Example Solve the equation $\cos x = 0.34$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution

Example Solve the equation $\cos x = 0.34$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 1st quadrant; one in 4th quadrant.

Example Solve the equation $\cos x = 0.34$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 1st quadrant; one in 4th quadrant.

By calculator: $\cos^{-1} 0.34 = 1.22388$

Example Solve the equation $\cos x = 0.34$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 1st quadrant; one in 4th quadrant.

By calculator: $\cos^{-1} 0.34 = 1.22388$

Solution in 1st quadrant: $x_1 = 1.22$.

Example Solve the equation $\cos x = 0.34$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 1st quadrant; one in 4th quadrant.

By calculator: $\cos^{-1} 0.34 = 1.22388$

Solution in 1st quadrant: $x_1 = 1.22$.

Solution in 4th quadrant: $x_2 = 2\pi - 1.22388 = 5.06$

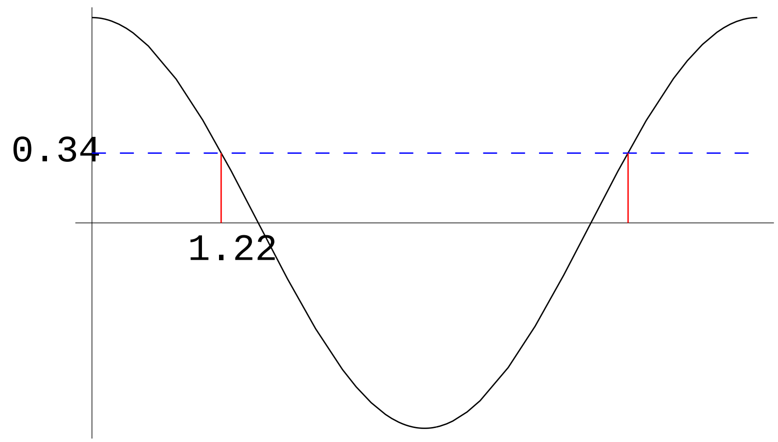
Example Solve the equation $\cos x = 0.34$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 1st quadrant; one in 4th quadrant.

By calculator: $\cos^{-1} 0.34 = 1.22388$

Solution in 1st quadrant: $x_1 = 1.22$.

Solution in 4th quadrant: $x_2 = 2\pi - 1.22388 = 5.06$



Example Solve the equation $\cos x = 0.34$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 1st quadrant; one in 4th quadrant.

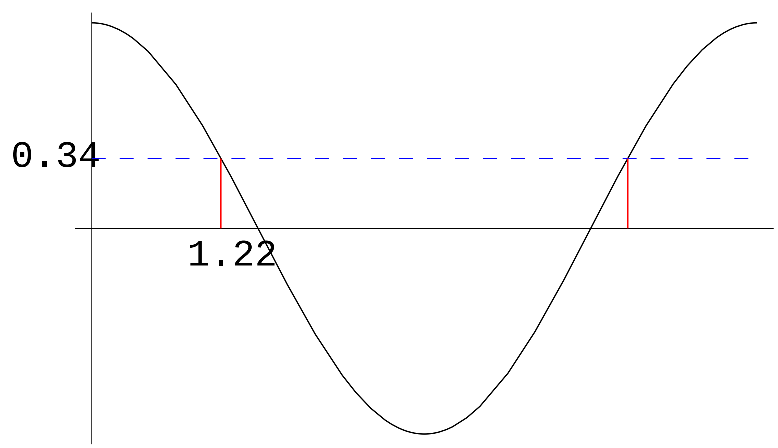
By calculator: $\cos^{-1} 0.34 = 1.22388$

Solution in 1st quadrant: $x_1 = 1.22$.

