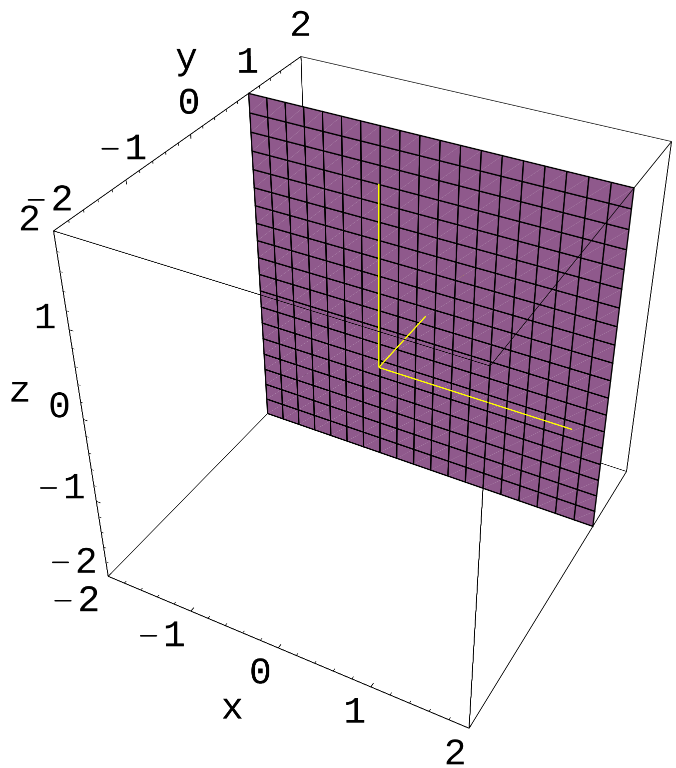

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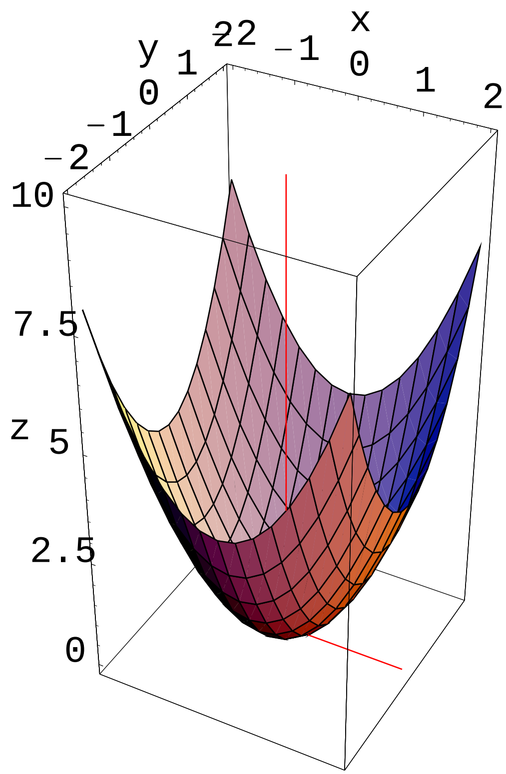

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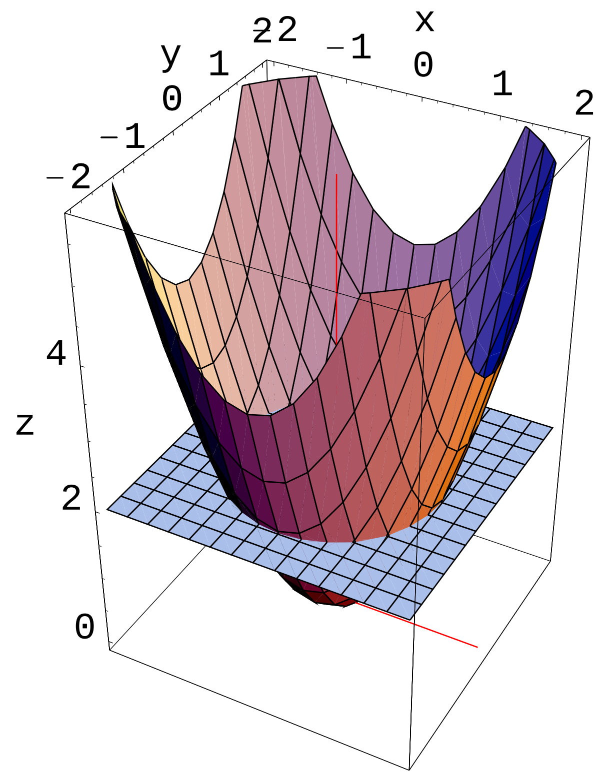

