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	$-1 < x < 0$	$x = 0$	$0 < x < 2$
$y = e^x - 1$	$-$	0	$+$

For $-1 \leq x \leq 0$, graph is below the x-axis.

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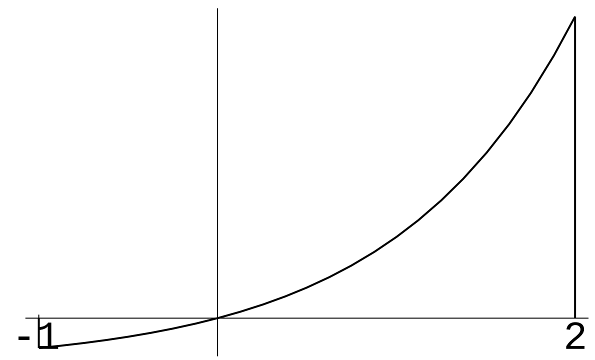

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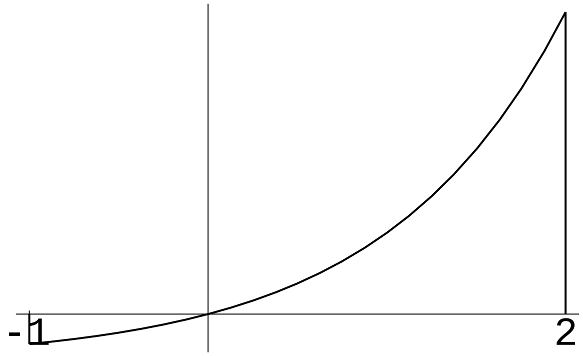

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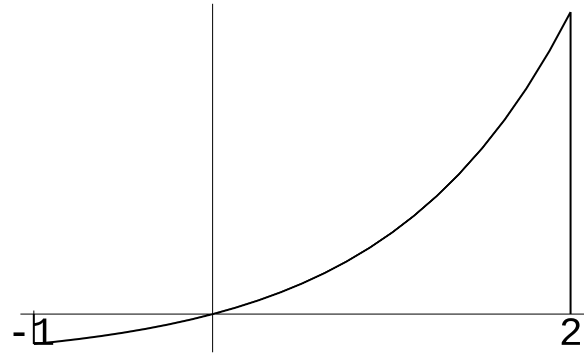

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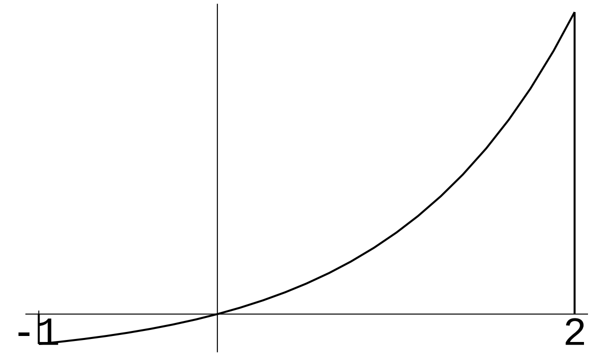

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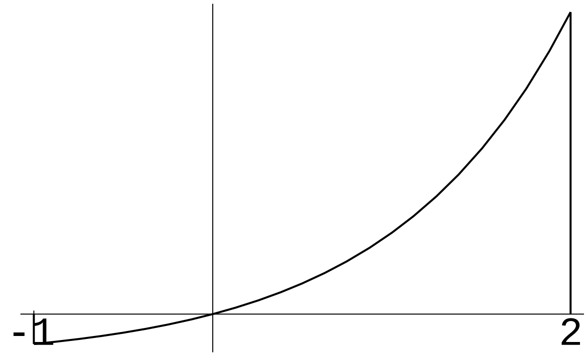

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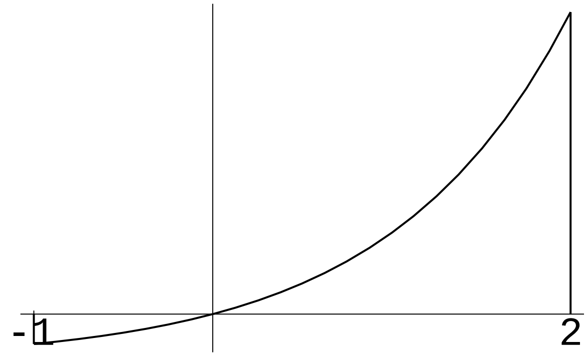

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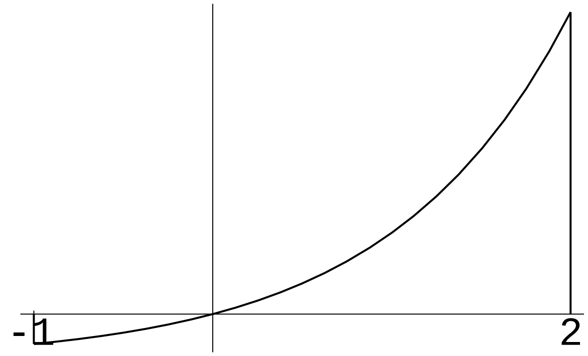