Solution in 4th quadrant: $x_2 = 2\pi - 1.22388 = 5.06$

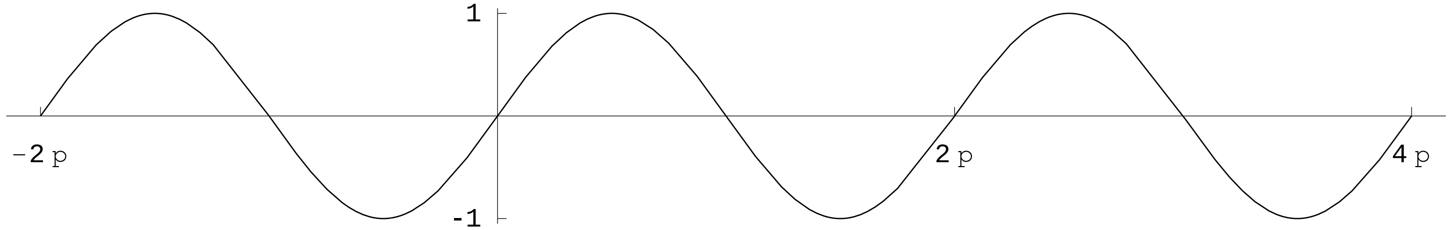
Identity

$\cos(2\pi - x) = \cos x$

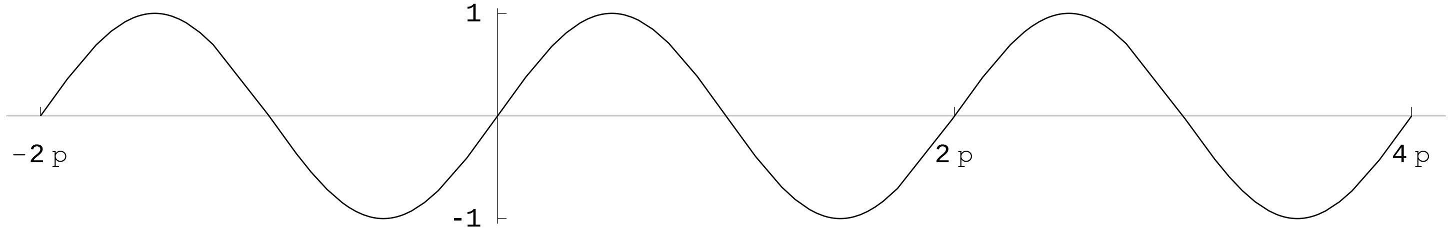


- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \qquad x \mapsto \sin x$

- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$

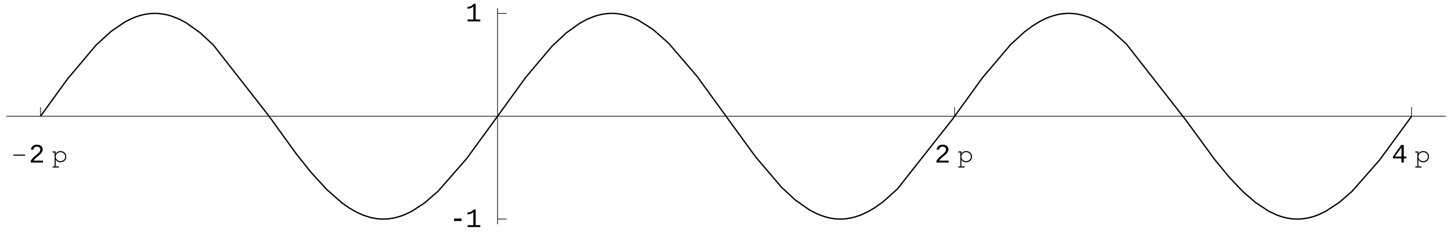


- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



Domain $= \mathbb{R}$, Range $= [-1, 1]$, *not injective*

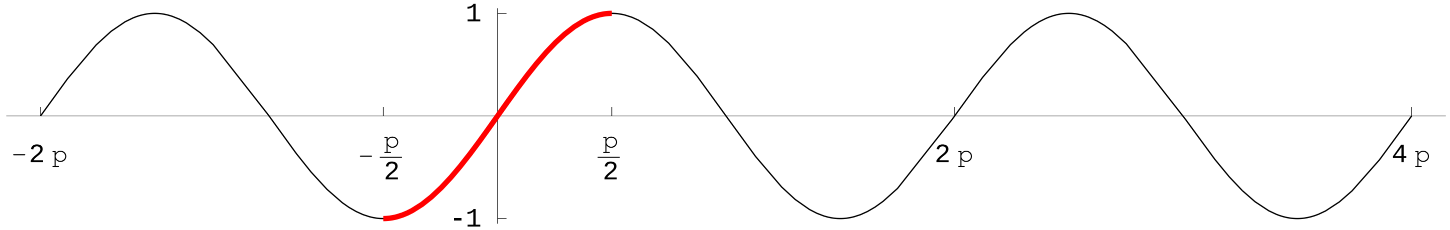
- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

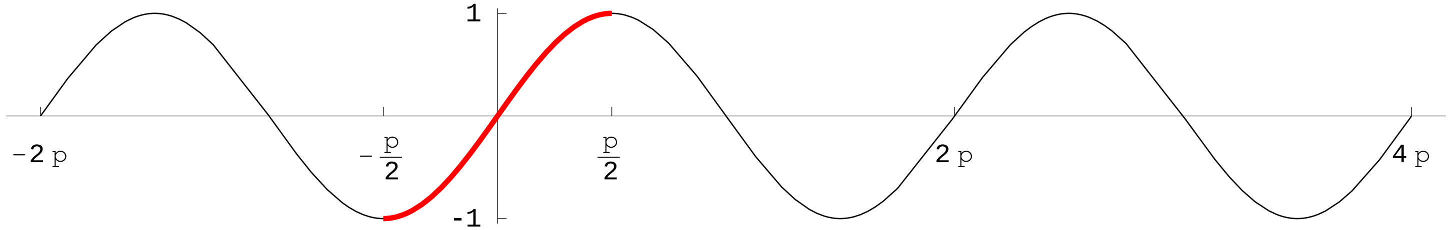
- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$

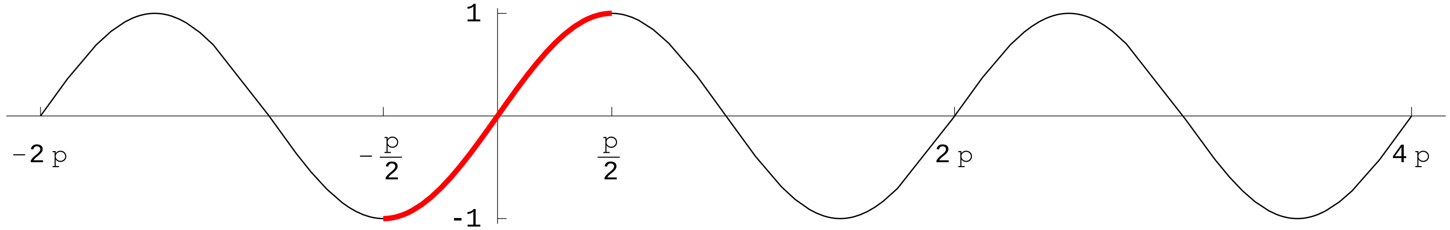


$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Injective $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$

- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



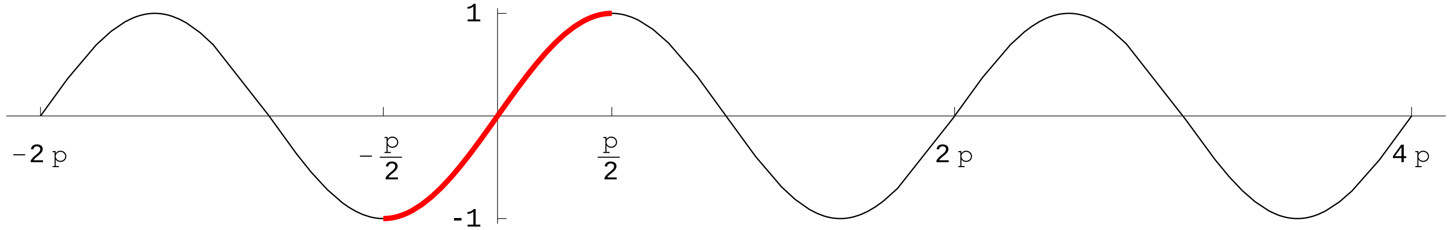
$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Injective $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$