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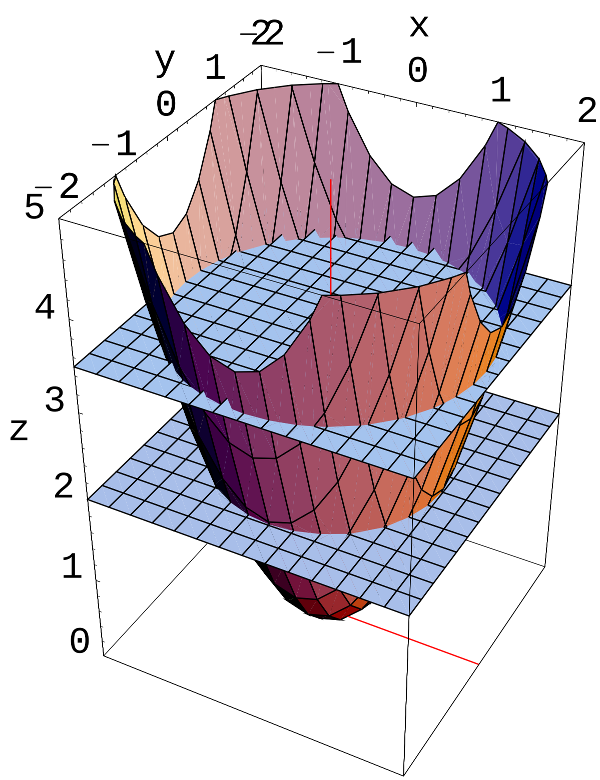

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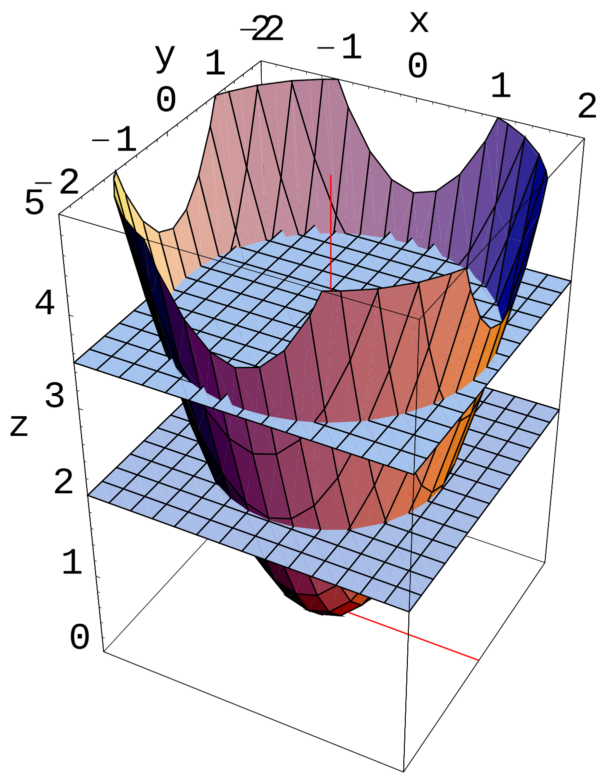

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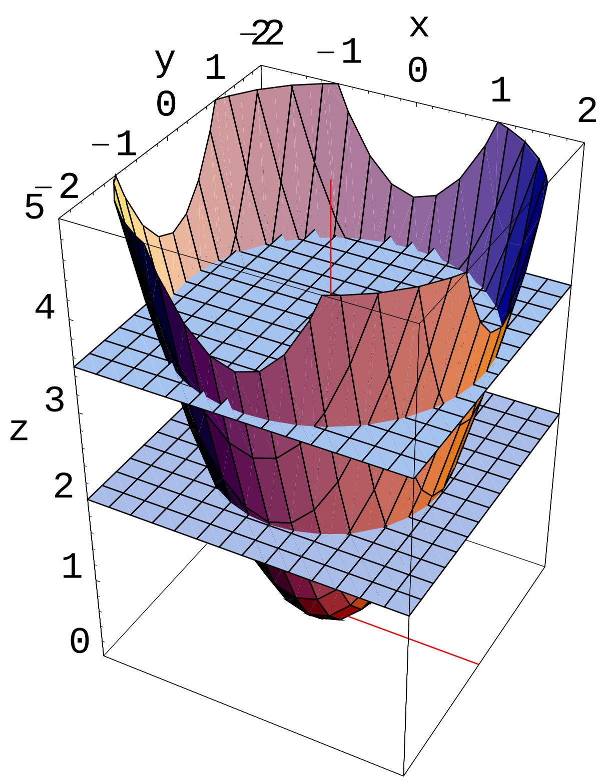

