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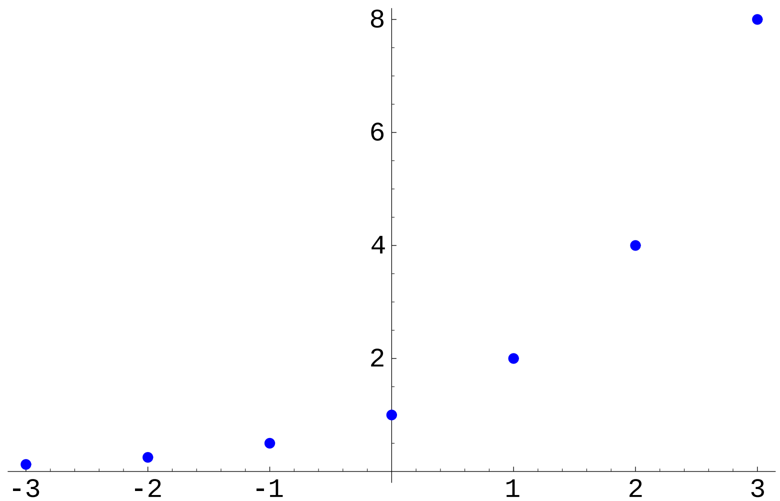

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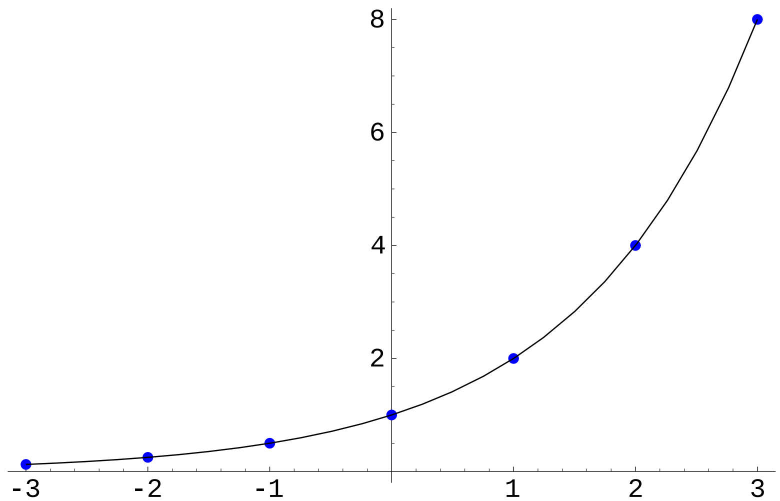

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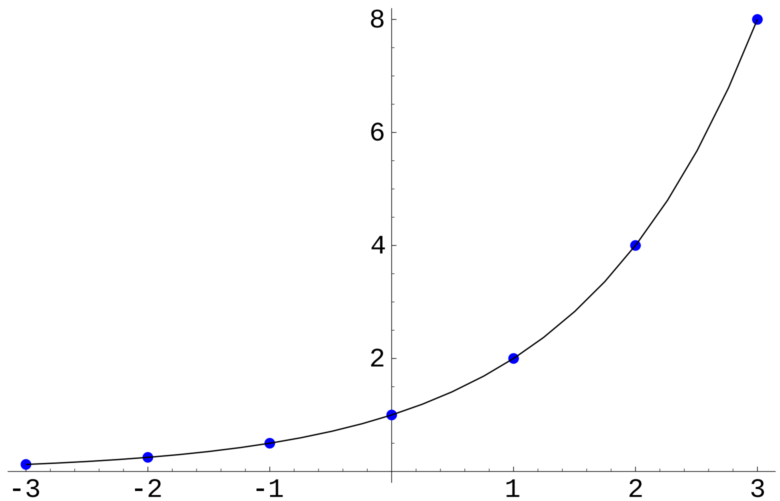


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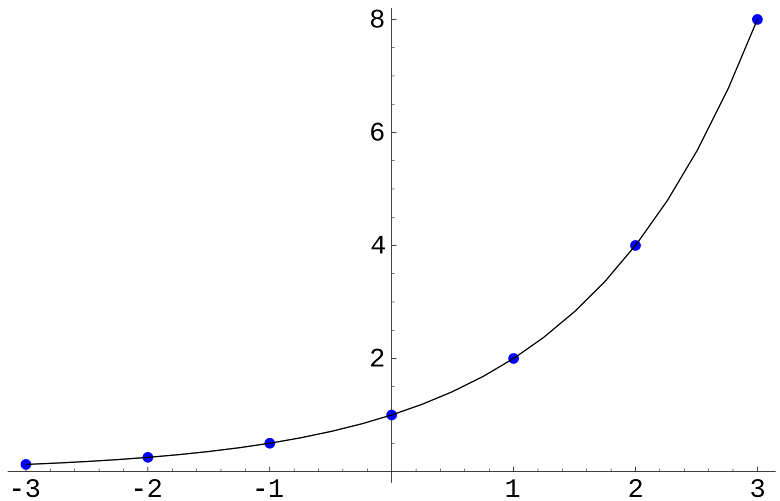