Inverse $\sin^{-1} : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

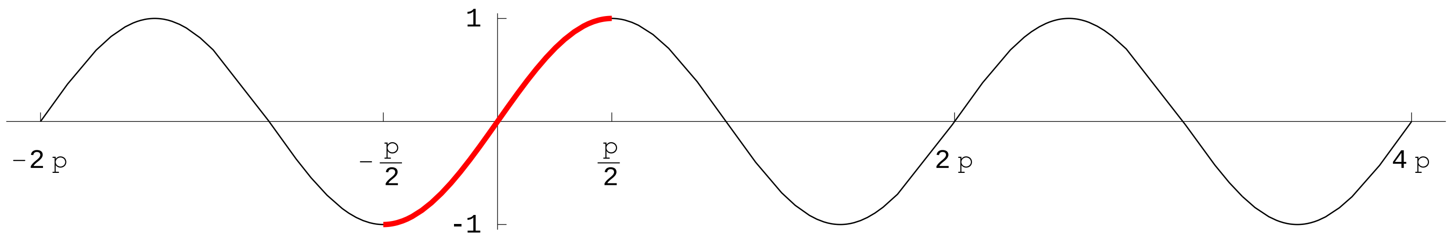
- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Injective $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$

Inverse $\sin^{-1} : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

- Cosine function: restrict domain to $[0, \pi]$

- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



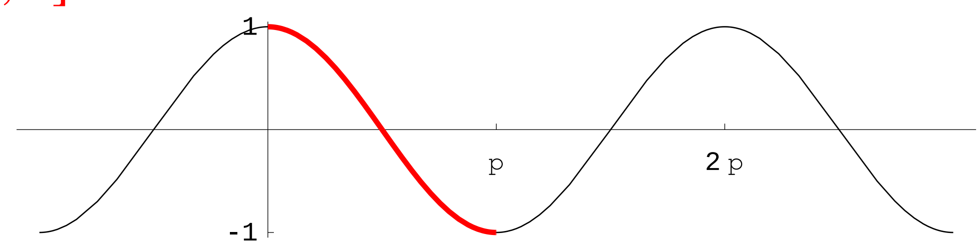
$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

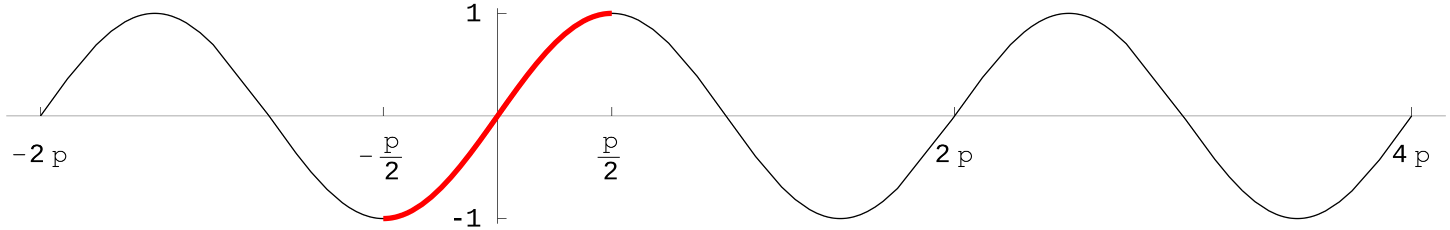
Injective $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$

Inverse $\sin^{-1} : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

- Cosine function: restrict domain to $[0, \pi]$



- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

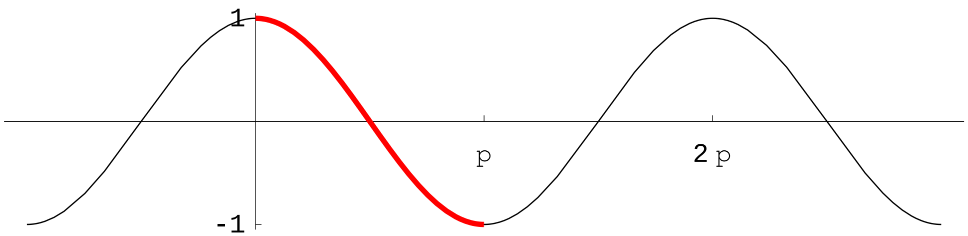
- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

Injective $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$

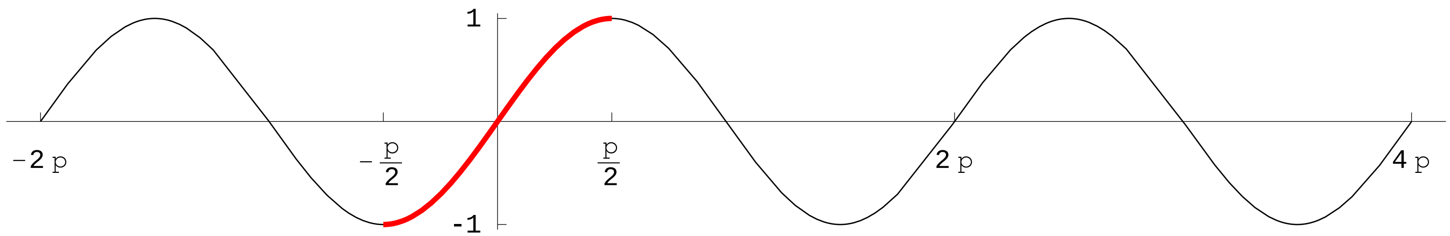
Inverse $\sin^{-1} : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

- Cosine function: restrict domain to $[0, \pi]$

Injective $\cos : [0, \pi] \longrightarrow [-1, 1]$



- Sine function $\sin : \mathbb{R} \longrightarrow \mathbb{R}, \quad x \mapsto \sin x$



