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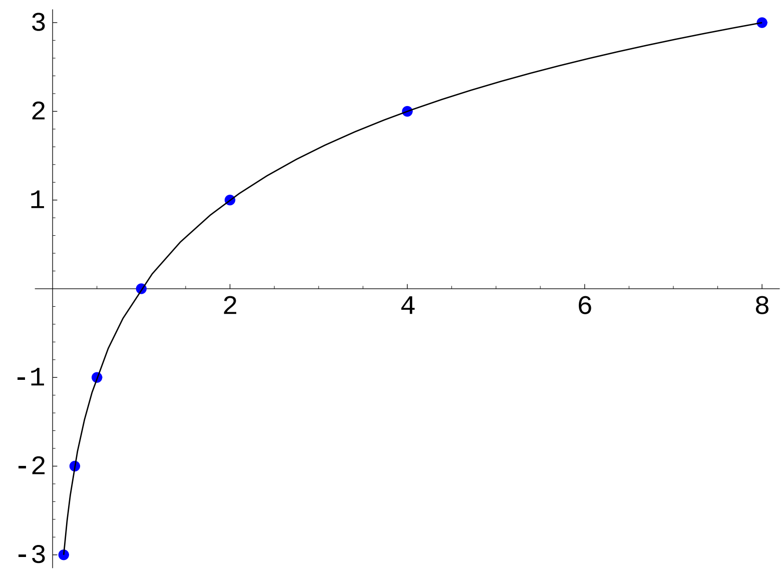

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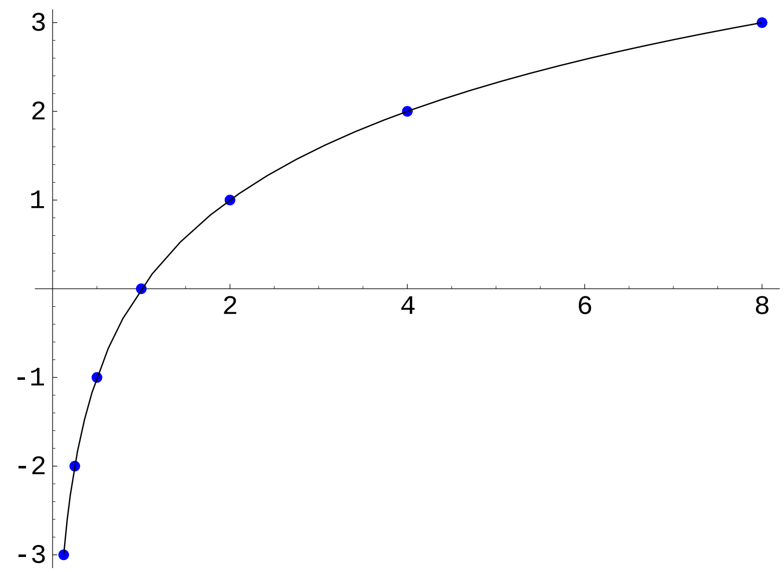

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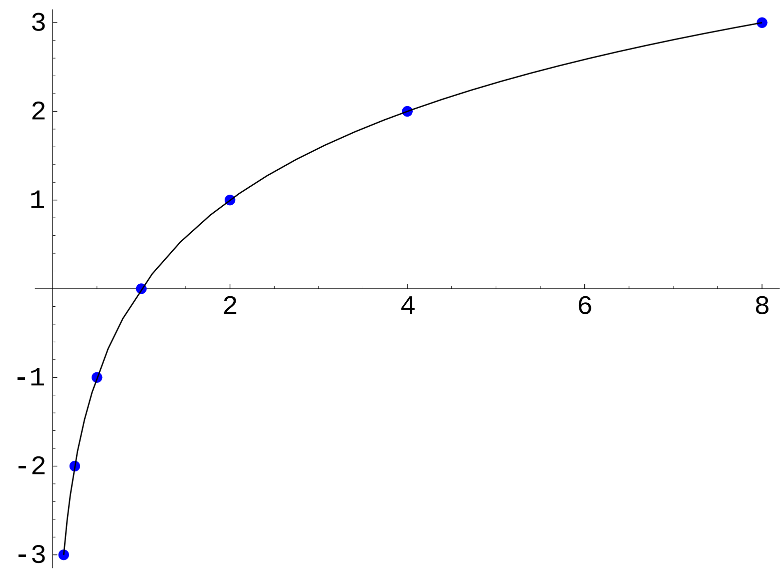

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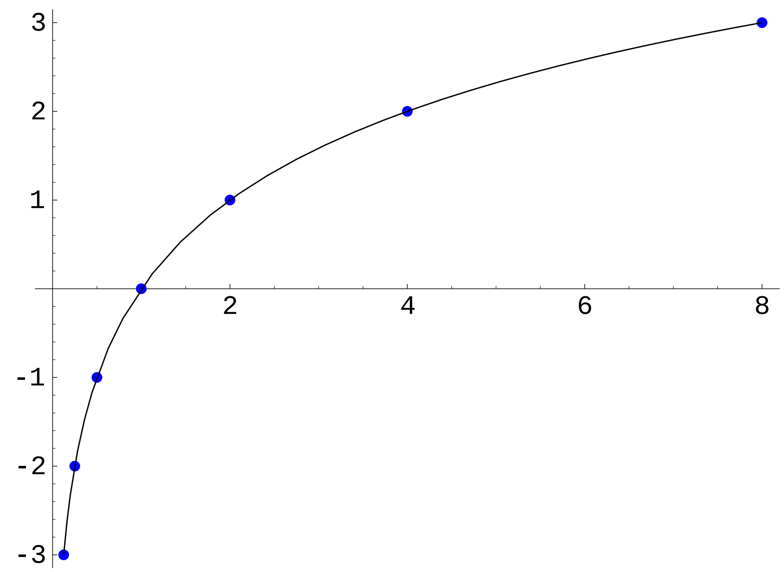