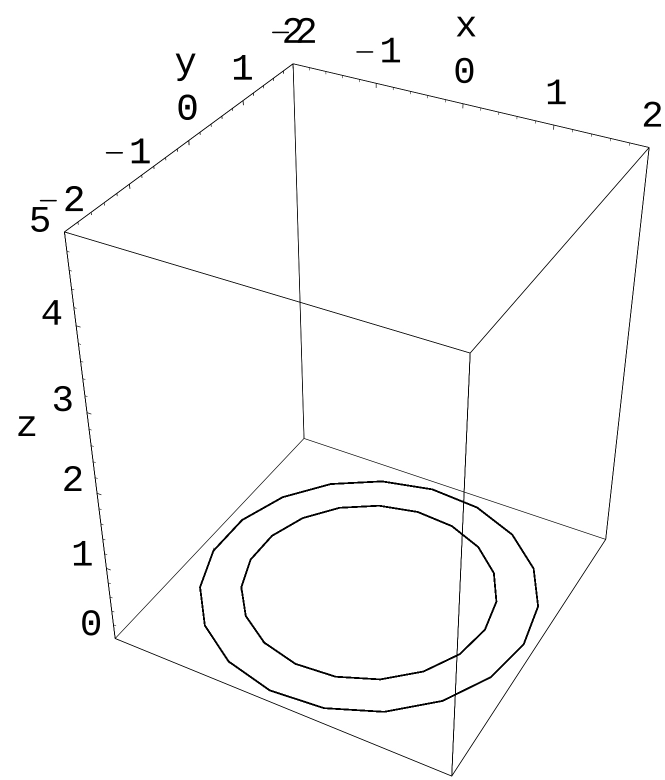

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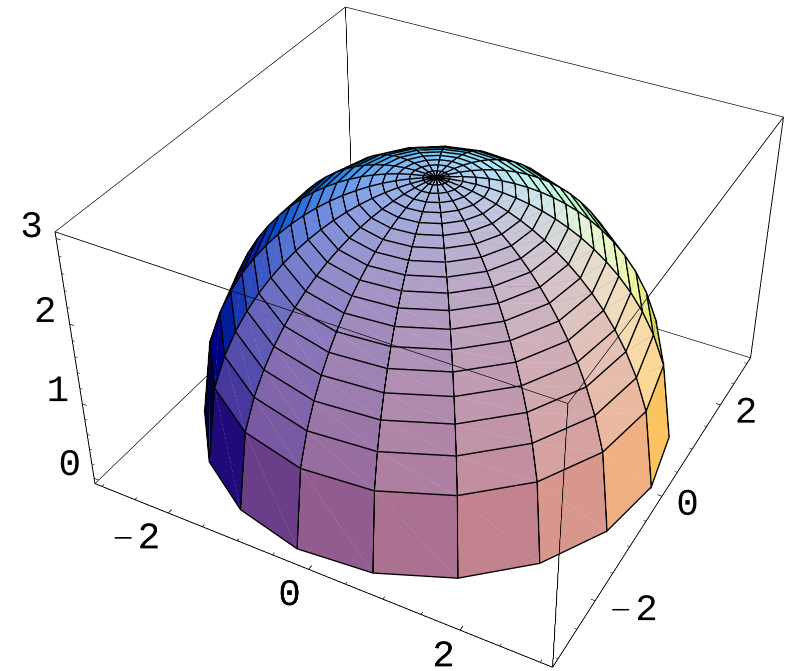

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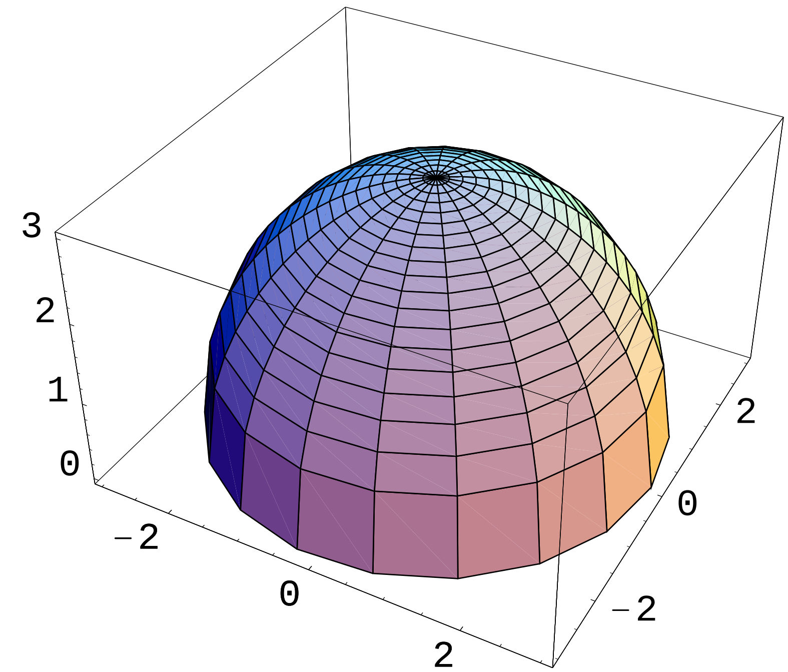

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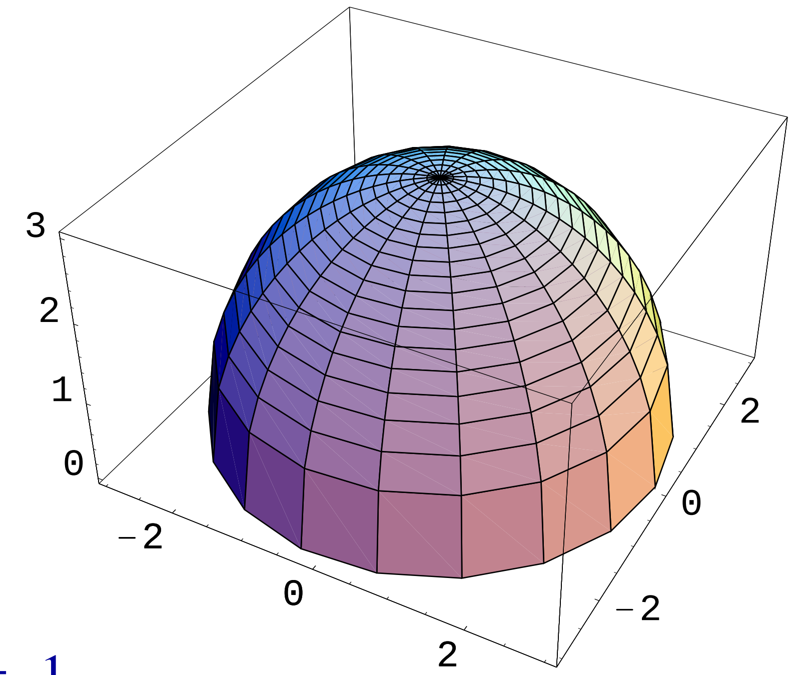

