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Wrong solution

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$$= \cdots$$

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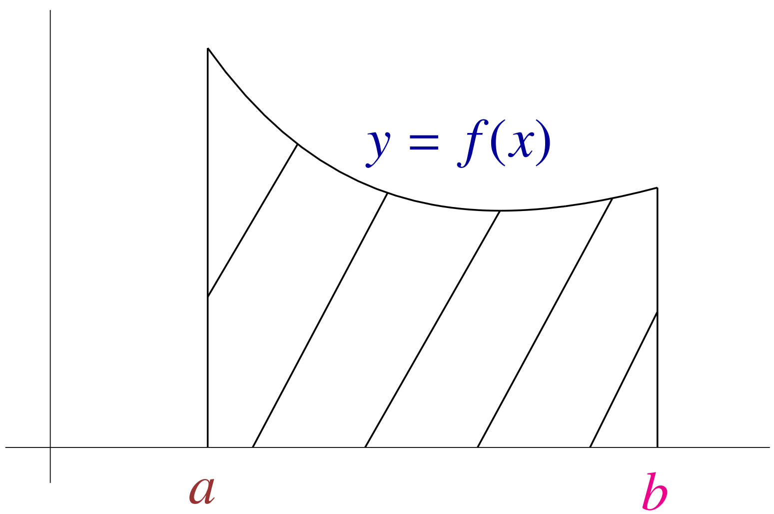
- $$\begin{aligned} \int_0^2 3x^4 \, dx &= \left[3 \cdot \frac{x^5}{5}\right]_0^2 \\ &= \frac{3}{5} \cdot 2^5 - 0 \\ &= \cdots \end{aligned}$$

Application of Definite Integrals

- Suppose $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and *nonnegative*. Then

$$\int_a^b f(x)\,\mathrm{d}x$$

is the area under the graph of f and above the interval $[a, b]$ on the x-axis.



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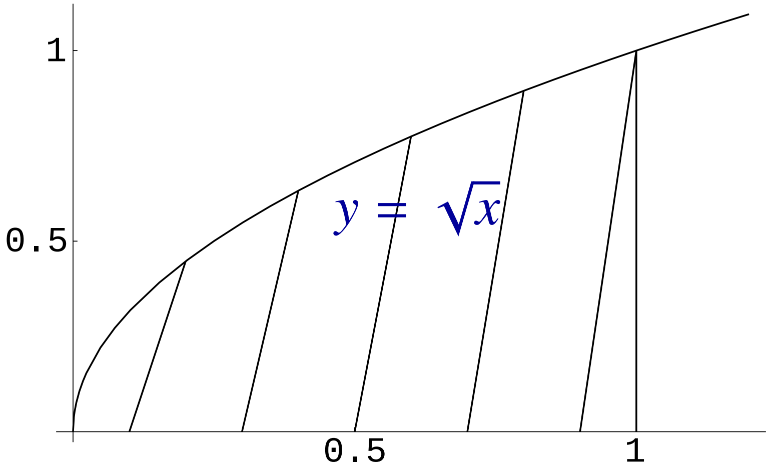

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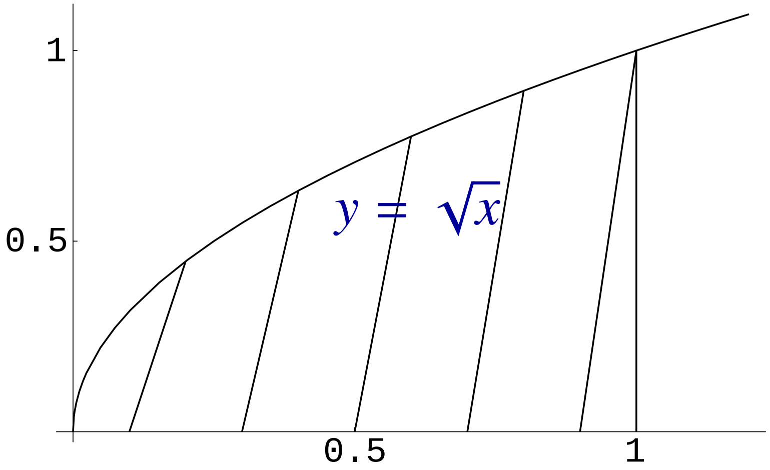