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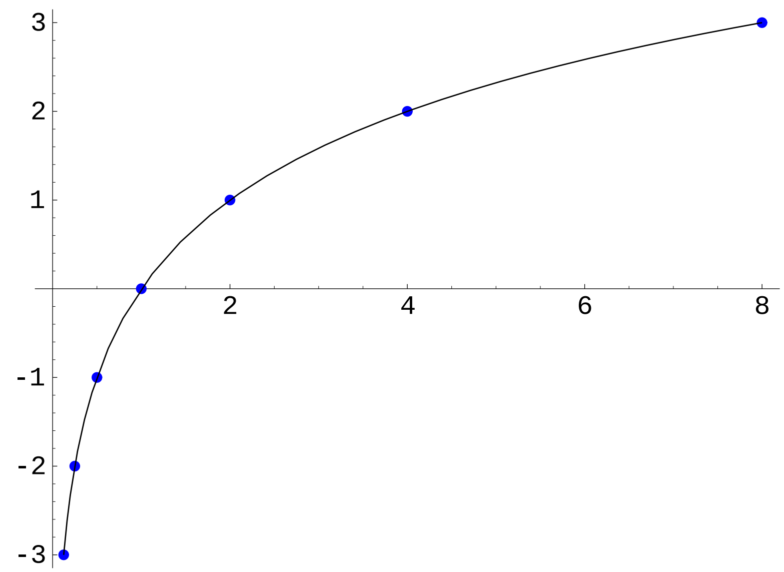

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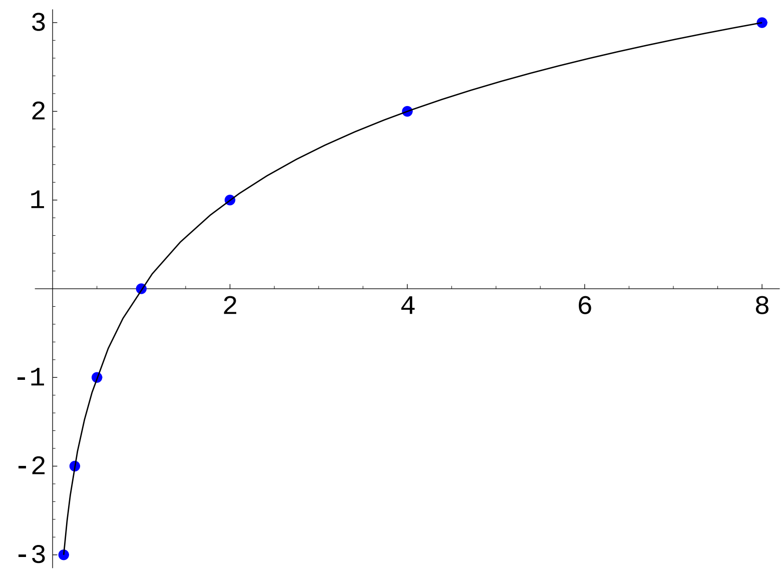

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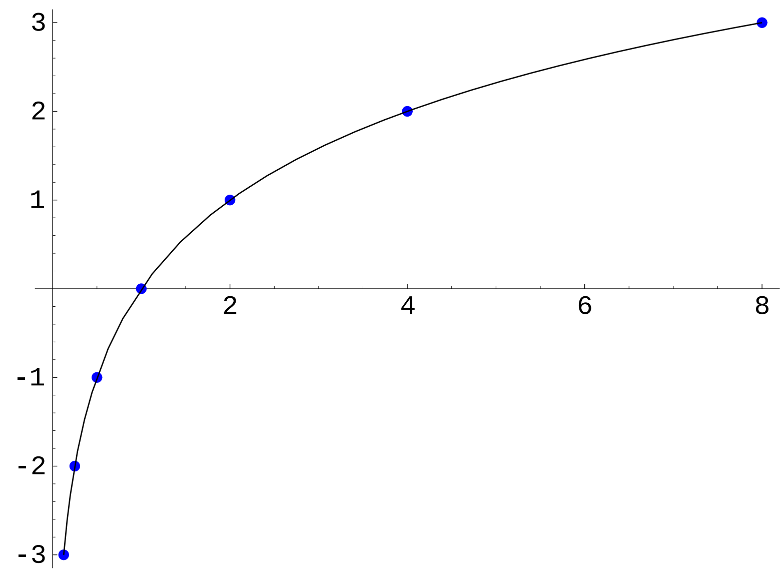

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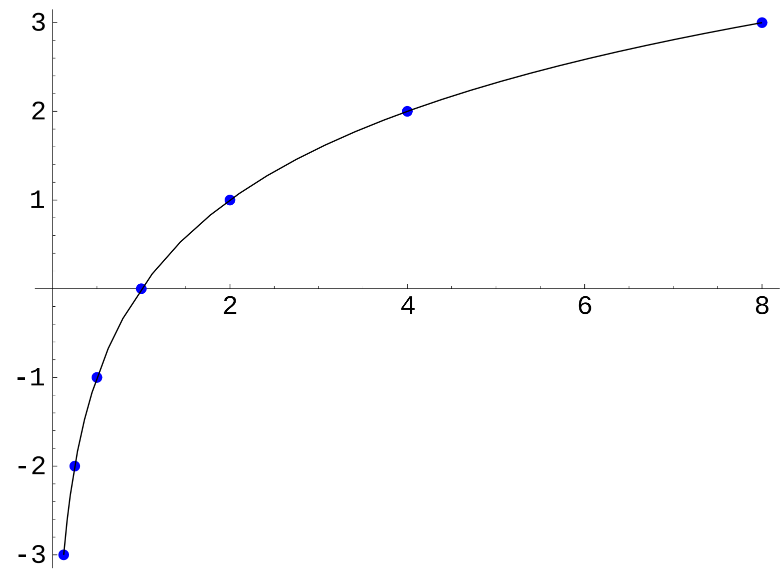