$\text{Domain} = \mathbb{R}, \quad \text{Range} = [-1, 1],$ *not injective*

- Restrict domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

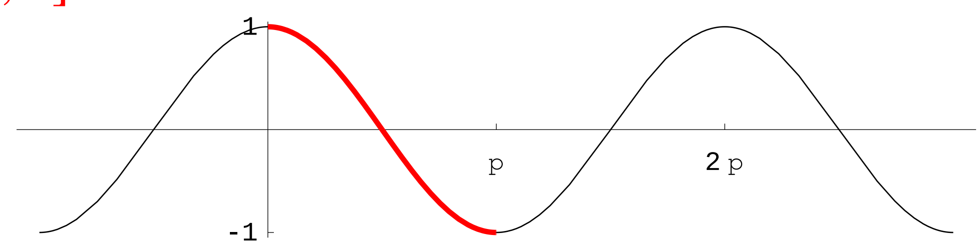
Injective $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1, 1]$

Inverse $\sin^{-1} : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$

- Cosine function: restrict domain to $[0, \pi]$

Injective $\cos : [0, \pi] \longrightarrow [-1, 1]$

Inverse $\cos^{-1} : [-1, 1] \longrightarrow [0, \pi]$



Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution

Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 3rd quadrant; one in 4th quadrant.

Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

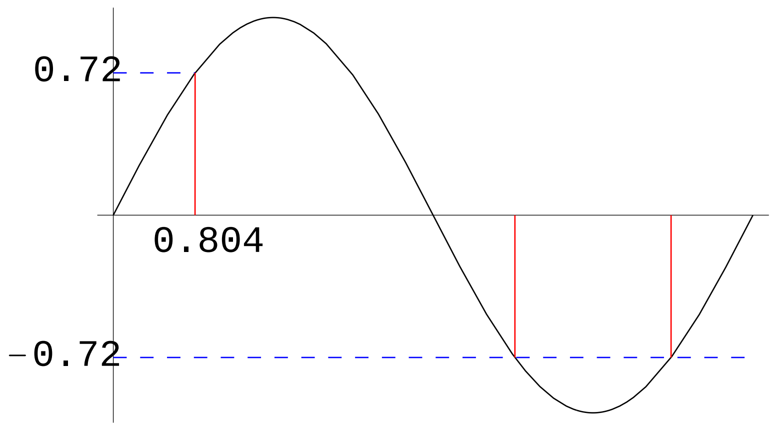
Solution One solution in 3rd quadrant; one in 4th quadrant.

By calculator: $\sin^{-1} 0.72 = 0.803802 \leftarrow$ *use positive value*

Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 3rd quadrant; one in 4th quadrant.

By calculator: $\sin^{-1} 0.72 = 0.803802 \leftarrow$ *use positive value*

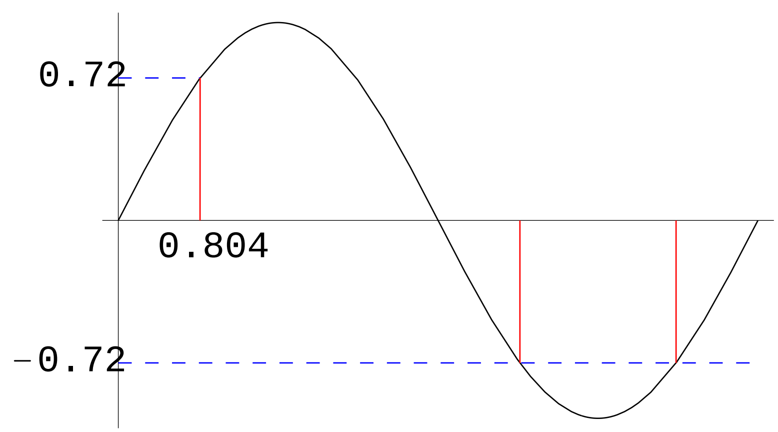


Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 3rd quadrant; one in 4th quadrant.

By calculator: $\sin^{-1} 0.72 = 0.803802 \leftarrow$ *use positive value*

Solution in 3rd quadrant: $x_1 = \pi + 0.803802 = 3.95$



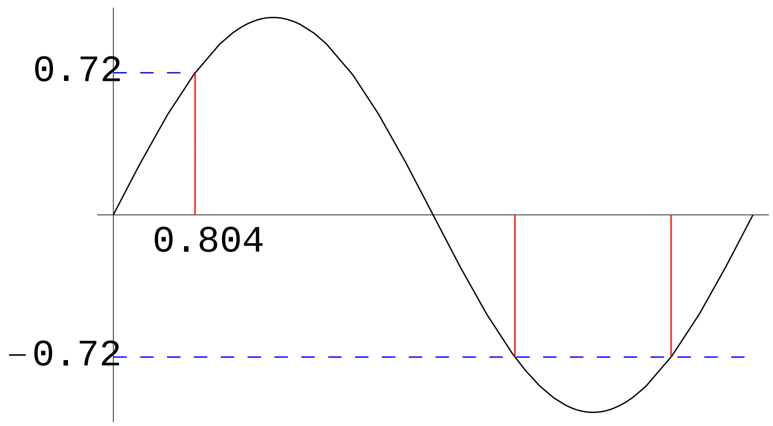
Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 3rd quadrant; one in 4th quadrant.

By calculator: $\sin^{-1} 0.72 = 0.803802 \leftarrow$ *use positive value*

Solution in 3rd quadrant: $x_1 = \pi + 0.803802 = 3.95$

Solution in 4th quadrant: $x_2 = 2\pi - 0.803802 = 5.48$



Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 3rd quadrant; one in 4th quadrant.