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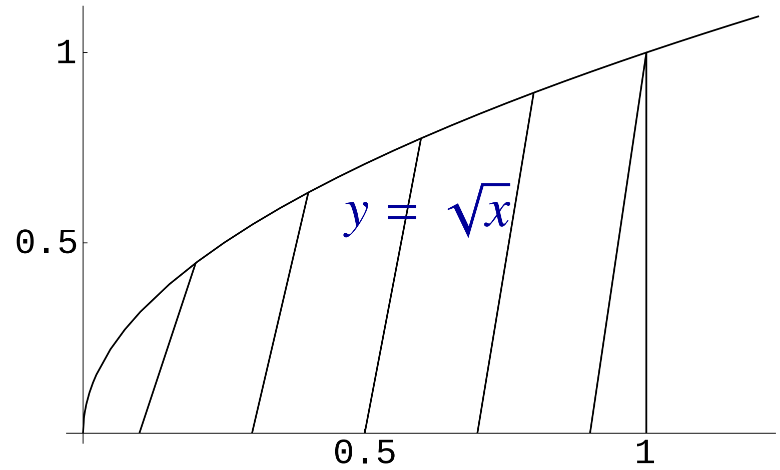

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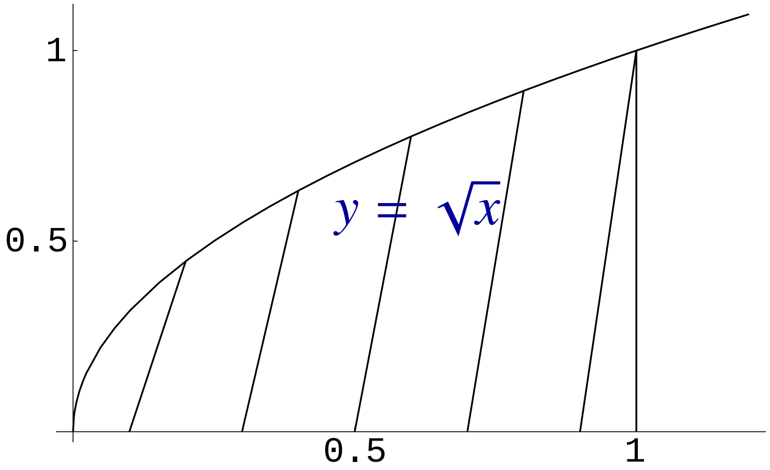

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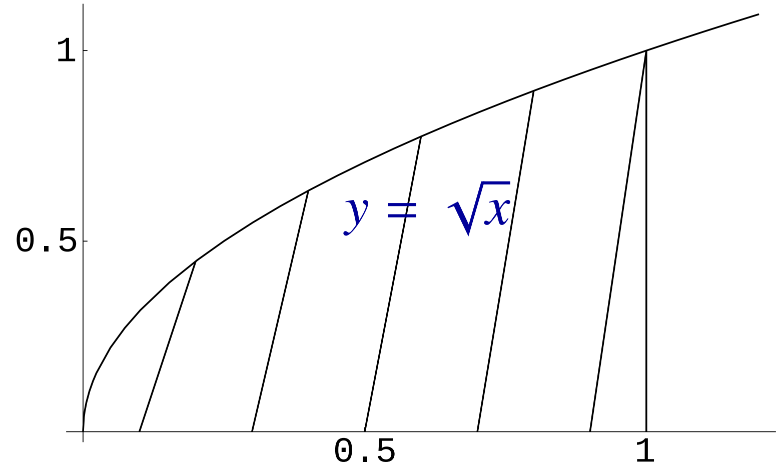