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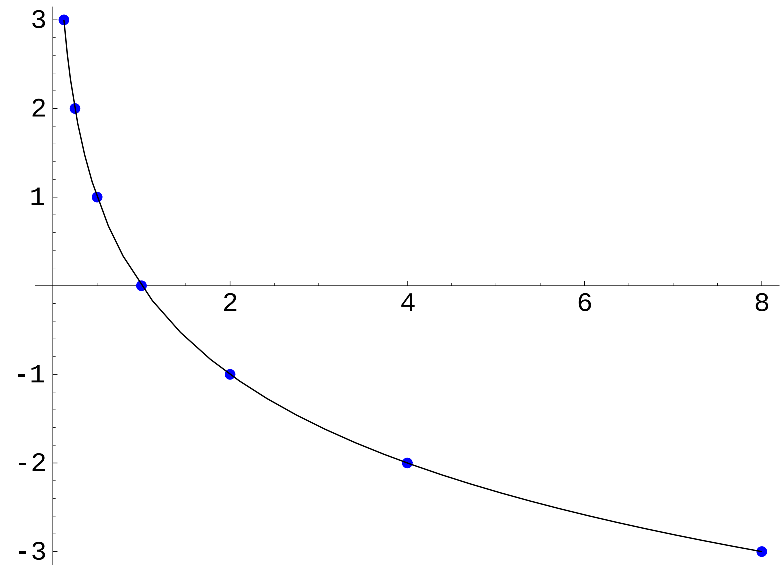

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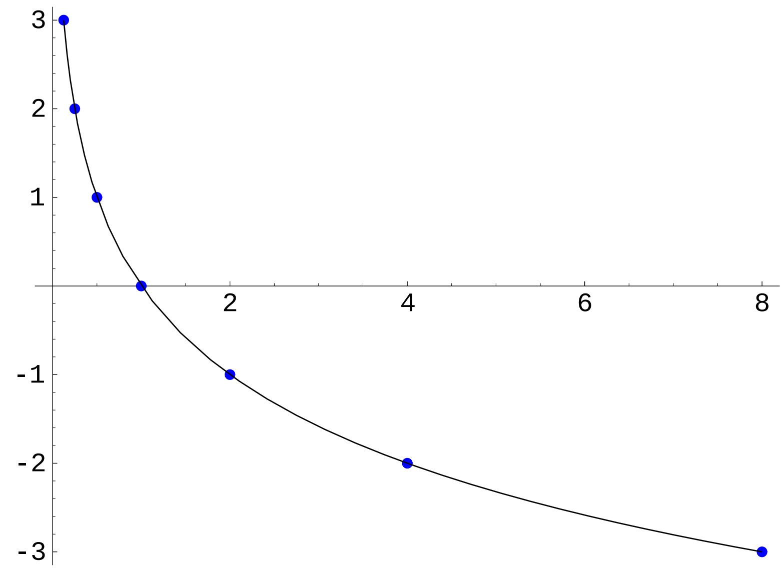

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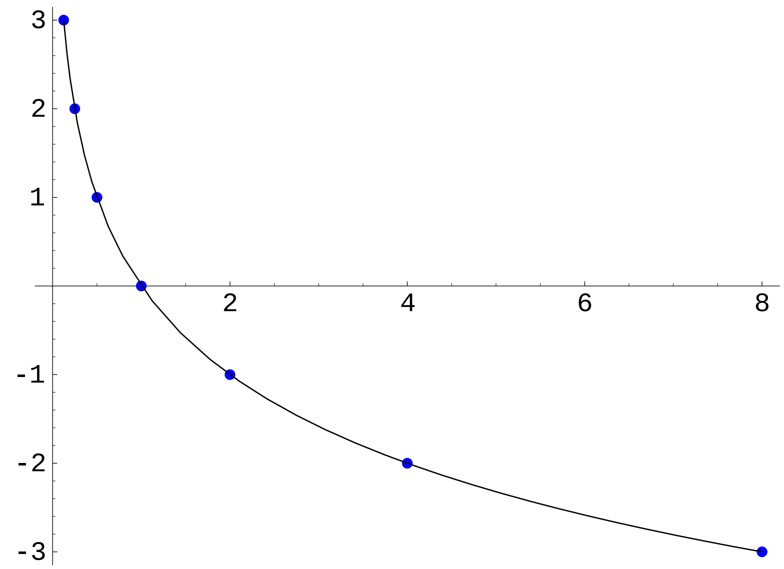

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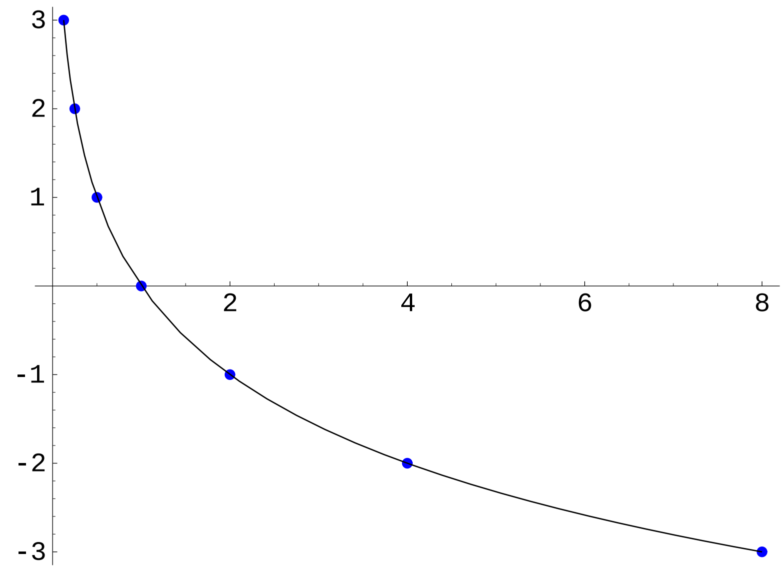