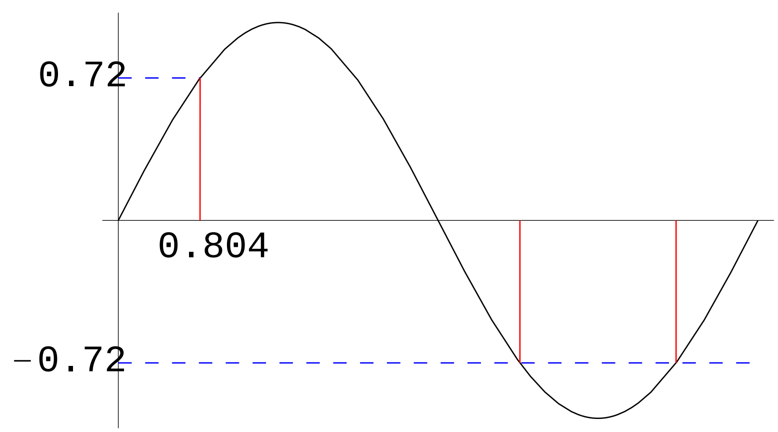
By calculator: $\sin^{-1} 0.72 = 0.803802 \leftarrow$ *use positive value*

Solution in 3rd quadrant: $x_1 = \pi + 0.803802 = 3.95$

Solution in 4th quadrant: $x_2 = 2\pi - 0.803802 = 5.48$

Identity

$\sin(\pi + x) = -\sin x$



Example Solve the equation $\sin x = -0.72$ for $x \in [0, 2\pi]$; give your answer correct to 3 significant figures.

Solution One solution in 3rd quadrant; one in 4th quadrant.

By calculator: $\sin^{-1} 0.72 = 0.803802$ ← *use positive value*

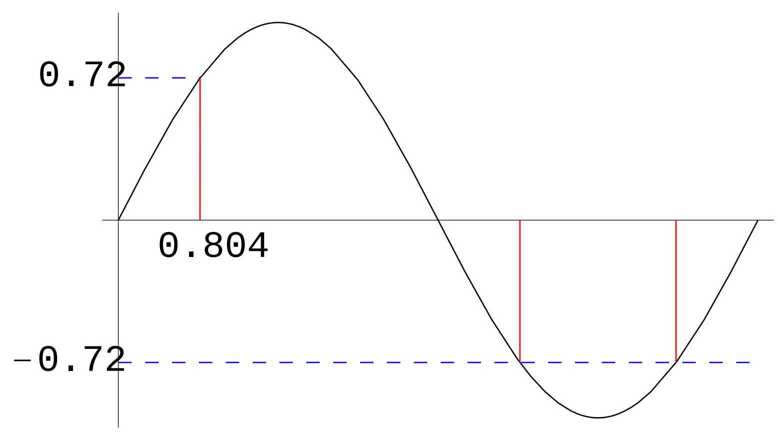
Solution in 3rd quadrant: $x_1 = \pi + 0.803802 = 3.95$

Solution in 4th quadrant: $x_2 = 2\pi - 0.803802 = 5.48$

Identity

$\sin(\pi + x) = -\sin x$

$\sin(2\pi - x) = -\sin x$



Continuity

The sine and cosine functions are *continuous on* $\mathbb{R}$

Continuity

The sine and cosine functions are *continuous on* $\mathbb{R}$

$$\lim_{x \to a} \sin x = \sin a$$

$$\lim_{x \to a} \cos x = \cos a$$

for all $a \in \mathbb{R}$.

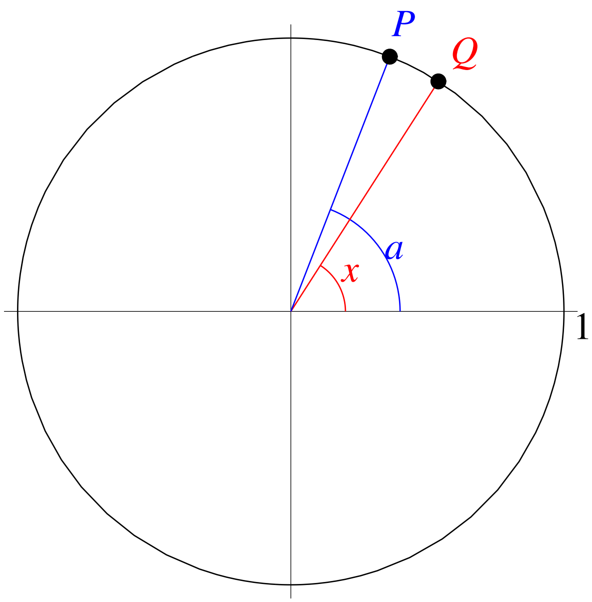
Continuity

The sine and cosine functions are *continuous on* $\mathbb{R}$

$$\lim_{x \to a} \sin x = \sin a$$

$$\lim_{x \to a} \cos x = \cos a$$

for all $a \in \mathbb{R}$.



Reason If x is close to a, then the point Q lying on the unit circle that corresponds to x is close to the point P that corresponds to a. See animation

An important limit

$$(**) \quad \lim_{x\to 0}\frac{\sin x}{x} = 1$$

Proof See lecture notes.

An important limit

$$(**) \quad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Proof See lecture notes.

Remark

- $(**)$ means that *if x is small, $\sin x$ is approximately equal to x.* Show xls

An important limit

$$(**) \quad \lim_{x \to 0} \frac{\sin x}{x} = 1$$

Proof See lecture notes.

Remark

- $(**)$ means that *if x is small, $\sin x$ is approximately equal to x.* Show xls

- *If x is in degrees*, the result is different: $\lim_{x \to 0} \frac{\sin x^\circ}{x} = \frac{\pi}{180}$

- The following result will be used in deriving the formula for $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x$

Example $\displaystyle \lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

- The following result will be used in deriving the formula for $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x$

Example $\displaystyle\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$

Analysis

- Can’t use substitution.

- The following result will be used in deriving the formula for $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x$

Example $\displaystyle\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$

Analysis

- Can't use substitution.
- To get the limit, we try to "*cancel*" the factor h in the denominator.