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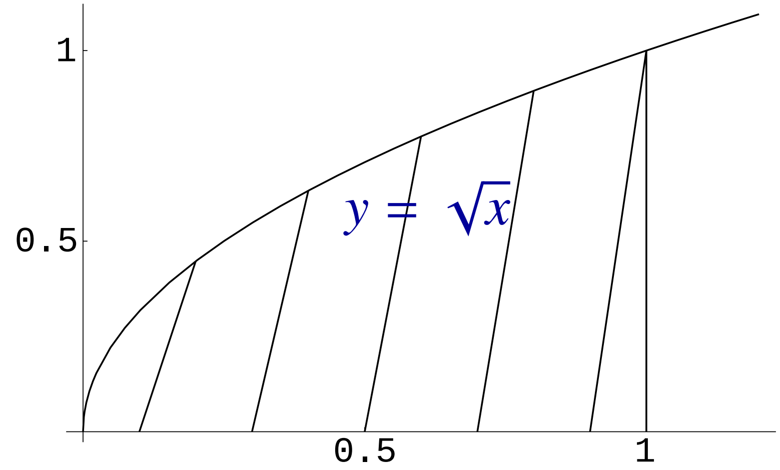

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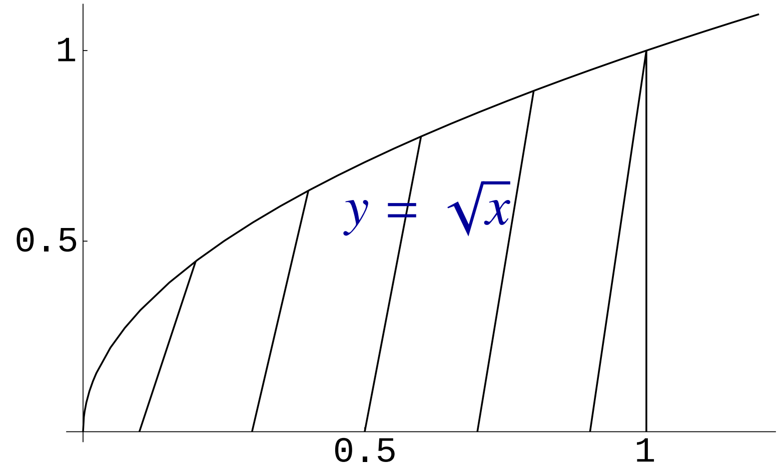

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- Suppose $f, g : [a, b] \longrightarrow \mathbb{R}$ are continuous functions with

$$f(x) \leq g(x) \quad \textit{for all} \;\; x \in [a, b]$$

Then the area of the region bounded by the two graphs and the lines $x = a$ and $x = b$ is

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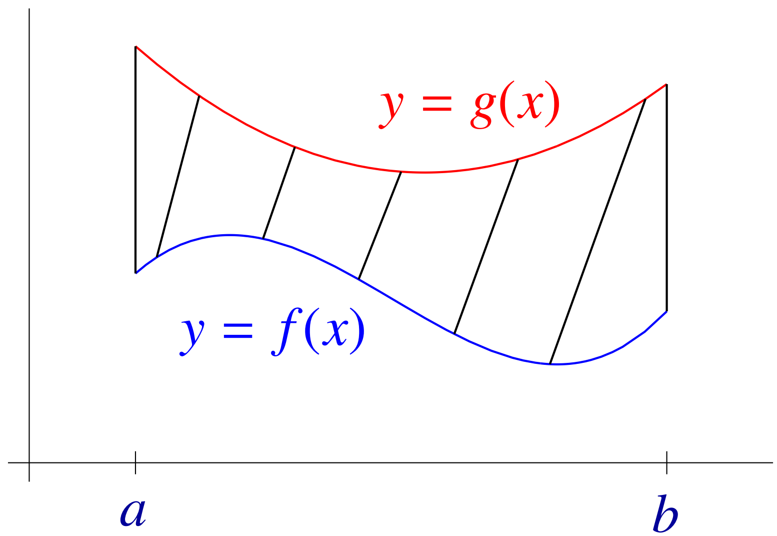