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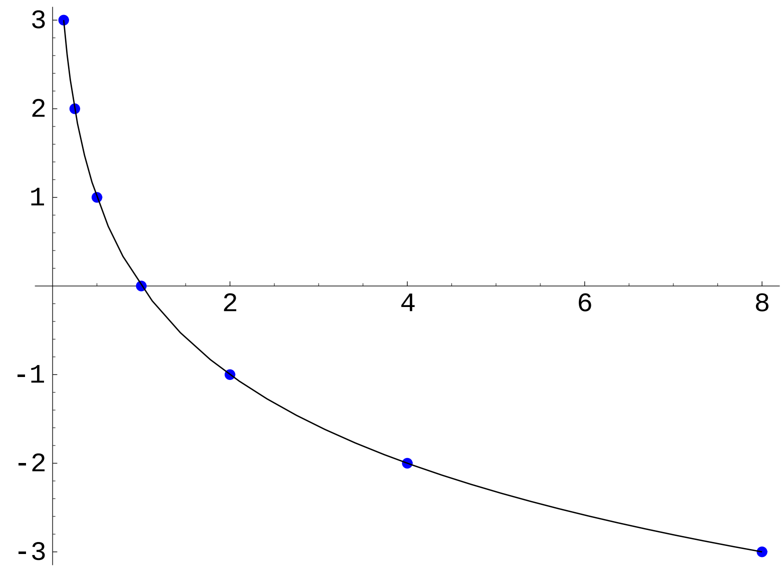

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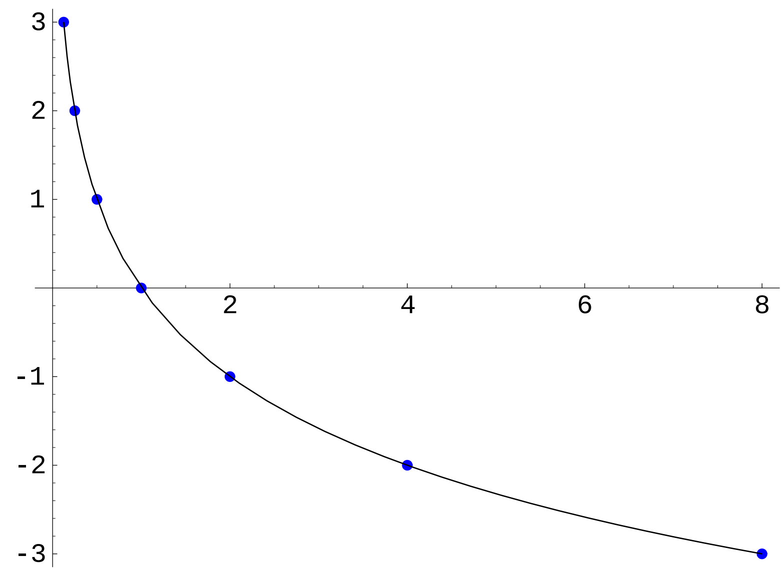

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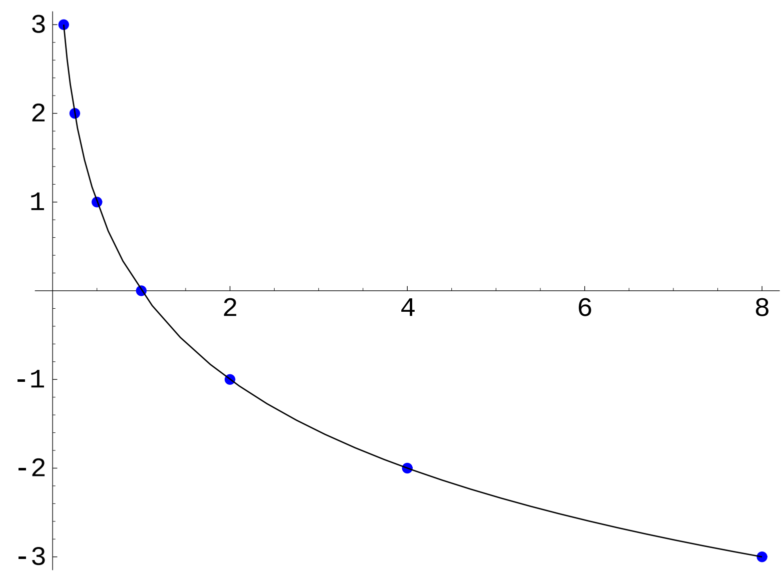

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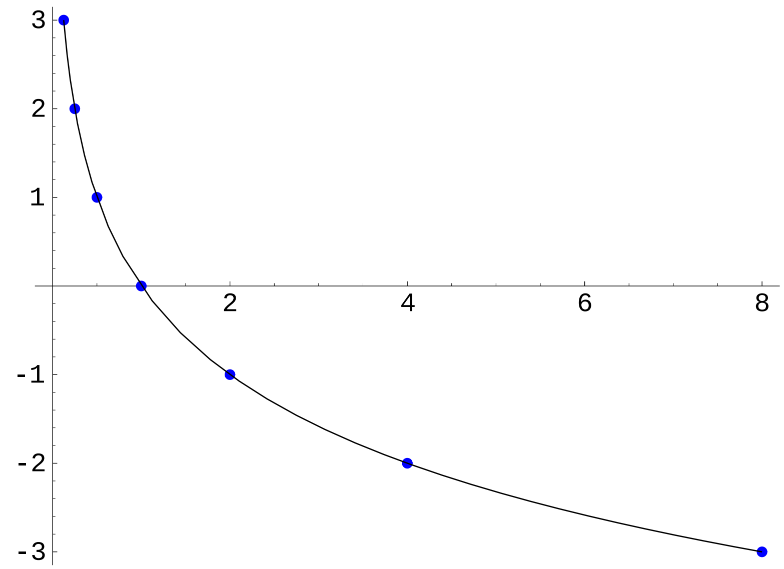