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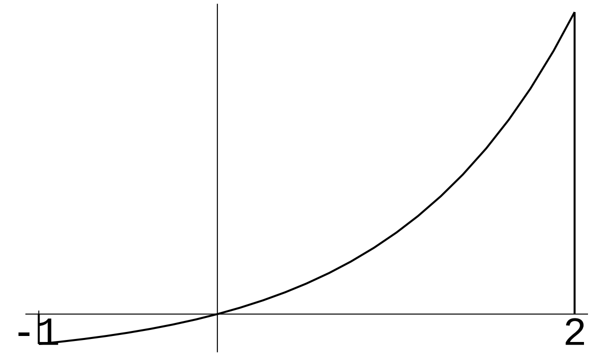

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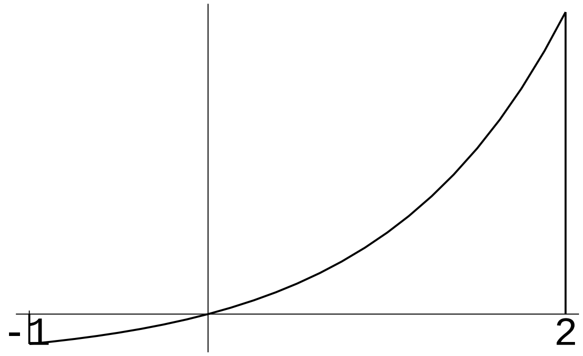

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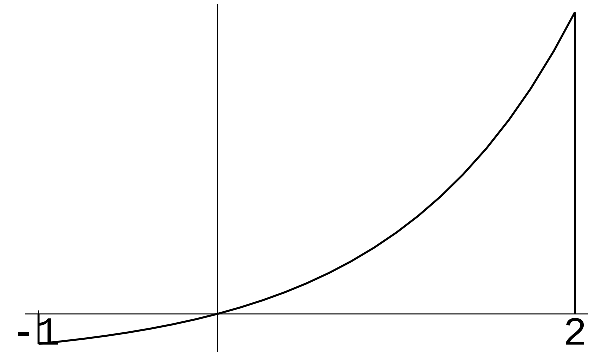

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- Up to this moment, can do simple integration using
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Chain rule in alternative form

$$\frac{\mathrm{d}}{\mathrm{d}x}F\big(g(x)\big) = F'\big(g(x)\big) \cdot g'(x)$$

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Therefore $\int 2x\cos x^2\,\mathrm{d}x = \sin x^2 + C$

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Example Find $\displaystyle\int (x^2 + 1)^2 \cdot 2x\, \mathrm{d}x$.

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- $\int 2x(x^2 + 1) \, \mathrm{d}x$

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- Alternative method $\displaystyle\int (x^2+1)^2 \cdot 2x\,\mathrm{d}x = \int 2x(x^4+2x^2+1)\,\mathrm{d}x$

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- Change the integral $\int \sqrt{x^2+1} \cdot 2x \, \mathrm{d}x$, can use substitution method but not method by expansion.

Example $$\int (x^2+1)^2 \cdot 2x\,\mathrm{d}x = \int u^2\,\mathrm{d}u \qquad \textit{where } u = x^2+1$$

$$= \frac{u^3}{3} + C = \frac{(x^2+1)^3}{3} + C$$

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(4) Substitute back $u = g(x)$ to express the answer in x.

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consider $\displaystyle\int \frac{1}{x} \cdot \ln x \, \mathrm{d}x$ *or* $\displaystyle\int \ln x \cdot \frac{1}{x} \, \mathrm{d}x$

Solution

Example Find $\int \frac{\ln x}{x}\, dx$.

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- Let $u = \ln x$.

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- Let $u = x^3$.
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- $\int x^2 e^{x^3}\,dx = \int \frac{1}{3} e^{x^3} \cdot 3x^2\,dx$

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Remark Can use $u = e^{x^3}$.

Example Find $\int x^2 e^{x^3}\,\mathrm{d}x$. *consider* $\int x^2 \cdot e^{x^3}\,\mathrm{d}x$ *or* $\int e^{x^3} \cdot x^2\,\mathrm{d}x$

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- Let $u = x^3$.
- Then $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ and so $\mathrm{d}u = 3x^2\,\mathrm{d}x$.
- $$\int x^2 e^{x^3}\,\mathrm{d}x = \int \frac{1}{3} e^{x^3} \cdot 3x^2\,\mathrm{d}x$$

$$= \int \frac{1}{3} e^{u}\,\mathrm{d}u \qquad \textit{substitution}$$

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Remark Can use $u = e^{x^3}$. Get $\int x^2 e^{x^3}\,\mathrm{d}x = \int \frac{1}{3}\,\mathrm{d}u$

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