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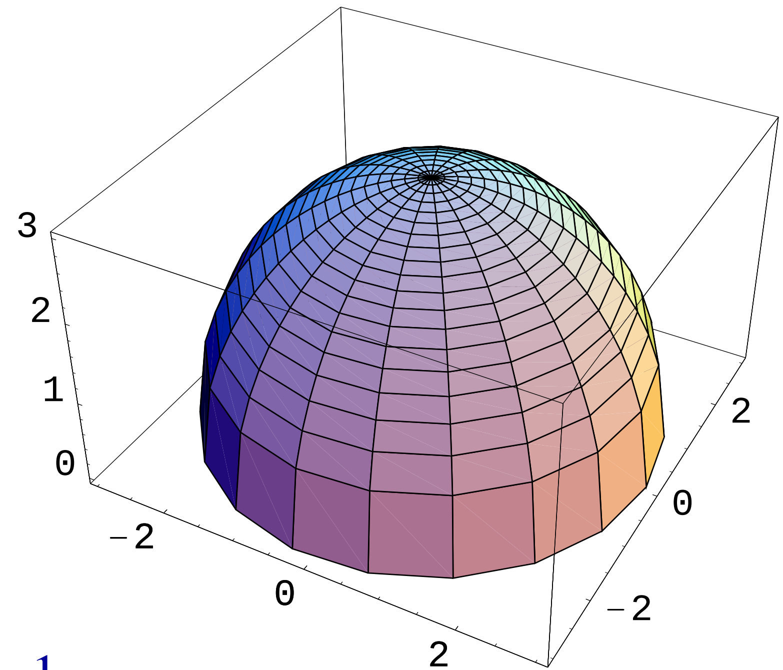

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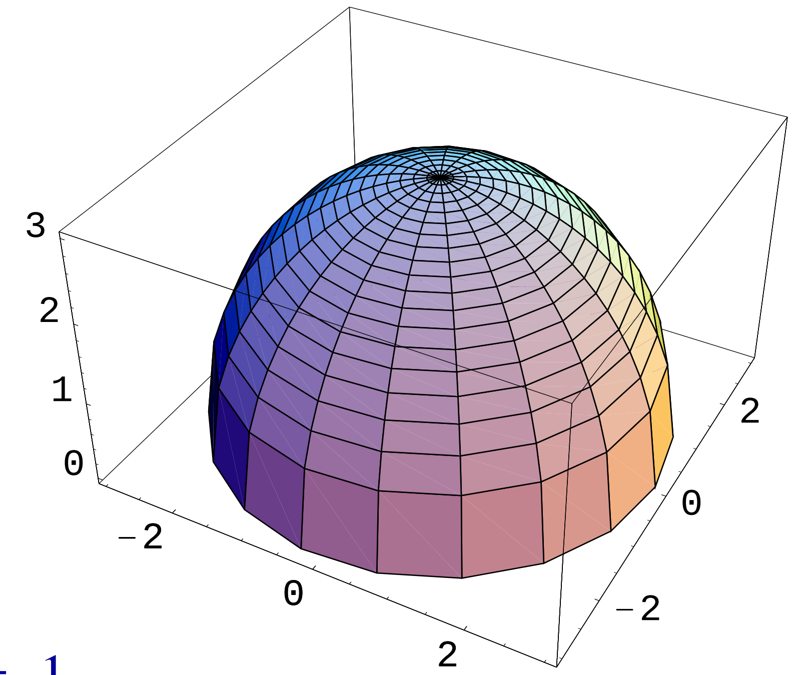

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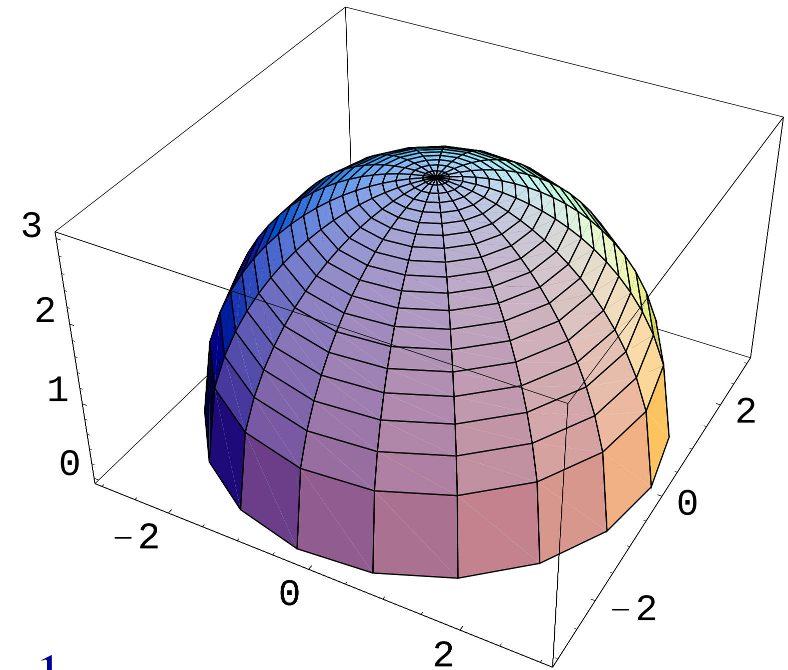