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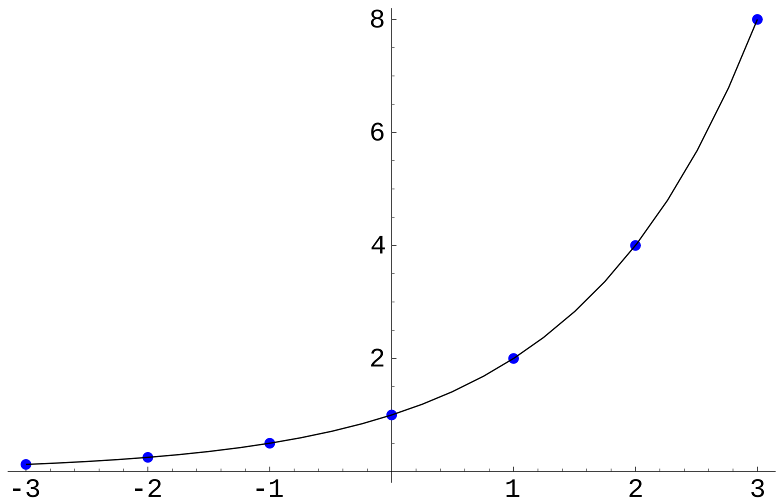


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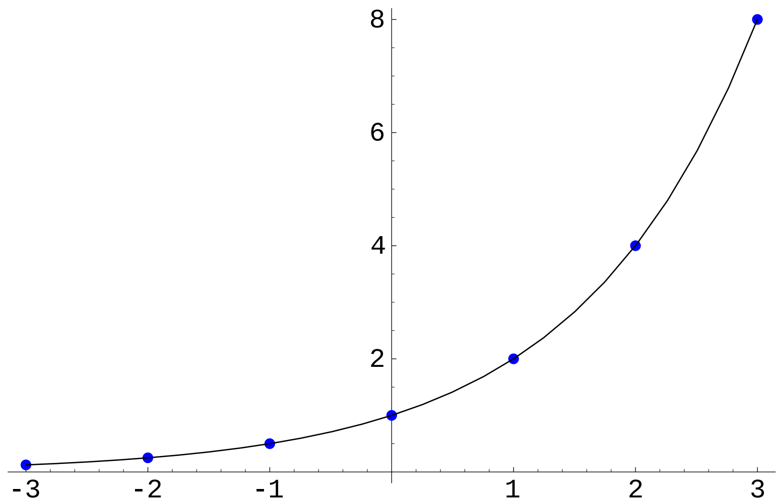


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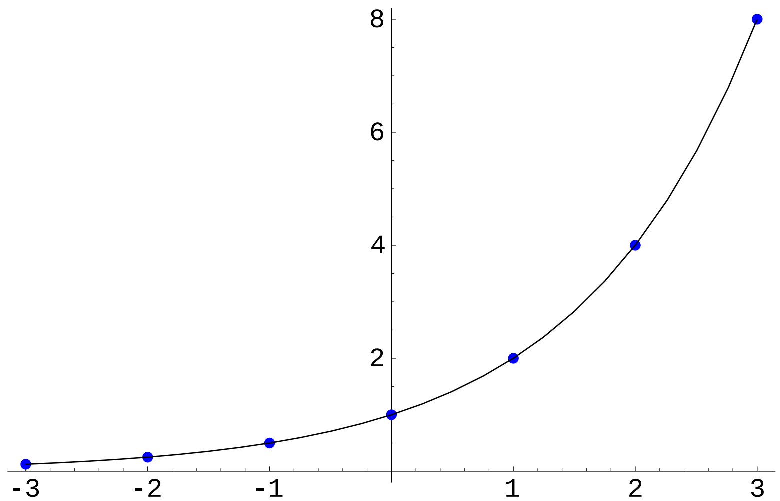