- The following result will be used in deriving the formula for $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x$

Example $\displaystyle\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$

Analysis

- Can't use substitution.
- To get the limit, we try to "*cancel*" the factor h in the denominator.
- However, the numerator is not a polynomial.
  - ◇ Instead of making a factor h in the numerator,

- The following result will be used in deriving the formula for $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x$

Example $\displaystyle\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$

Analysis

- Can't use substitution.
- To get the limit, we try to "*cancel*" the factor h in the denominator.
- However, the numerator is not a polynomial.
  - ◇ Instead of making a factor h in the numerator,
  - ◇ we try to make a factor $\sin h$

- The following result will be used in deriving the formula for $\dfrac{\mathrm{d}}{\mathrm{d}x}\sin x$

Example $\displaystyle\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$

Analysis

- Can't use substitution.
- To get the limit, we try to "*cancel*" the factor h in the denominator.
- However, the numerator is not a polynomial.
  - ◇ Instead of making a factor h in the numerator,
  - ◇ we try to make a factor $\sin h$

  and then apply $\boxed{\displaystyle\lim_{h\to 0}\frac{\sin h}{h} = 1}$

Example

$$\lim_{h \to 0} \frac{\cos h - 1}{h} \quad =$$

Example

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$$

Example

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$$

$$= \lim_{h\to 0}\frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$(*) \qquad \sin^2 x + \cos^2 x = 1$$

Example

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \qquad \text{by } (*)$$

$$(*) \qquad \sin^2 x + \cos^2 x = 1$$

Example

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \qquad \text{by } (*)$$

$$= \lim_{h\to 0} \left(-\sin h \times \frac{\sin h}{h} \times \frac{1}{\cos h + 1} \right)$$

$$(*) \quad \sin^2 x + \cos^2 x = 1$$

$$(**) \quad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Example

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \qquad \text{by } (*)$$

$$= \lim_{h\to 0} \left(-\sin h \times \frac{\sin h}{h} \times \frac{1}{\cos h + 1} \right)$$

$$= -\sin 0 \times 1 \times \frac{1}{\cos 0 + 1} \qquad \text{by } (**)$$

$$(*) \quad \sin^2 x + \cos^2 x = 1$$

$$(**) \quad \lim_{x\to 0} \frac{\sin x}{x} = 1$$

Example

$$\lim_{h\to 0} \frac{\cos h - 1}{h} = \lim_{h\to 0} \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \lim_{h\to 0} \frac{-\sin^2 h}{h(\cos h + 1)} \qquad \text{by } (*)$$

$$= \lim_{h\to 0} \left(-\sin h \times \frac{\sin h}{h} \times \frac{1}{\cos h + 1} \right)$$

$$= -\sin 0 \times 1 \times \frac{1}{\cos 0 + 1} \qquad \text{by } (**)$$

$$= 0$$

Differentiation of Trigonometric Functions

(1) $$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$$

Differentiation of Trigonometric Functions

(1) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x}$$

Proof

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

Differentiation of Trigonometric Functions

(1) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

Proof

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h}$$

Differentiation of Trigonometric Functions

(1) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{(\sin x \cdot \cos h - \sin x) + \cos x \cdot \sin h}{h}
\end{aligned}$$

Differentiation of Trigonometric Functions

(1) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{(\sin x \cdot \cos h - \sin x) + \cos x \cdot \sin h}{h} \\
&= \lim_{h\to 0}\left[\frac{\sin x\,(\cos h - 1)}{h} + \frac{\cos x \cdot \sin h}{h}\right]
\end{aligned}$$

Differentiation of Trigonometric Functions

(1) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x}$$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

Proof

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cdot\cos h + \cos x\cdot\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{(\sin x\cdot\cos h - \sin x) + \cos x\cdot\sin h}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin x\,(\cos h - 1)}{h} + \frac{\cos x\cdot\sin h}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin x \times \frac{\cos h - 1}{h}\right] + \lim_{h\to 0}\left[\cos x \times \frac{\sin h}{h}\right]$$

Differentiation of Trigonometric Functions

(1) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

$$\sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

Proof

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cdot\cos h + \cos x\cdot\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{(\sin x\cdot\cos h - \sin x) + \cos x\cdot\sin h}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin x\,(\cos h - 1)}{h} + \frac{\cos x\cdot\sin h}{h}\right]$$

$$= \lim_{h\to 0}\left[\sin x \times \frac{\cos h - 1}{h}\right] + \lim_{h\to 0}\left[\cos x \times \frac{\sin h}{h}\right]$$

$$= (\sin x)(0) +$$

Differentiation of Trigonometric Functions

(1) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x}$$

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{(\sin x \cdot \cos h - \sin x) + \cos x \cdot \sin h}{h} \\
&= \lim_{h\to 0}\left[\frac{\sin x\,(\cos h - 1)}{h} + \frac{\cos x \cdot \sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\sin x \times \frac{\cos h - 1}{h}\right] + \lim_{h\to 0}\left[\cos x \times \frac{\sin h}{h}\right] \\
&= (\sin x)(0) + (\cos x)(1)
\end{aligned}$$

Differentiation of Trigonometric Functions

(1) $$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$

$$\sin(A + B) = \sin A \cdot \cos B + \cos A \cdot \sin B$$

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x \cdot \cos h + \cos x \cdot \sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{(\sin x \cdot \cos h - \sin x) + \cos x \cdot \sin h}{h} \\
&= \lim_{h\to 0}\left[\frac{\sin x\,(\cos h - 1)}{h} + \frac{\cos x \cdot \sin h}{h}\right] \\
&= \lim_{h\to 0}\left[\sin x \times \frac{\cos h - 1}{h}\right] + \lim_{h\to 0}\left[\cos x \times \frac{\sin h}{h}\right] \\
&= (\sin x)(0) + (\cos x)(1) = \cos x
\end{aligned}$$

(2) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x}$$

(2) $$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

Proof

- Can be proved by definition, similar to (1).

(2) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x}$$

Proof

- Can be proved by definition, similar to (1).
- Can also be proved using (1) and the chain rule (see Chapter 9).