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Get a function of y $\quad z = f(1, y) = \sqrt{9 - 1^2 - y^2}$

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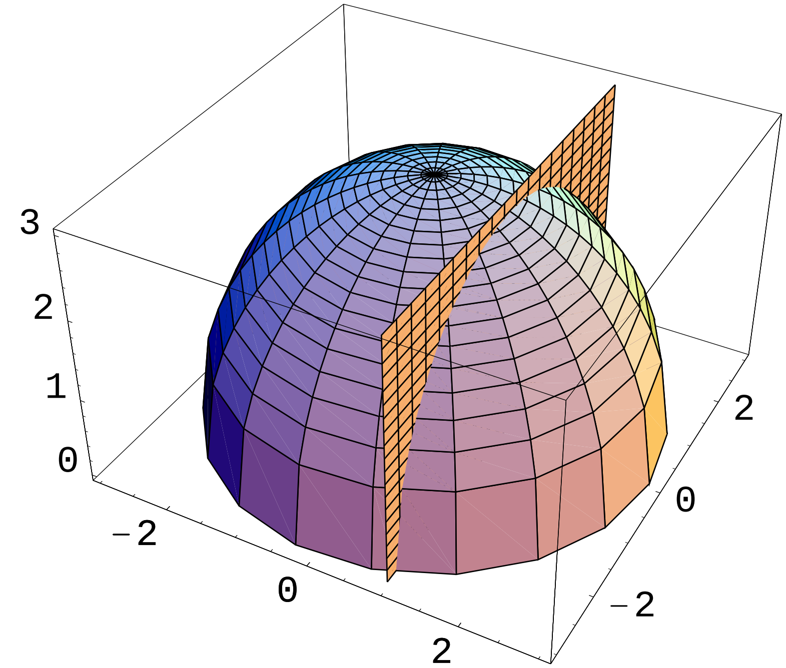

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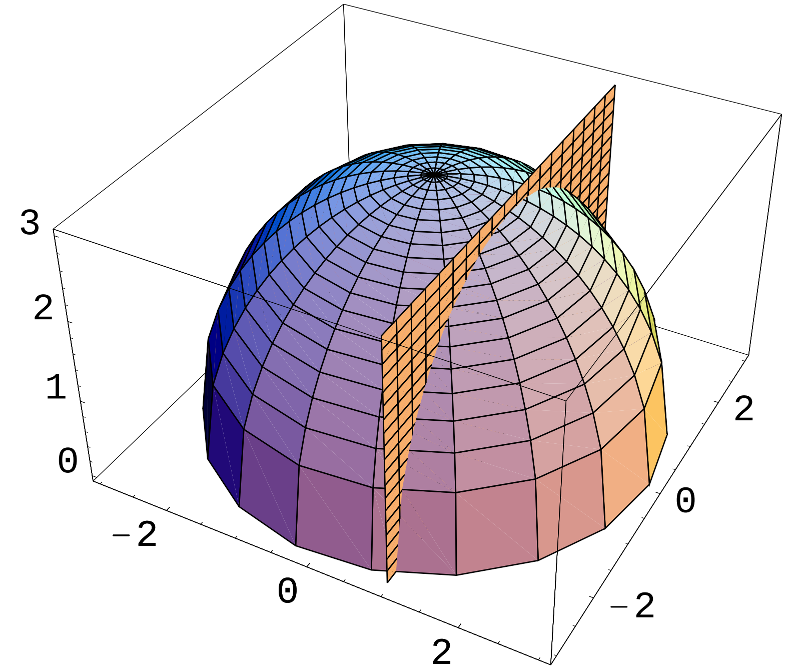

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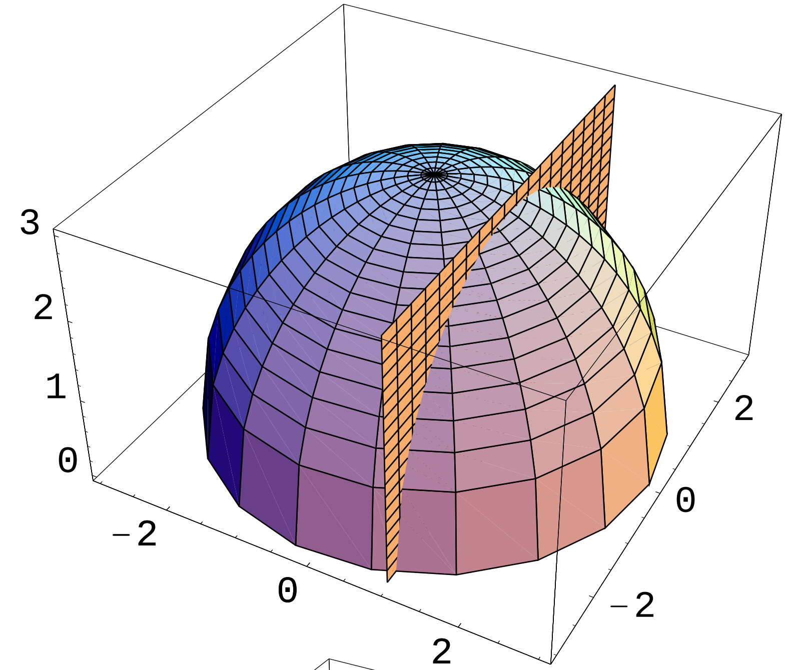