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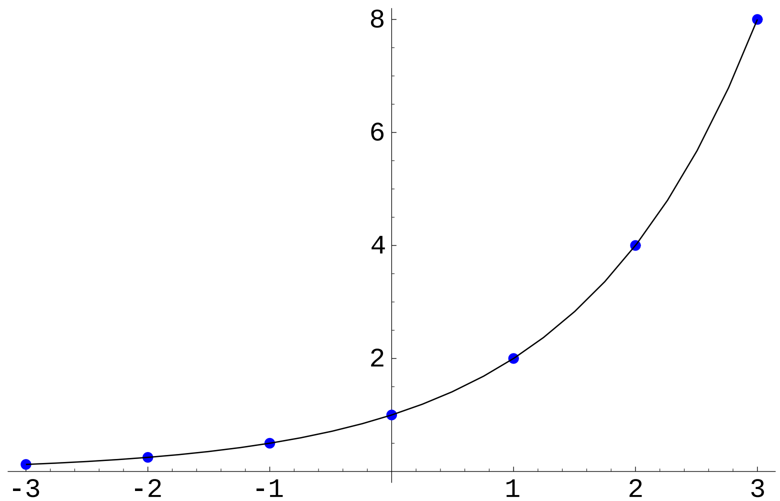


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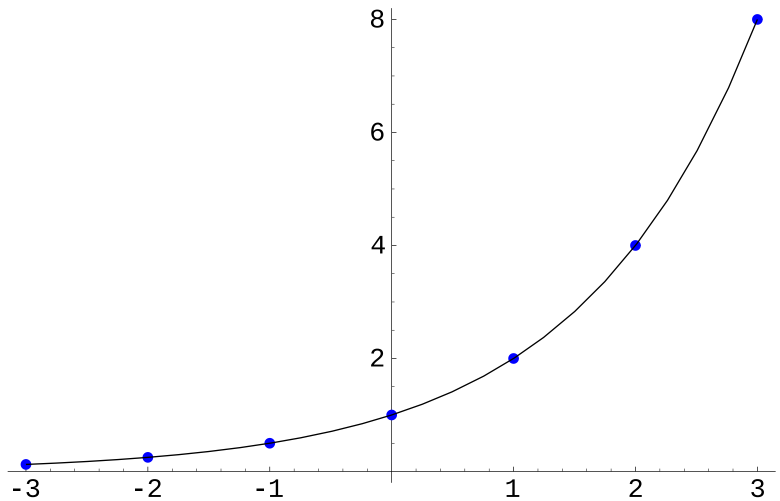


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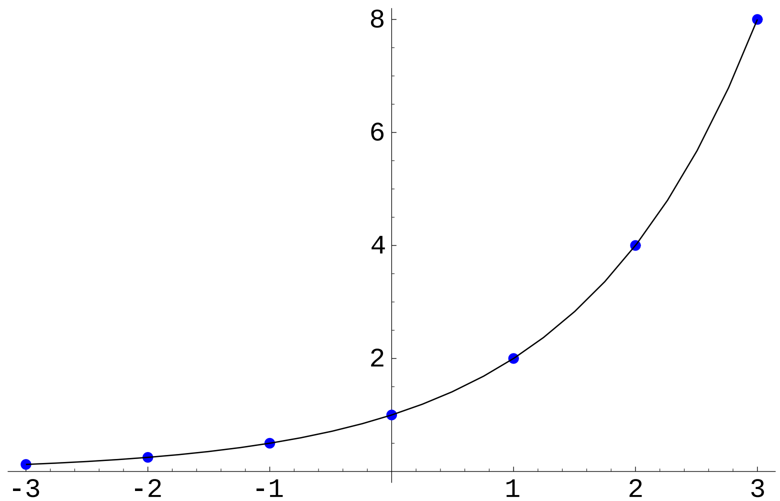


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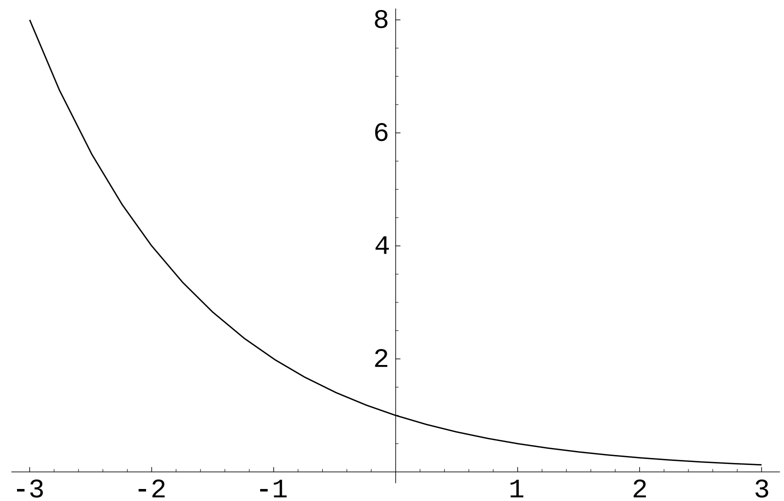