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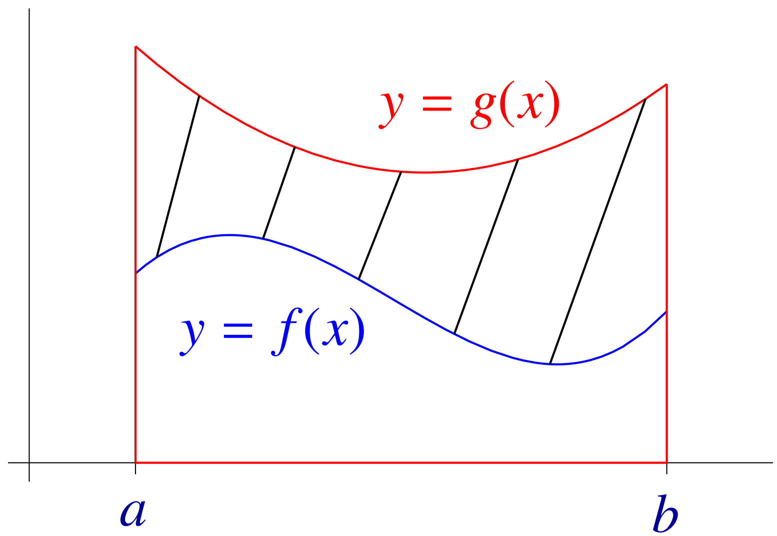

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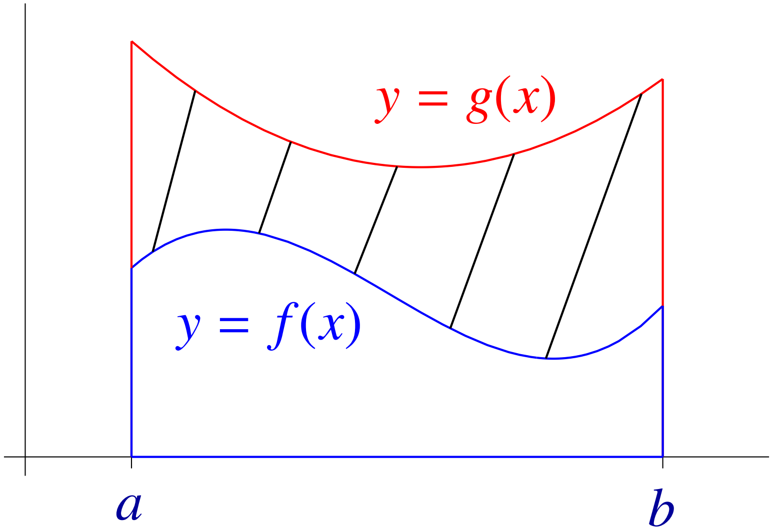

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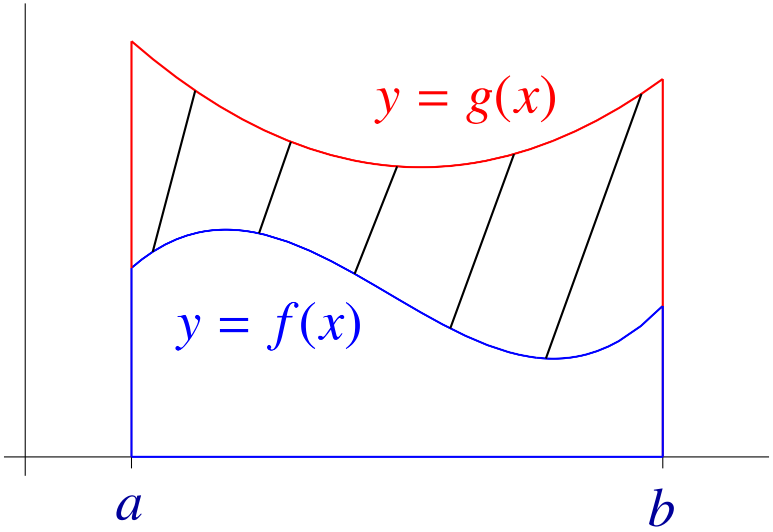