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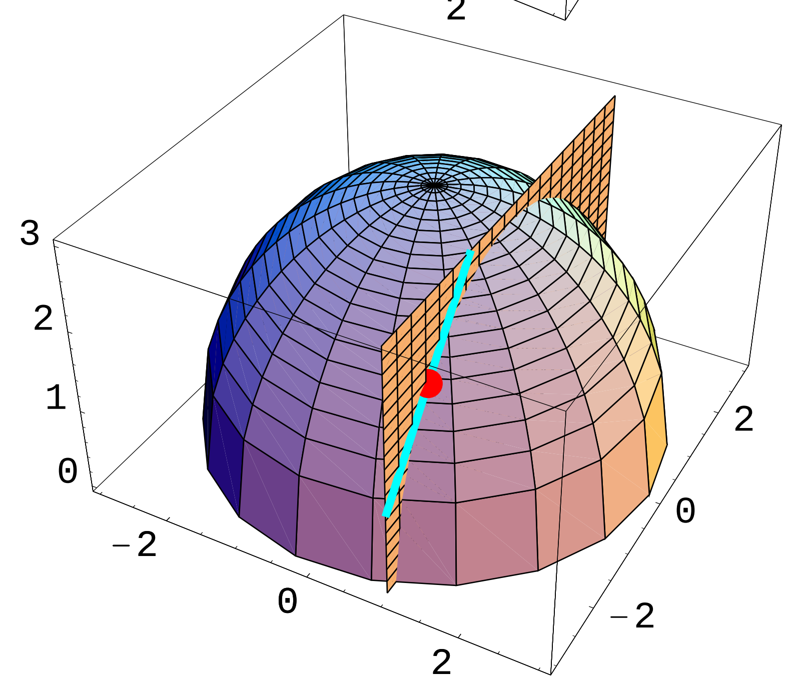

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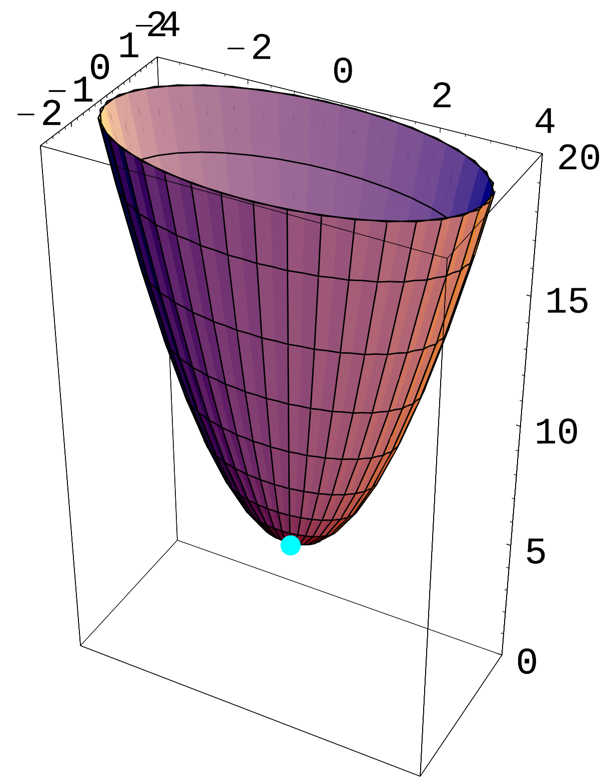

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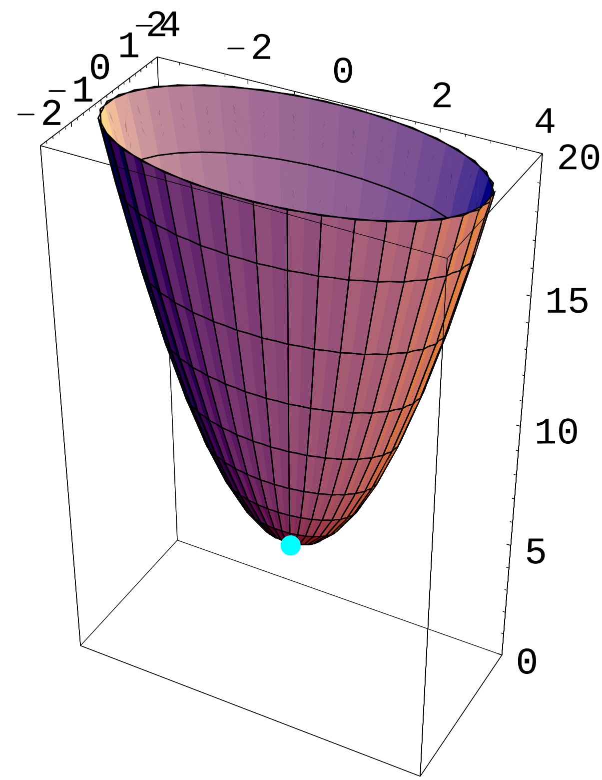