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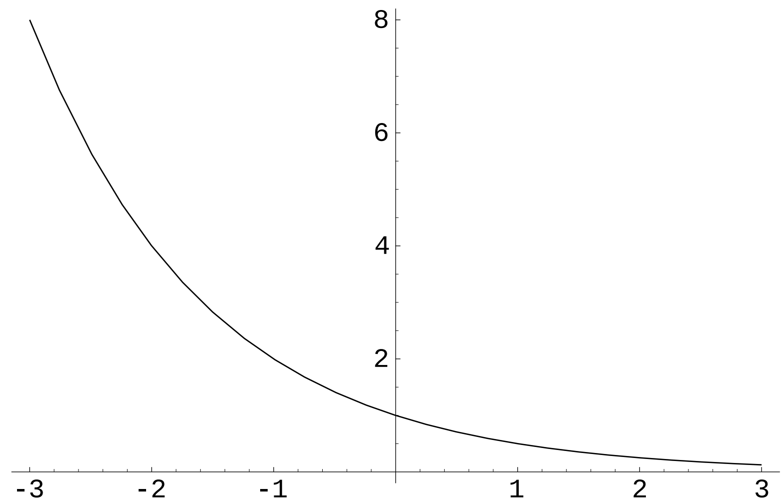

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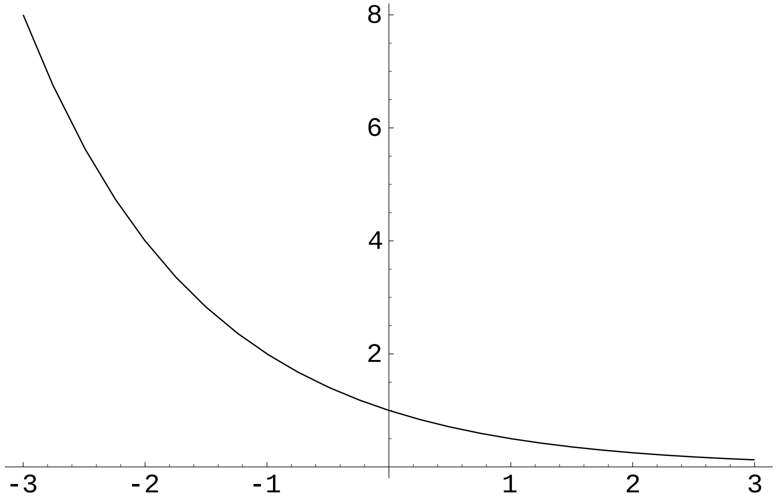


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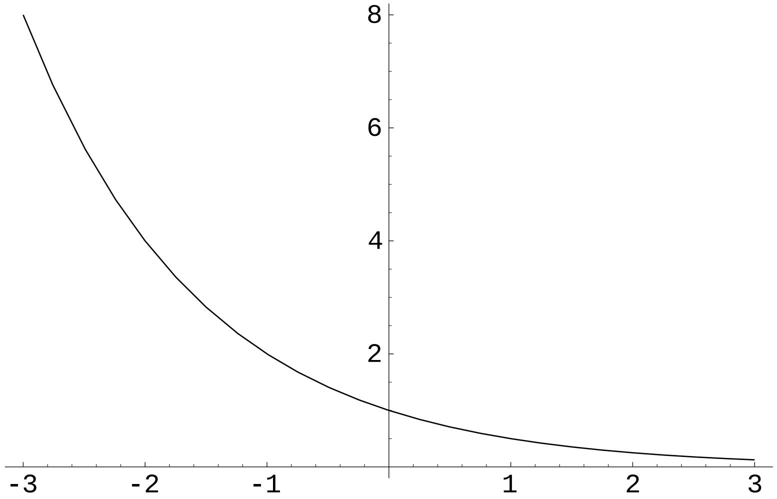

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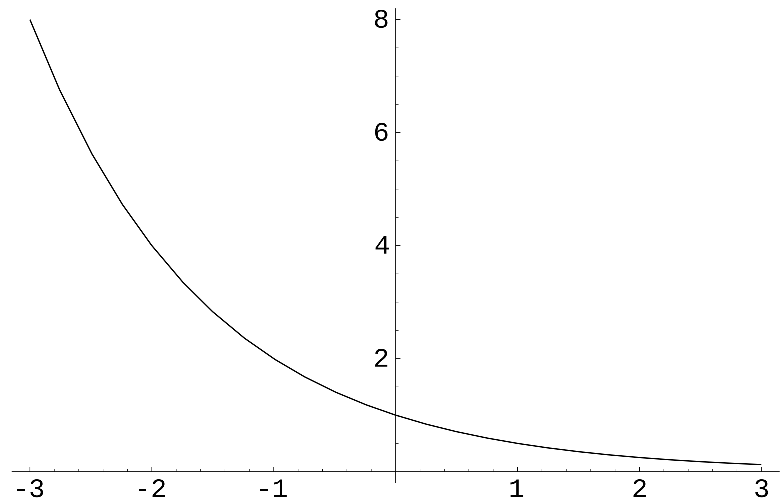