req'd area = $A_g - A_f$

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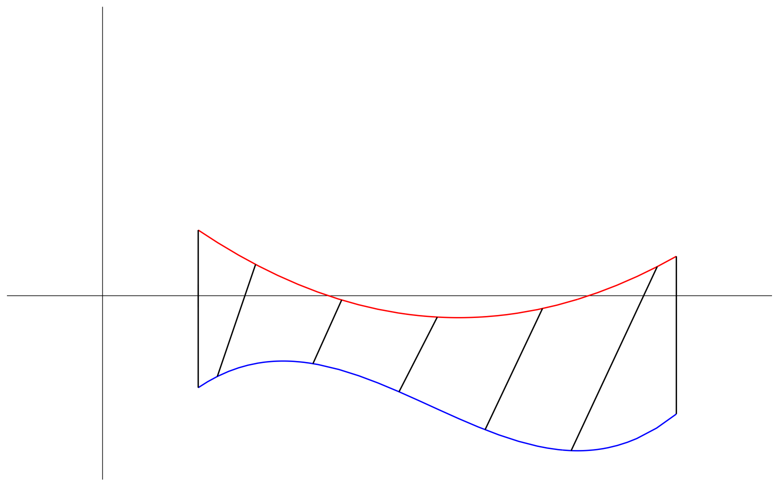

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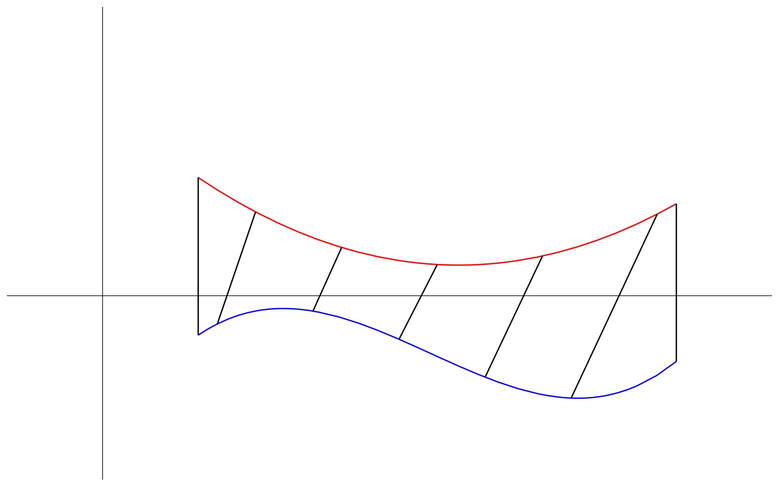

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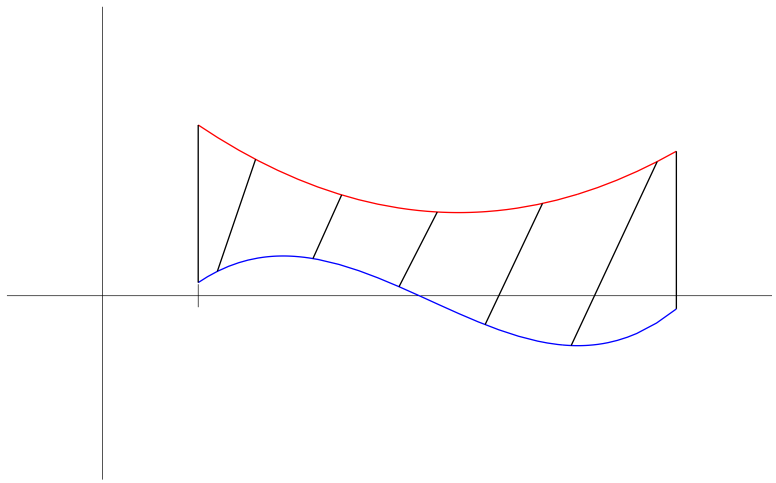

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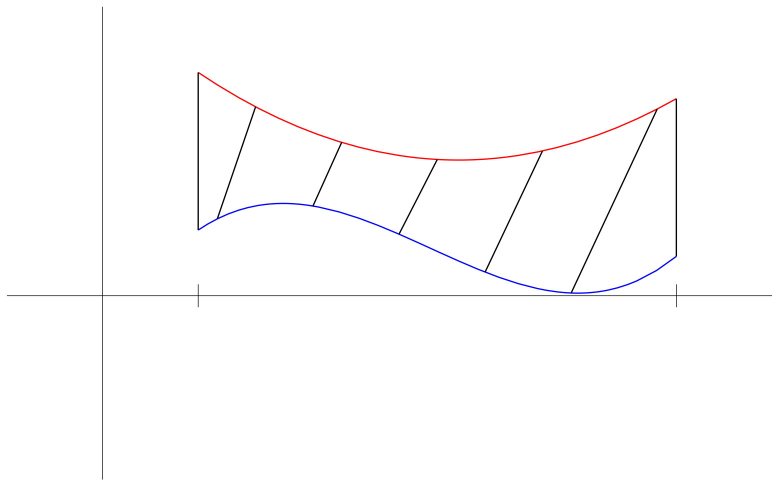