Remark Local minimum point of the graph $(0, 0, 5)$

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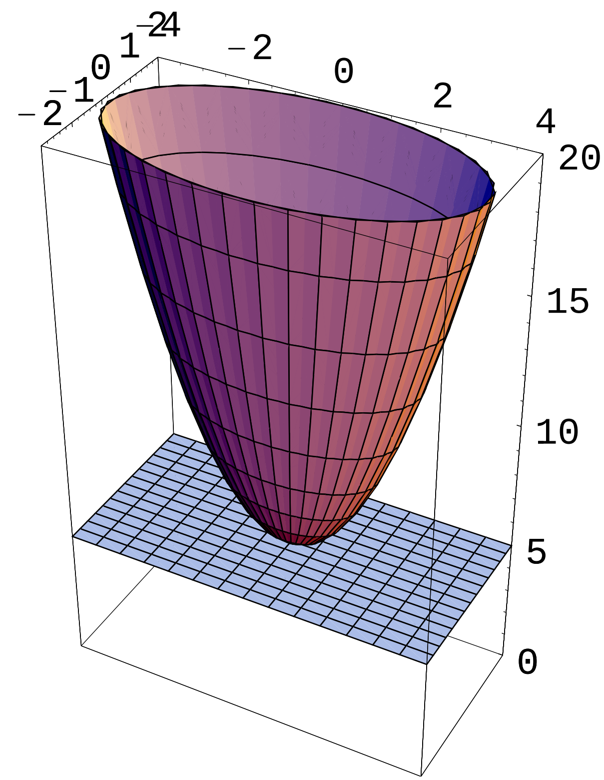

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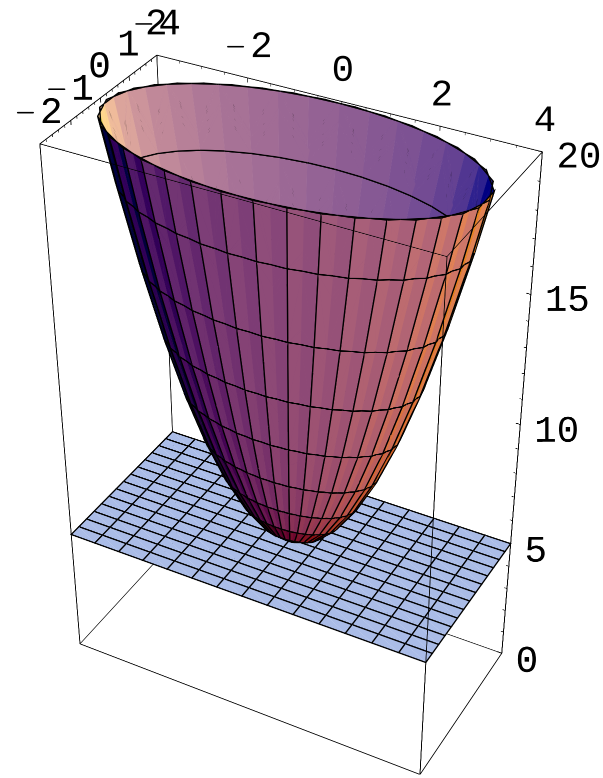

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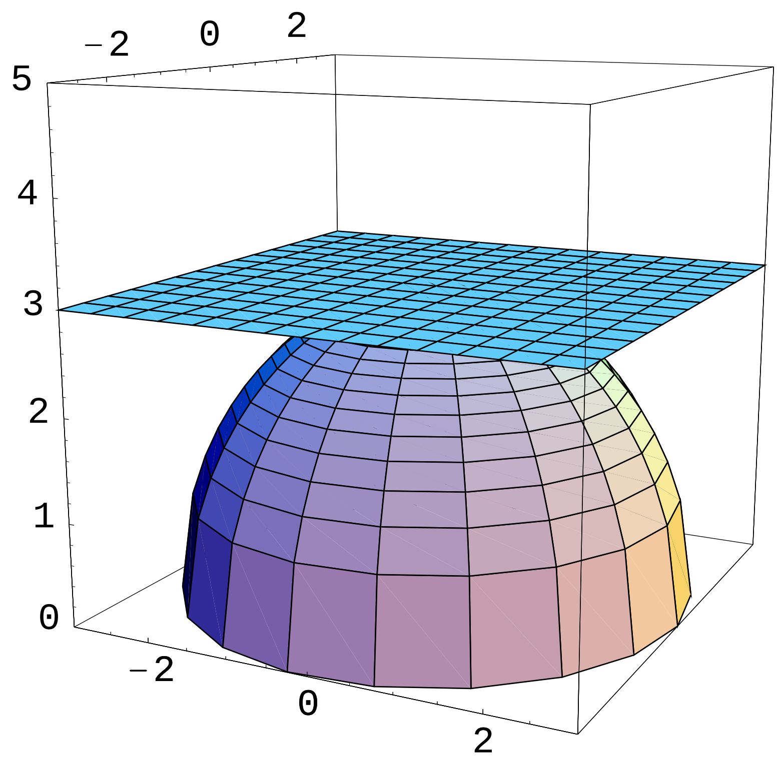