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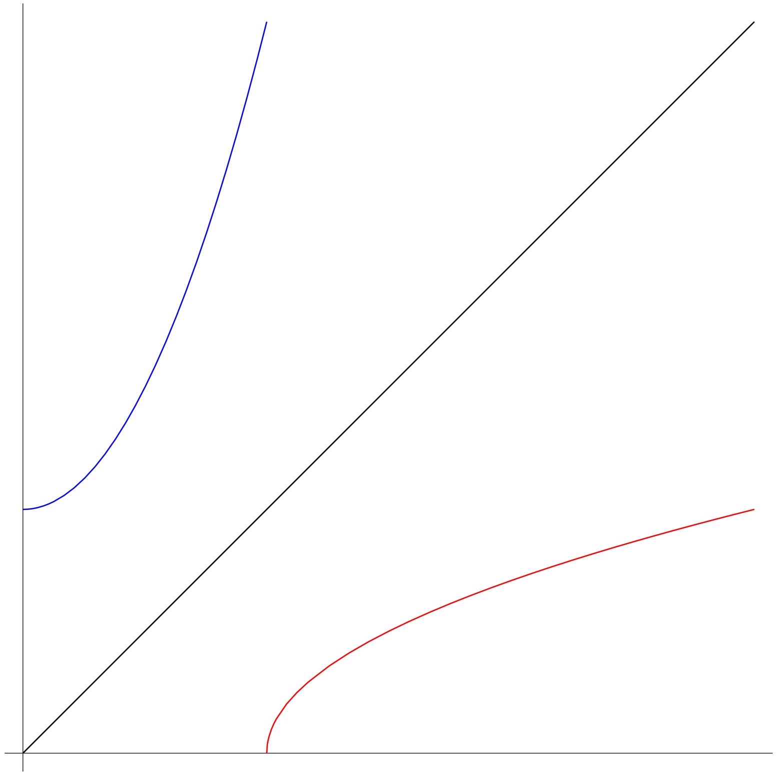

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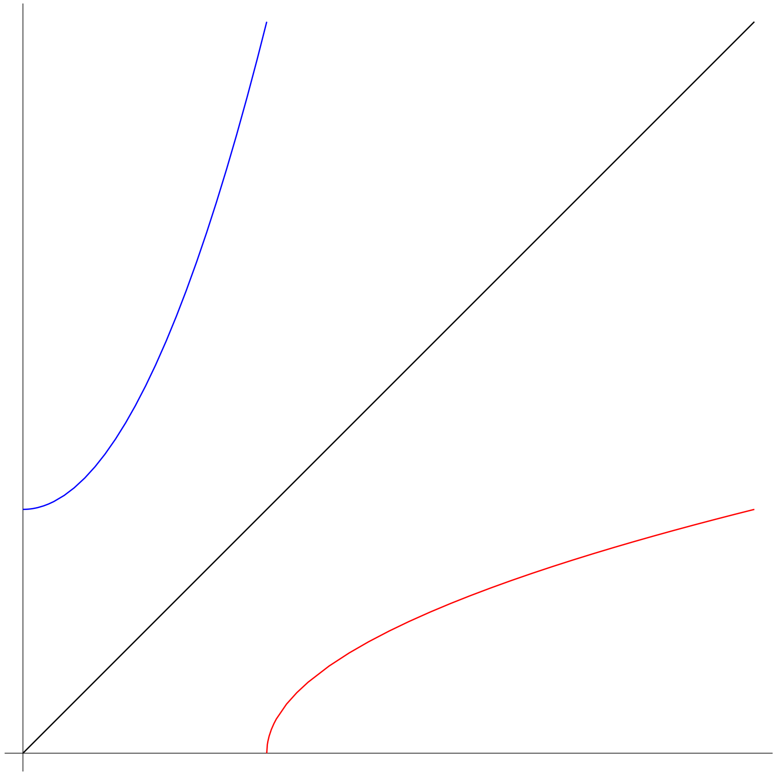

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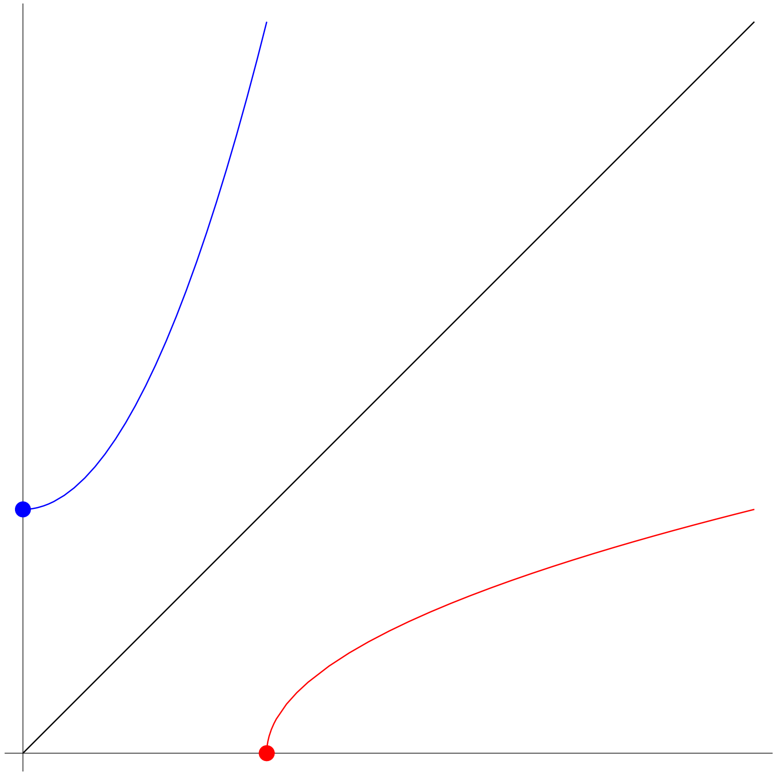