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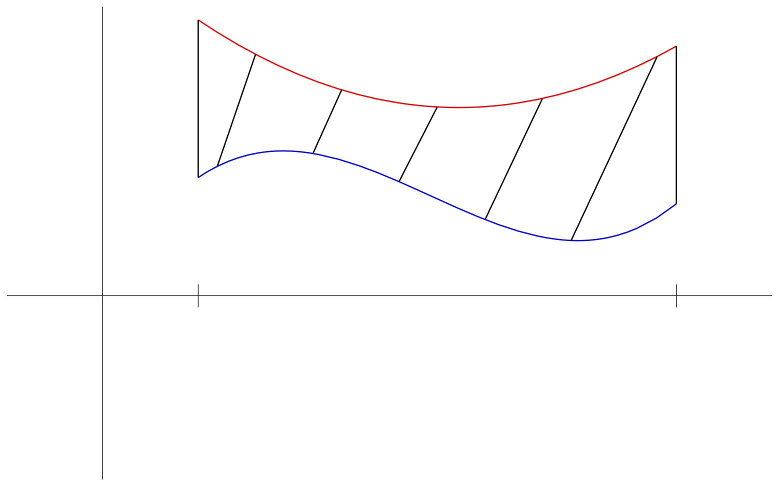

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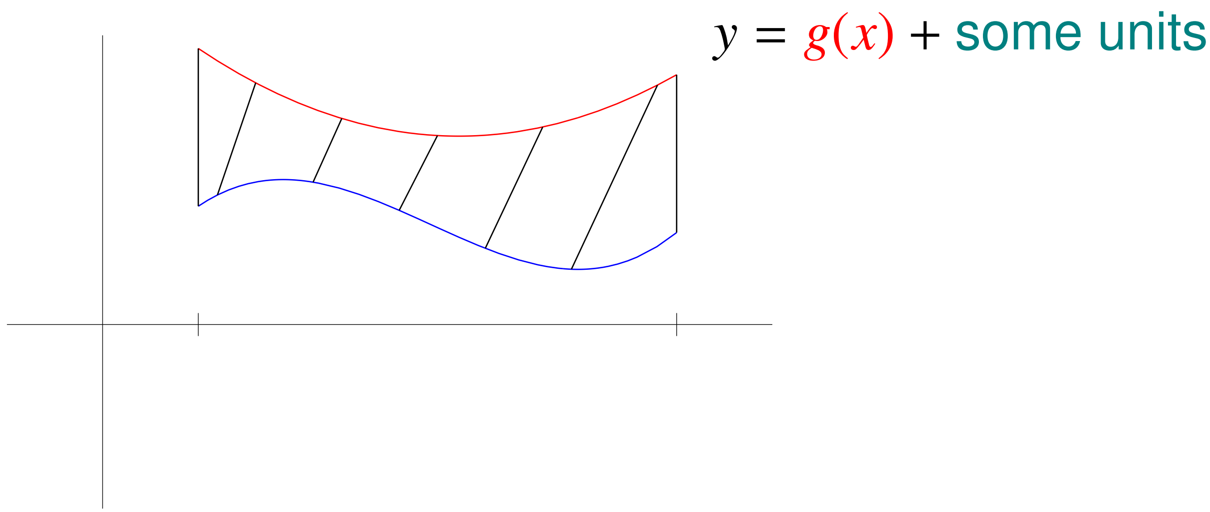