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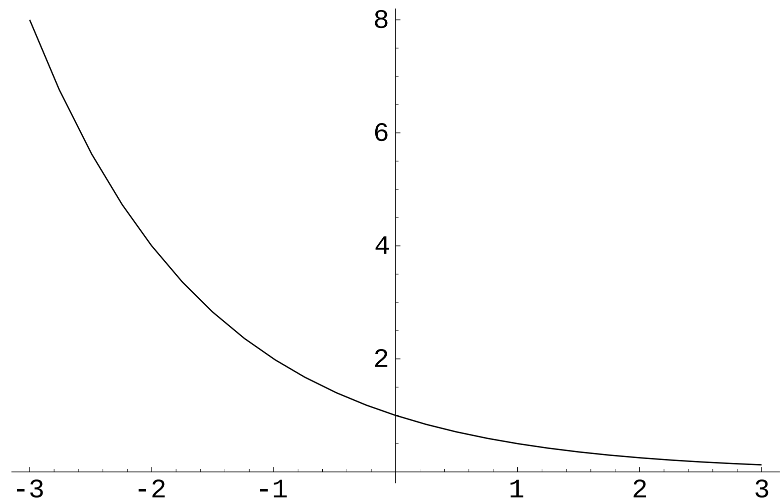


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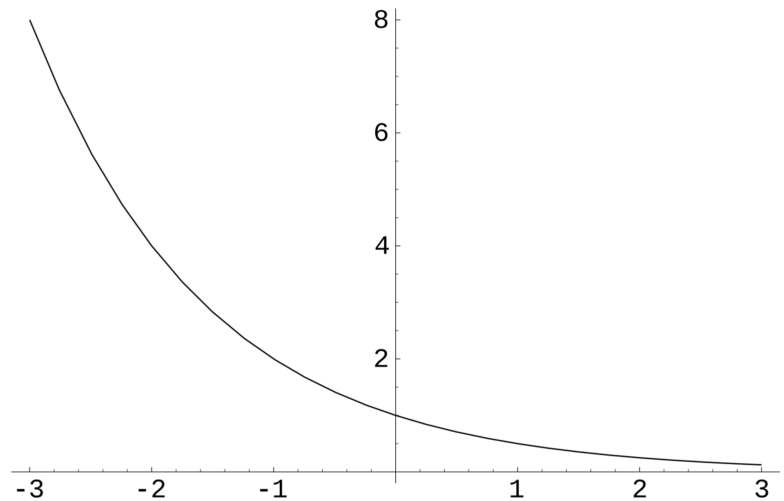


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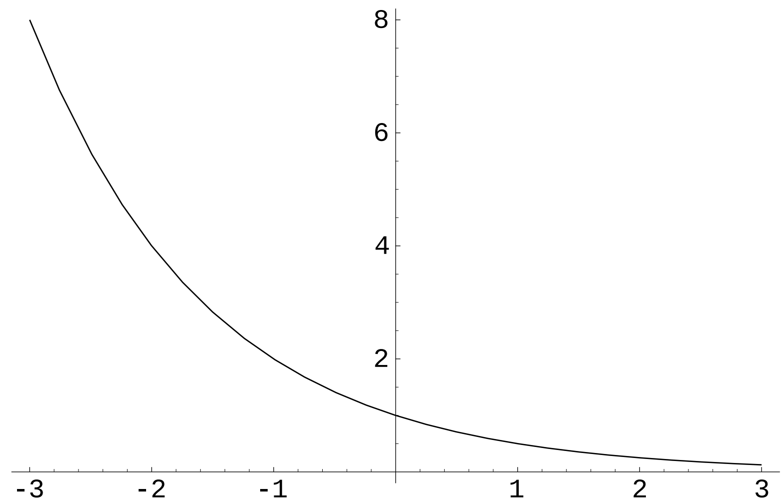


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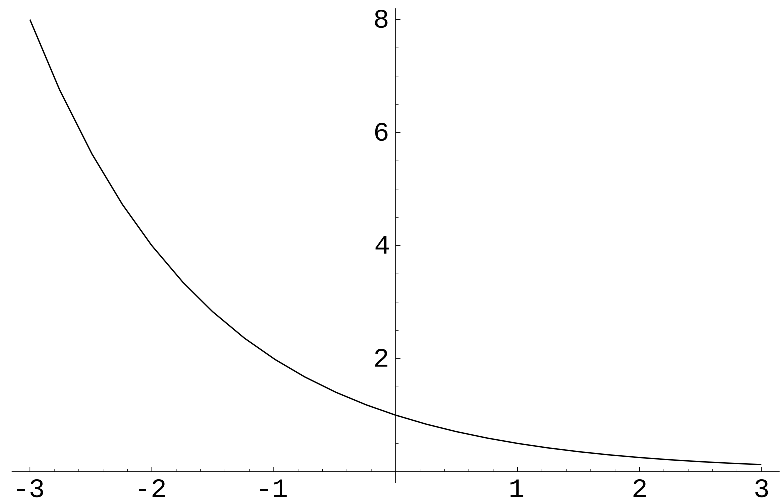