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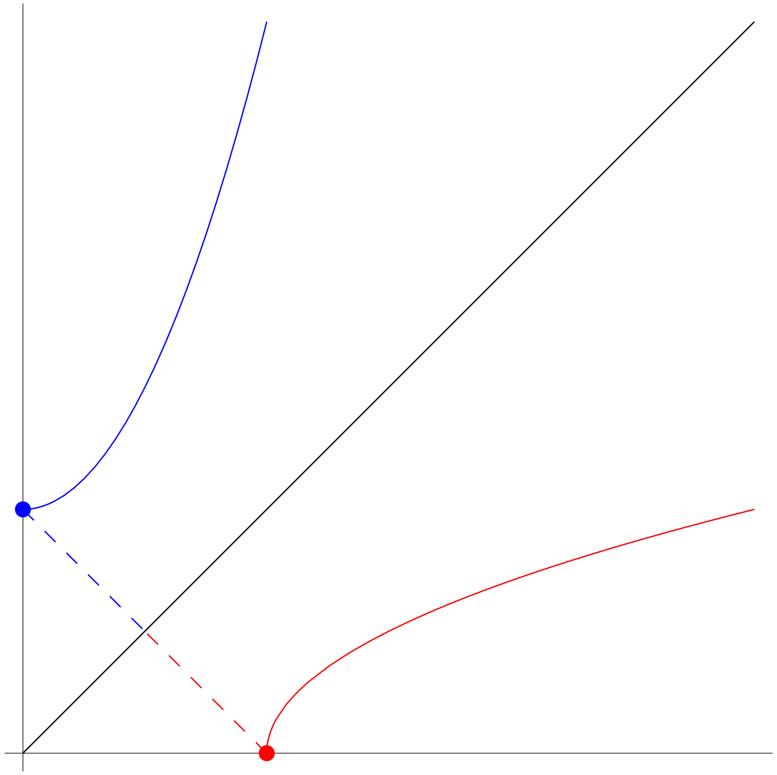

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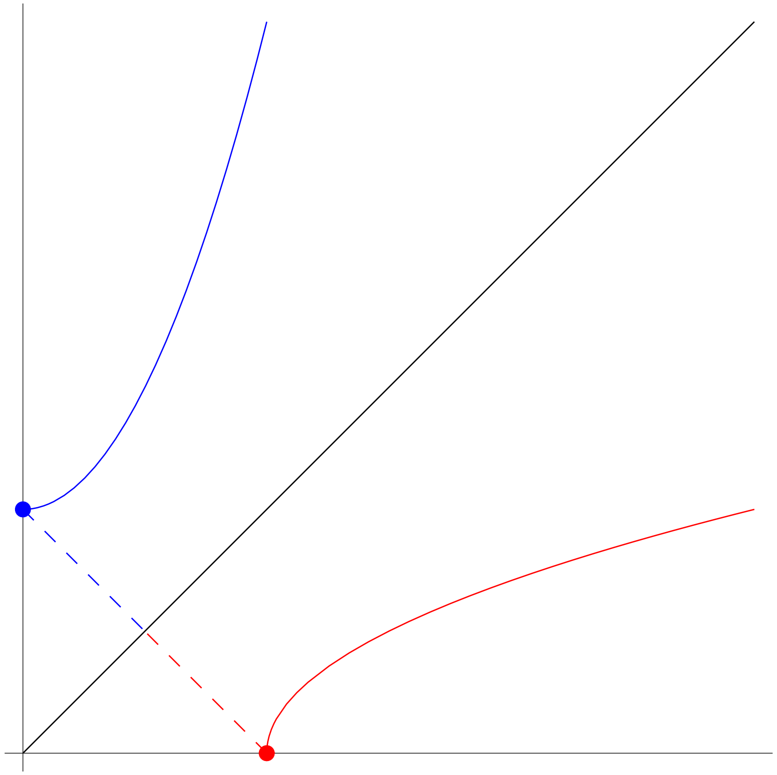
  $\because\ f(a) = b \Longleftrightarrow f^{-1}(b) = a$



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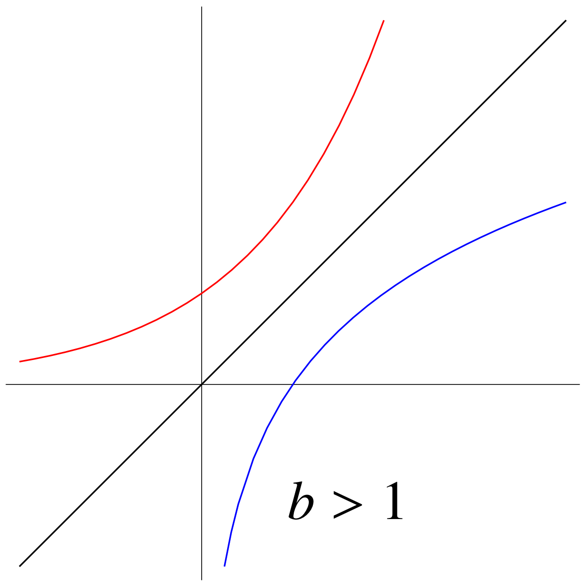