(3) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x}$$ where $\sec x := \dfrac{1}{\cos x}$

(3) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x}$$ where $\sec x := \dfrac{1}{\cos x}$

Proof

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)$$

(3) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x}$$ where $\sec x := \dfrac{1}{\cos x}$

Proof

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos^2 x}$$

(3) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x}$$ where $\sec x := \dfrac{1}{\cos x}$

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos^2 x} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}
\end{aligned}$$

(3) $$\boxed{\frac{d}{dx}\tan x = \sec^2 x}$$ where $\sec x := \dfrac{1}{\cos x}$

Proof

$$\begin{aligned}
\frac{d}{dx}\tan x &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \frac{d}{dx}\sin x - \sin x \cdot \frac{d}{dx}\cos x}{\cos^2 x} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{1}{\cos^2 x}
\end{aligned}$$

(3) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x}$$ where $\sec x := \dfrac{1}{\cos x}$

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos^2 x} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2
\end{aligned}$$

(3) $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x}$$ where $\sec x := \dfrac{1}{\cos x}$

Proof

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{\cos x}\right) \\
&= \frac{\cos x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\cos x}{\cos^2 x} \\
&= \frac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 \\
&= (\sec x)^2
\end{aligned}$$

Example Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(1) $y = 2\sin x - 7\cos x$

Example Find $\dfrac{dy}{dx}$ for the following:

(1) $y = 2\sin x - 7\cos x$

Solution $$\frac{dy}{dx} = \frac{d}{dx}(2\sin x - 7\cos x)$$

Example Find $\frac{dy}{dx}$ for the following:

(1) $y = 2\sin x - 7\cos x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(2\sin x - 7\cos x)$$

$$= 2 \cdot \frac{d}{dx}\sin x - 7 \cdot \frac{d}{dx}\cos x$$

Example Find $\frac{dy}{dx}$ for the following:

(1) $y = 2\sin x - 7\cos x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(2\sin x - 7\cos x)$$

$$= 2 \cdot \frac{d}{dx}\sin x - 7 \cdot \frac{d}{dx}\cos x$$

$$= 2\cos x - 7(-\sin x)$$

Example Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(1) $y = 2\sin x - 7\cos x$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(2\sin x - 7\cos x)$$

$$= 2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x - 7 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\cos x$$

$$= 2\cos x - 7(-\sin x)$$

$$= 2\cos x + 7\sin x$$

Example Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(2) $y = \tan x - x$

Example Find $\frac{dy}{dx}$ for the following:

(2) $y = \tan x - x$

Solution $$\frac{dy}{dx} = \frac{d}{dx}(\tan x - x)$$

Example Find $\dfrac{dy}{dx}$ for the following:

(2) $y = \tan x - x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x - x)$$

$$= \frac{d}{dx}\tan x - \frac{d}{dx}x$$

Example Find $\frac{dy}{dx}$ for the following:

(2) $y = \tan x - x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x - x)$$

$$= \frac{d}{dx}\tan x - \frac{d}{dx}x$$

$$= \sec^2 x - 1$$

Example Find $\frac{dy}{dx}$ for the following:

(2) $y = \tan x - x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x - x)$$

$$= \frac{d}{dx}\tan x - \frac{d}{dx}x$$

$$= \sec^2 x - 1$$

$$= \frac{1}{\cos^2 x} - 1$$

Example Find $\frac{dy}{dx}$ for the following:

(2) $y = \tan x - x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x - x)$$

$$= \frac{d}{dx}\tan x - \frac{d}{dx}x$$

$$= \sec^2 x - 1$$

$$= \frac{1}{\cos^2 x} - 1$$

$$= \frac{1 - \cos^2 x}{\cos^2 x}$$

Example Find $\frac{dy}{dx}$ for the following:

(2) $y = \tan x - x$

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}(\tan x - x) \\ &= \frac{d}{dx}\tan x - \frac{d}{dx}x \\ &= \sec^2 x - 1 \\ &= \frac{1}{\cos^2 x} - 1 \\ &= \frac{1 - \cos^2 x}{\cos^2 x} \\ &= \frac{\sin^2 x}{\cos^2 x}\end{aligned}$$

Example Find $\frac{dy}{dx}$ for the following:

(2) $y = \tan x - x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(\tan x - x)$$

$$= \frac{d}{dx}\tan x - \frac{d}{dx}x$$

$$= \sec^2 x - 1$$

$$= \frac{1}{\cos^2 x} - 1$$

$$= \frac{1 - \cos^2 x}{\cos^2 x}$$

$$= \frac{\sin^2 x}{\cos^2 x} = \tan^2 x$$

Example Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(3) $y = x\cos x$

Example Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(3) $y = x\cos x$

Solution $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(x\cos x)$$

Example Find $\frac{dy}{dx}$ for the following:

(3) $y = x\cos x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos x)$$

$$= x \cdot \frac{d}{dx}\cos x + \cos x \cdot \frac{d}{dx}x$$

Example Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(3) $y = x\cos x$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(x\cos x)$$

$$= x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\cos x + \cos x \cdot \frac{\mathrm{d}}{\mathrm{d}x}x$$

$$= x(-\sin x) + (\cos x)(1)$$

Example Find $\frac{dy}{dx}$ for the following:

(3) $y = x\cos x$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}(x\cos x)$$

$$= x \cdot \frac{d}{dx}\cos x + \cos x \cdot \frac{d}{dx}x$$

$$= x(-\sin x) + (\cos x)(1)$$

$$= \cos x - x\sin x$$

Example Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(4) $y = \dfrac{\sin x}{x^2 + 1}$

Example Find $\frac{dy}{dx}$ for the following:

(4) $y = \frac{\sin x}{x^2 + 1}$

Solution $$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{x^2 + 1}\right)$$

Example Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the following:

(4) $y = \dfrac{\sin x}{x^2 + 1}$

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x^2 + 1}\right)$$

$$= \frac{(x^2 + 1)\frac{\mathrm{d}}{\mathrm{d}x}\sin x - \sin x \cdot \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 1)}{(x^2 + 1)^2}$$

Example Find $\frac{dy}{dx}$ for the following:

(4) $y = \dfrac{\sin x}{x^2 + 1}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{x^2 + 1}\right)$$

$$= \frac{(x^2 + 1)\frac{d}{dx}\sin x - \sin x \cdot \frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)(\cos x) - (\sin x)(2x)}{(x^2 + 1)^2}$$

Example Find $\frac{dy}{dx}$ for the following:

(4) $y = \frac{\sin x}{x^2 + 1}$

Solution

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\sin x}{x^2 + 1}\right)$$

$$= \frac{(x^2 + 1)\frac{d}{dx}\sin x - \sin x \cdot \frac{d}{dx}(x^2 + 1)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)(\cos x) - (\sin x)(2x)}{(x^2 + 1)^2}$$

$$= \frac{(x^2 + 1)\cos x - 2x\sin x}{(x^2 + 1)^2}$$

Chapter 8: Exponential and Logarithmic Functions

- Exponential Functions
- Logarithmic Functions
- Differentiation of Exponential and Logarithmic Functions

Objectives

- Definition and Properties of Exponential and Logarithmic Functions.
- Formulas for derivatives of Exponential and Logarithmic Functions.