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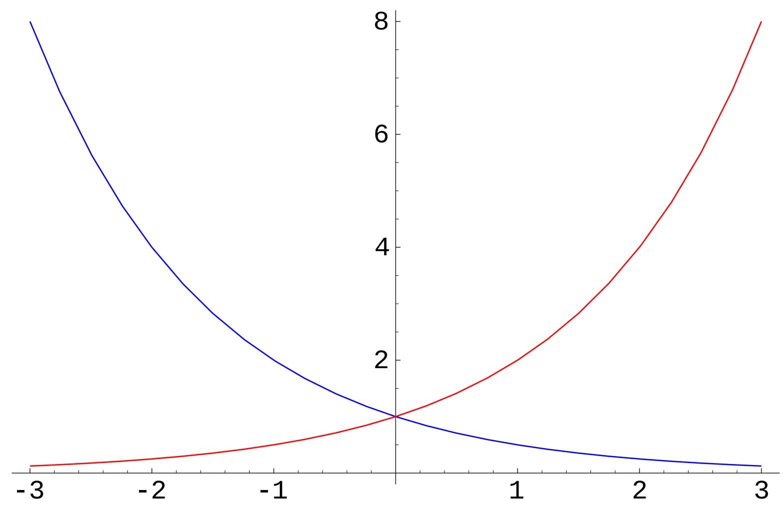

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  - By the Intermediate Value Theorem, $b^x$ *attains all positive values.*

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## Exponential Equations

$f(x) = b^x \quad (0 < b \neq 1)$

**Fact** Exponential functions are *injective*, that is, $x_1 \neq x_2 \Longrightarrow b^{x_1} \neq b^{x_2}$, or equivalently, $b^{x_1} = b^{x_2} \Longrightarrow x_1 = x_2$.

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Therefore $x = 2$ *or* $x = -\dfrac{4}{3}$.

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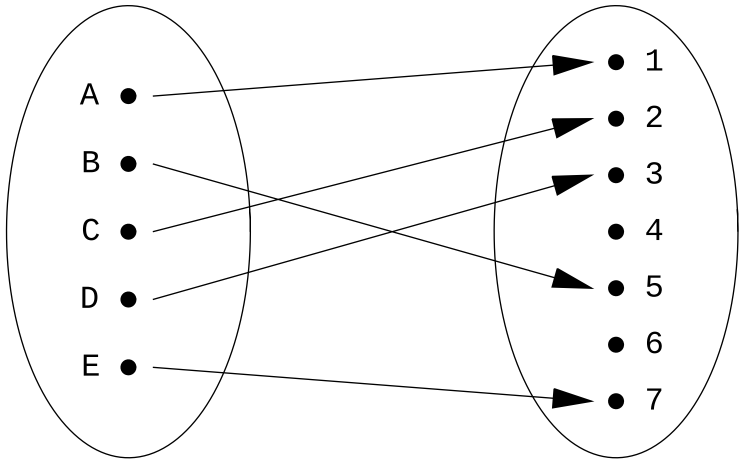
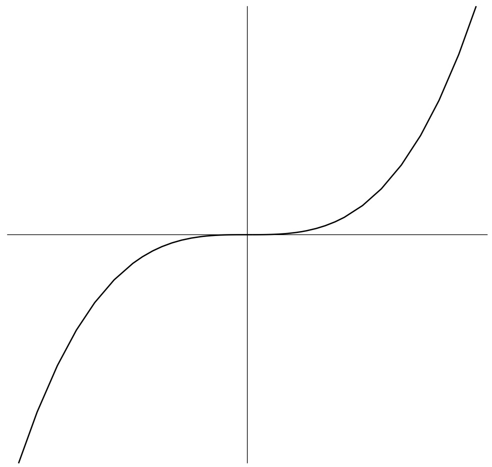
# Recall

- Let $f : A \longrightarrow B$ be an *injective* function.