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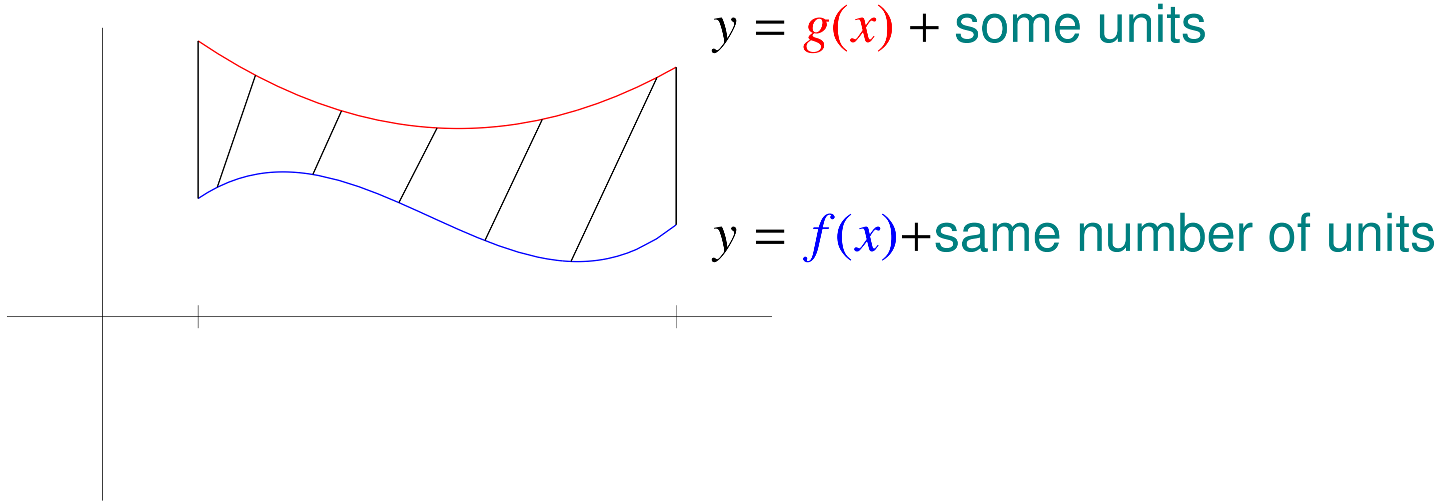

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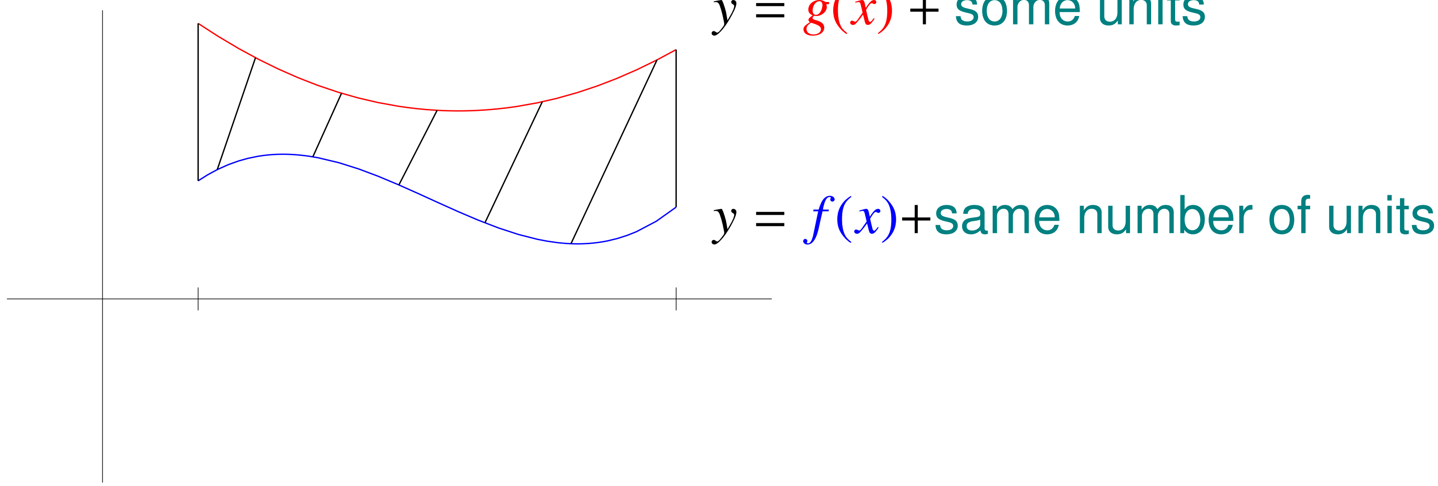