Exponential Functions

Definition Let $0 < b \neq 1$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = b^x$$

is called the *exponential function* with *base* b.

Exponential Functions

Definition Let $0 < b \neq 1$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = b^x$$

is called the *exponential function* with *base* b.

Remark

- If x is a rational number, eg $x = \frac{4}{3}$, then $b^x = b^{\frac{4}{3}} = \sqrt[3]{b^4}$

Exponential Functions

Definition Let $0 < b \neq 1$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = b^x$$

is called the *exponential function* with *base* b.

Remark

- If x is a rational number, eg $x = \frac{4}{3}$, then $b^x = b^{\frac{4}{3}} = \sqrt[3]{b^4}$
- If x is an irrational number, eg. $x = \sqrt{2}$, to define b^x, we use approximations,

Exponential Functions

Definition Let $0 < b \neq 1$. The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = b^x$$

is called the *exponential function* with *base* b.

Remark

- If x is a rational number, eg $x = \frac{4}{3}$, then $b^x = b^{\frac{4}{3}} = \sqrt[3]{b^4}$
- If x is an irrational number, eg. $x = \sqrt{2}$, to define b^x, we use approximations, *more precisely, use limits*

Example To assign a value to $3^{\sqrt{2}}$,

Example To assign a value to $3^{\sqrt{2}}$,

- $\sqrt{2} = 1.414213562373095\cdots$

Example To assign a value to $3^{\sqrt{2}}$,

- $\sqrt{2} = 1.414213562373095\cdots$
- $3^{\sqrt{2}}$ may be approximated by $3^{1.4} = 3^{\frac{14}{10}}$.

Example To assign a value to $3^{\sqrt{2}}$,

- $\sqrt{2} = 1.414213562373095\cdots$
- $3^{\sqrt{2}}$ may be approximated by $3^{1.4} = 3^{\frac{14}{10}}$.
- Better approximations are $3^{1.41}$ and so on.

Example To assign a value to $3^{\sqrt{2}}$,

- $\sqrt{2} = 1.414213562373095\cdots$
- $3^{\sqrt{2}}$ may be approximated by $3^{1.4} = 3^{\frac{14}{10}}$.
- Better approximations are $3^{1.41}$ and so on.
- Denote

$$a_1 = 1.4, \quad a_2 = 1.41, \quad a_3 = 1.414, \quad a_4 = 1.4142, \quad a_5 = 1.41421, \quad \ldots$$

Example To assign a value to $3^{\sqrt{2}}$,

- $\sqrt{2} = 1.414213562373095\cdots$
- $3^{\sqrt{2}}$ may be approximated by $3^{1.4} = 3^{\frac{14}{10}}$.
- Better approximations are $3^{1.41}$ and so on.
- Denote

$$a_1 = 1.4, \quad a_2 = 1.41, \quad a_3 = 1.414, \quad a_4 = 1.4142, \quad a_5 = 1.41421, \quad \ldots$$

- It can be shown that the sequence

$$3^{a_1},\ 3^{a_2},\ 3^{a_3},\ 3^{a_4}, \ldots$$

is convergent.

Example To assign a value to $3^{\sqrt{2}}$,

- $\sqrt{2} = 1.414213562373095\cdots$
- $3^{\sqrt{2}}$ may be approximated by $3^{1.4} = 3^{\frac{14}{10}}$.
- Better approximations are $3^{1.41}$ and so on.
- Denote

$$a_1 = 1.4, \quad a_2 = 1.41, \quad a_3 = 1.414, \quad a_4 = 1.4142, \quad a_5 = 1.41421, \quad \ldots$$

- It can be shown that the sequence

$$3^{a_1},\ 3^{a_2},\ 3^{a_3},\ 3^{a_4}, \ldots$$

is convergent.

n	3^{a_n}
1	4.65554
2	4.70697
3	4.7277
4	4.72873
5	4.72879
⋮	show more

Example To assign a value to $3^{\sqrt{2}}$,

- $\sqrt{2} = 1.414213562373095\cdots$
- $3^{\sqrt{2}}$ may be approximated by $3^{1.4} = 3^{\frac{14}{10}}$.
- Better approximations are $3^{1.41}$ and so on.
- Denote

$$a_1 = 1.4, \quad a_2 = 1.41, \quad a_3 = 1.414, \quad a_4 = 1.4142, \quad a_5 = 1.41421, \quad \ldots$$

- It can be shown that the sequence

$$3^{a_1},\ 3^{a_2},\ 3^{a_3},\ 3^{a_4}, \ldots$$

  is convergent.

- $3^{\sqrt{2}}$ is defined to be this limit.

n	3^{a_n}
1	4.65554
2	4.70697
3	4.7277
4	4.72873
5	4.72879
⋮	show more

Rules for Exponents

$$(1)\quad a^m a^n = a^{m+n}; \qquad (2)\quad \frac{a^m}{a^n} = a^{m-n};$$

$$(3)\quad (a^m)^n = a^{mn}; \qquad (4)\quad (ab)^n = a^n b^n;$$

$$(5)\quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}; \qquad (6)\quad a^1 = a;$$

$$(7)\quad a^0 = 1; \qquad (8)\quad a^{-n} = \frac{1}{a^n}$$

where *a and b are positive real numbers* and *m, n can be any real number*.