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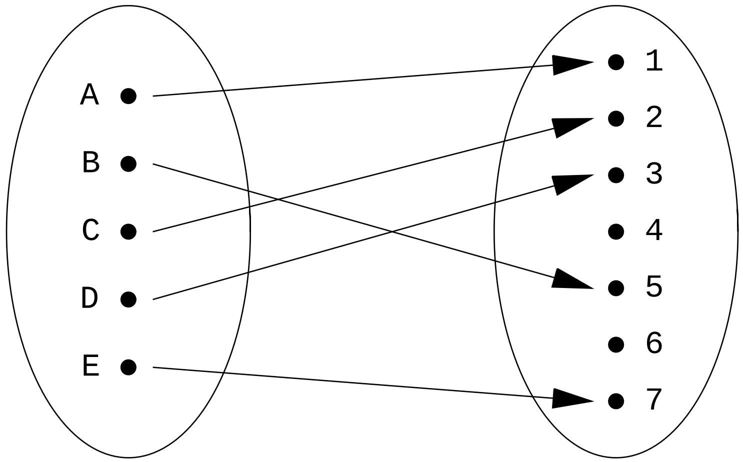


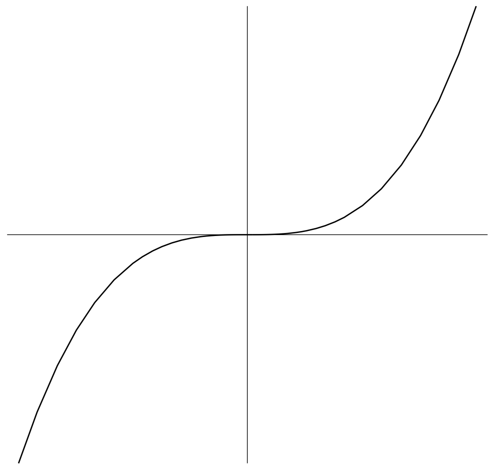

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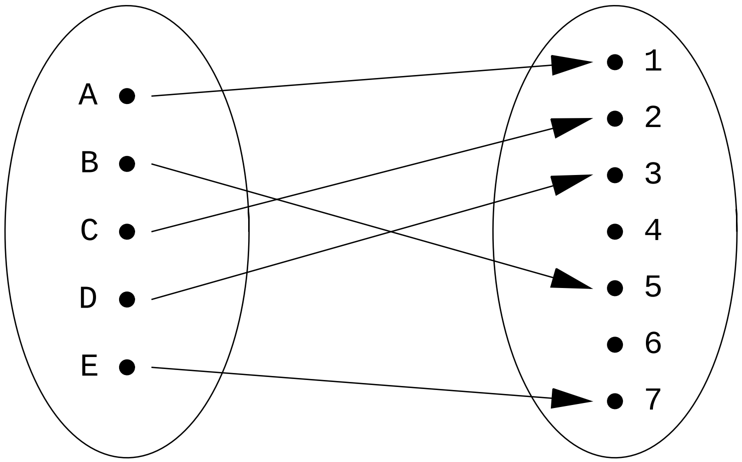


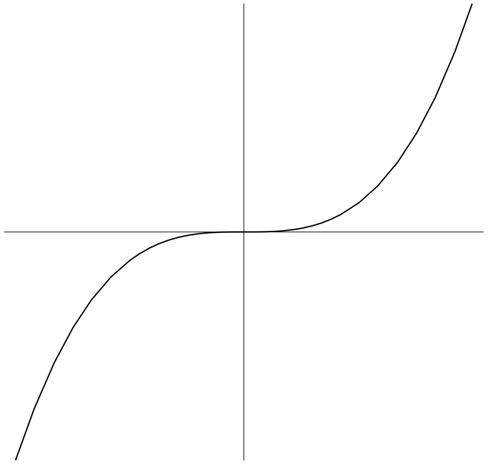

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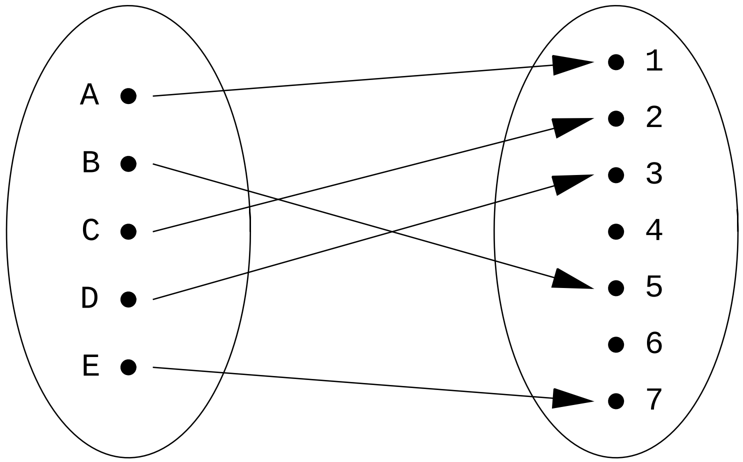