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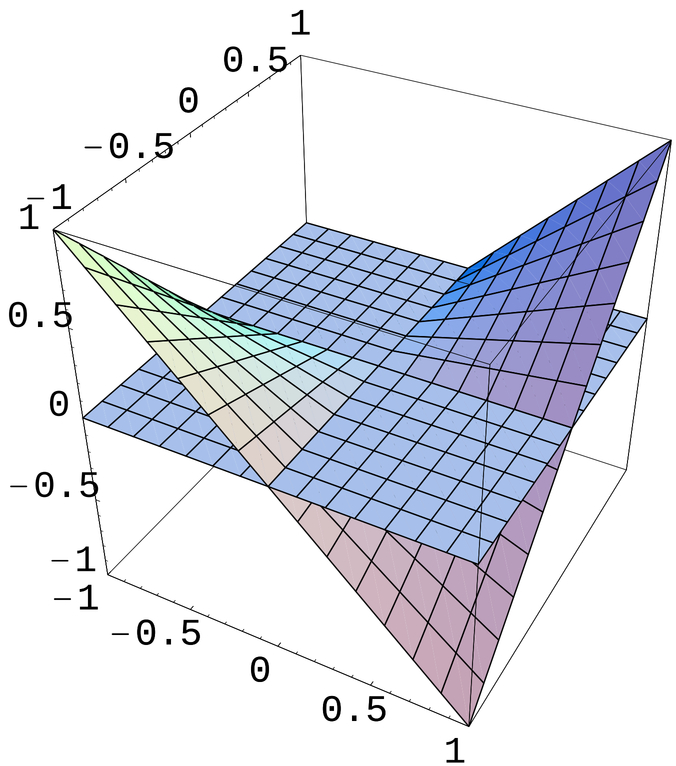

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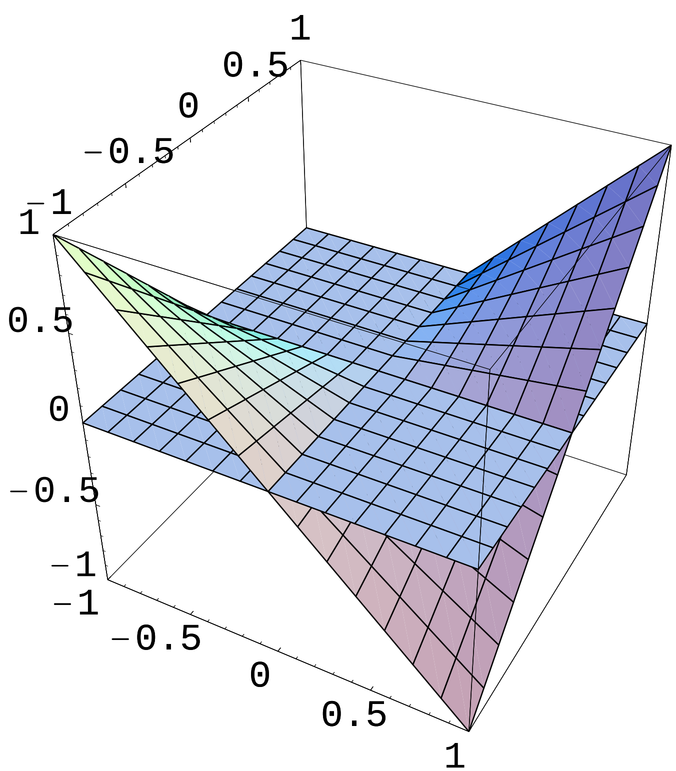

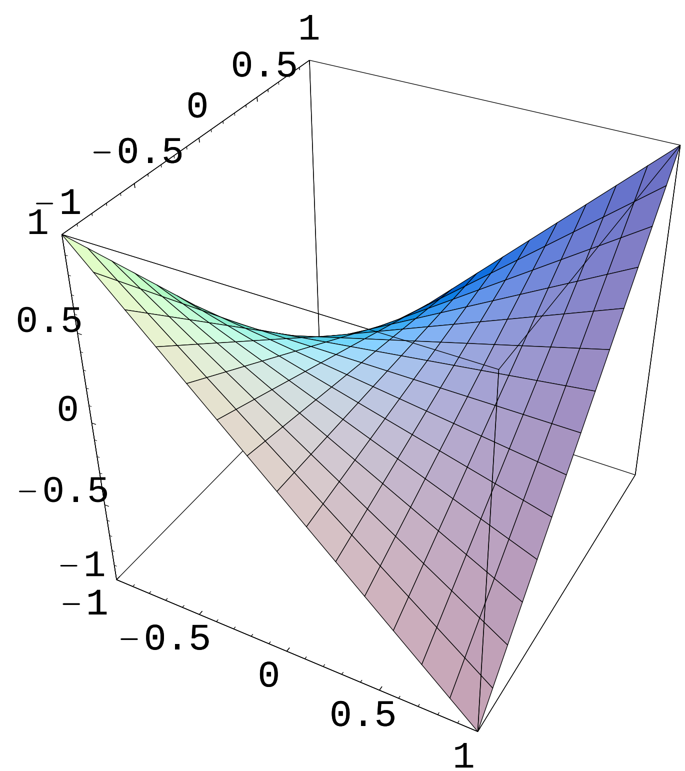

To determine nature of critical points

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Second Partial-derivatives Test (similar to one variable case)

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First Partial-derivatives ??? (no such test)