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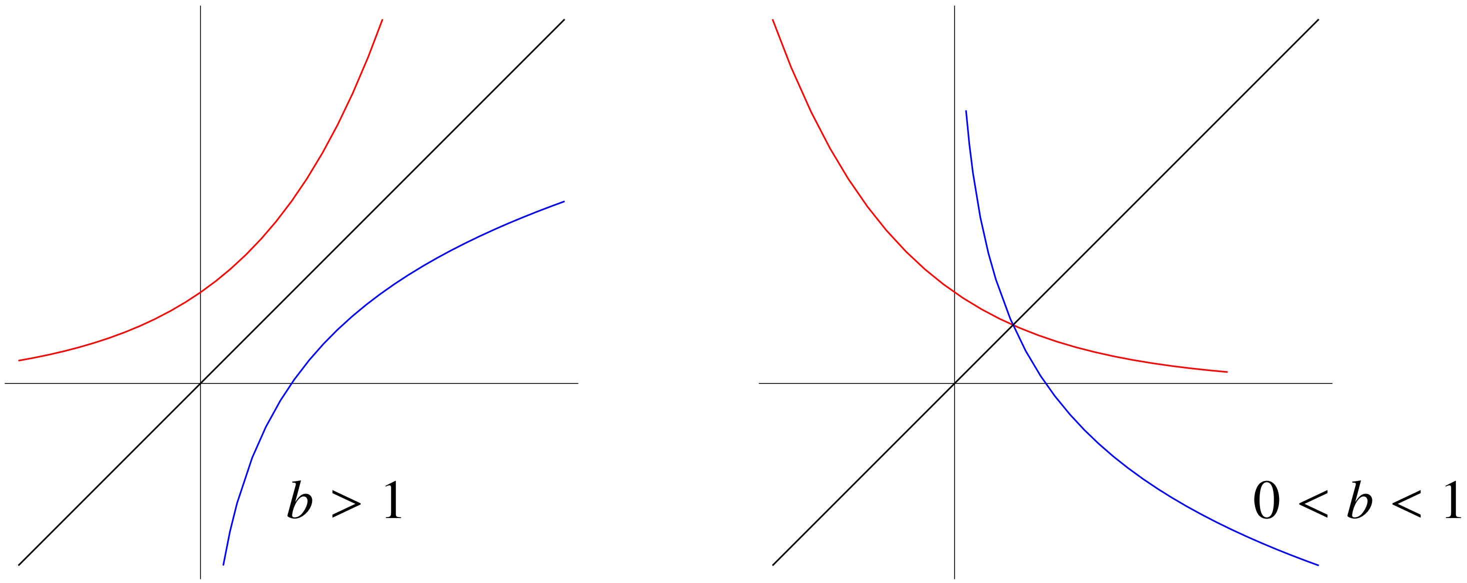

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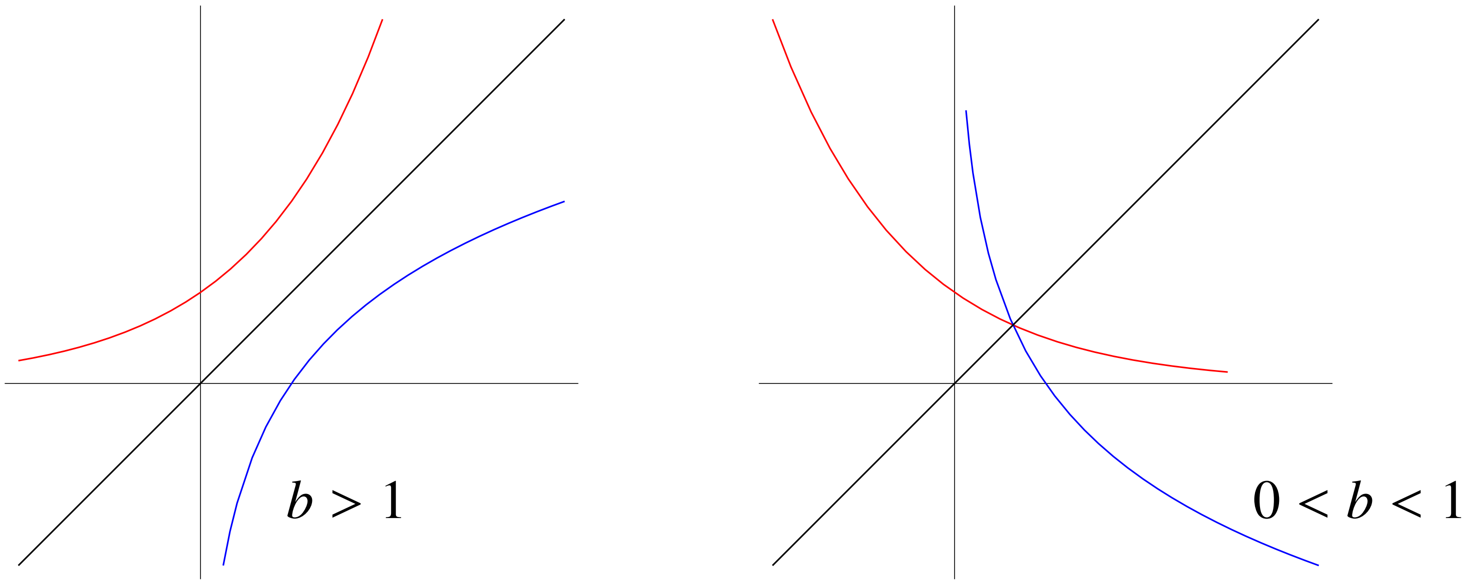