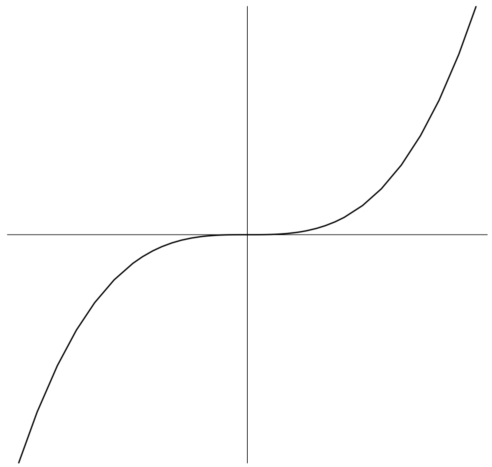

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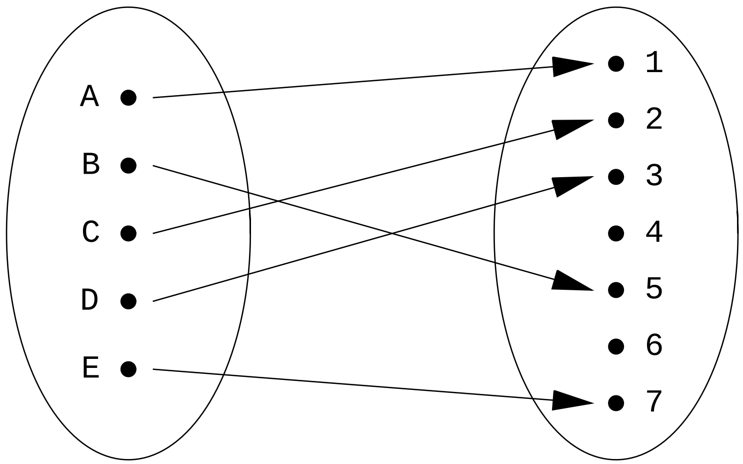


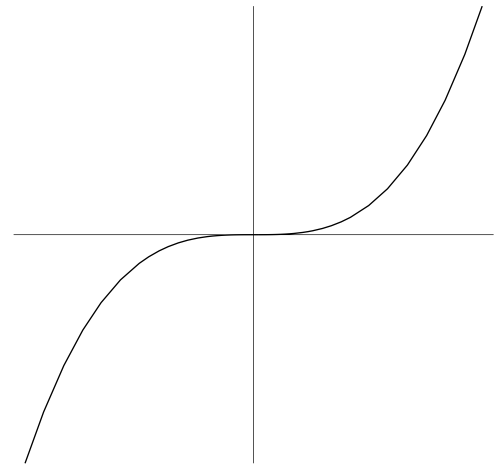

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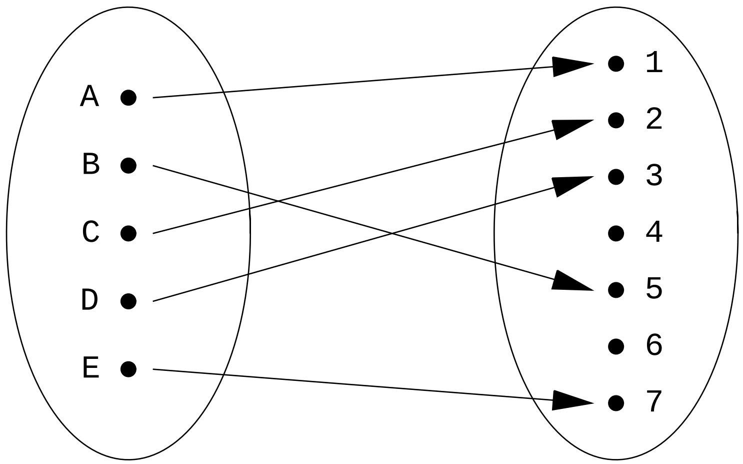


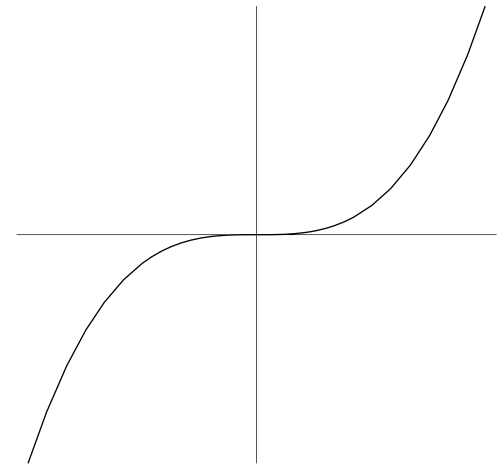

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## Logarithmic Functions

**Recall** Exponential function $f(x) = b^x$ with base $b$ (where $b > 0$ and $b \neq 1$)

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log(base) of number = exponent means base$^{\text{exponent}}$ = number

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- $\ln e = 1$ means $e^1 = e$
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**Remark** Can also use base e: $\log_2 3 = \dfrac{\ln 3}{\ln 2}$