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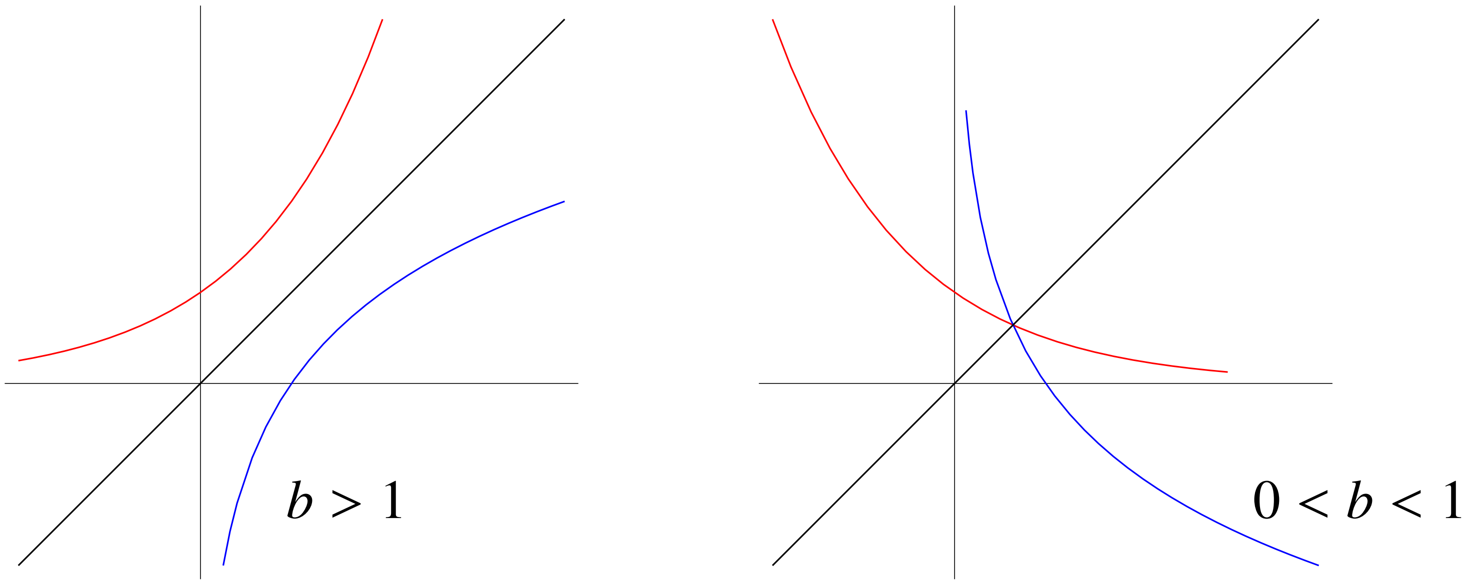

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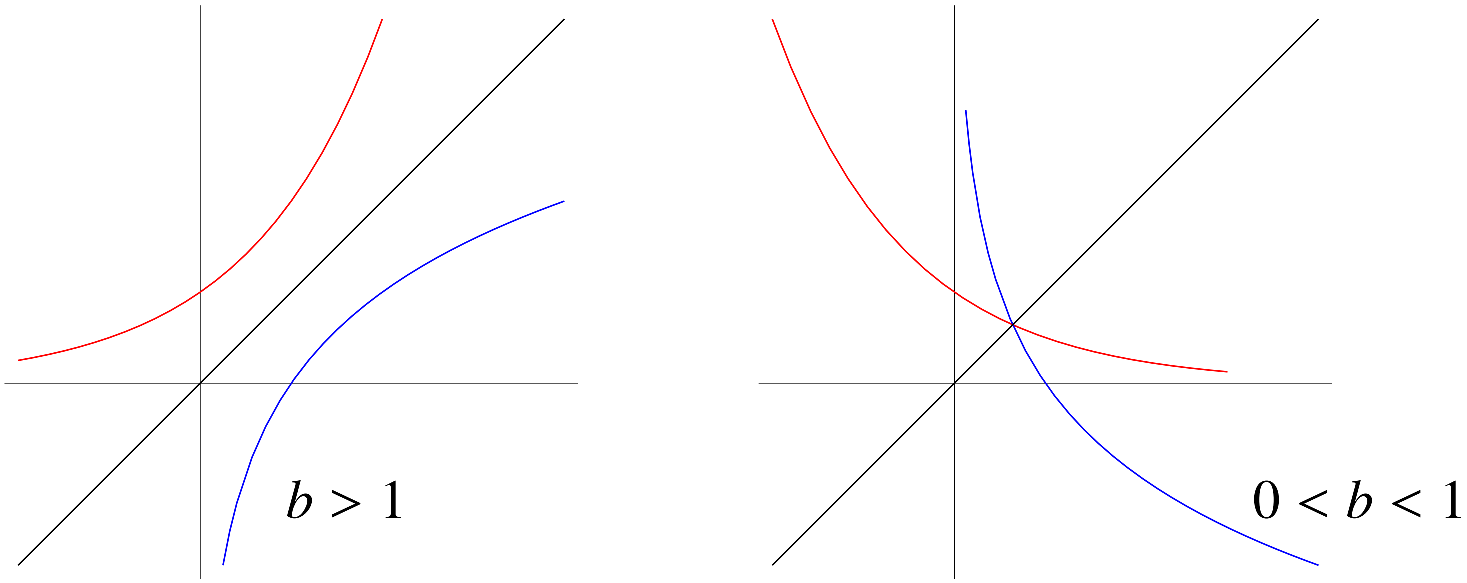

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logarithmic functions and exponential functions are injective

- $\log_b m = \log_b n \implies m = n$
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Example Solve the equation $\log_2 x = 5 - \log_2(x+4)$

Solution

$$\begin{aligned}
\log_2 x &= \log_2 2^5 - \log_2(x+4) \\
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Differentiation of the Natural Logarithmic Function

Formula $$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

Proof

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put $t=\frac{x}{h}$

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Example Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following

(1) $y = 2 + 3\ln x$

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Eg. Suppose x changes twice as fast as y, then

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Eg. Suppose x changes twice as fast as y, then y changes half as fast as x.

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Formula $$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$$

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Proof Put $y = \mathrm{e}^{x}$.

Formula

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Proof Put $y = \mathrm{e}^x$. Then $x = \ln y$

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Example Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \mathrm{e}^x - 2x$$

(a) Find and classify the critical point(s), if any, of f.

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critical point of f

	$x < \ln 2$	$\ln 2$	$x > \ln 2$
$f'(x)$		0	
f			

	$x < \ln 2$	$\ln 2$	$x > \ln 2$
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f			

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f	$\searrow$		$\nearrow$

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(b) $$f''(x) = \frac{d}{dx}(e^x - 2)$$

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$f'(x)$	$-$	0	$+$
f	$\searrow$		$\nearrow$

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(b)
$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^x - 2)$$
$$= \mathrm{e}^x$$

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Therefore, f has no inflection point.

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