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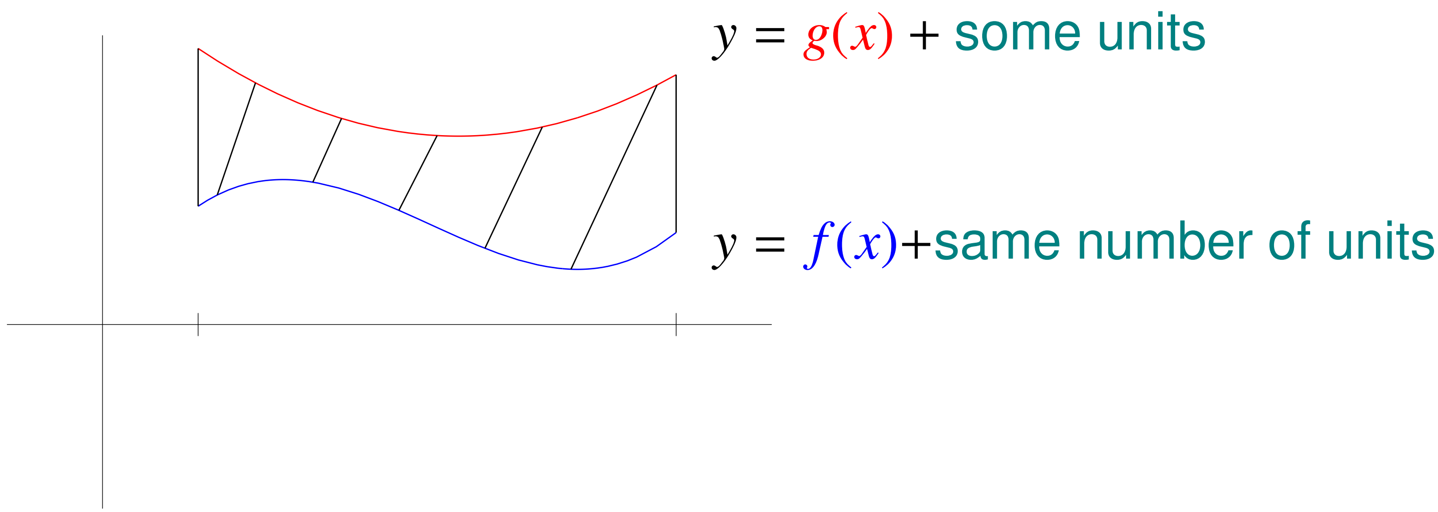

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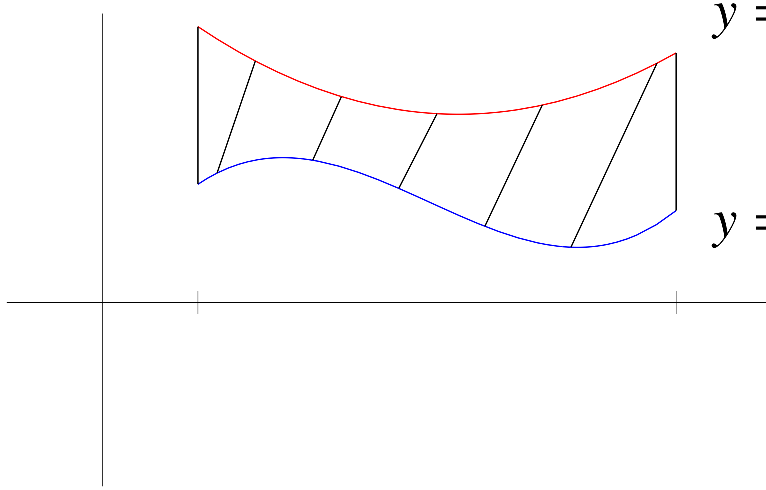

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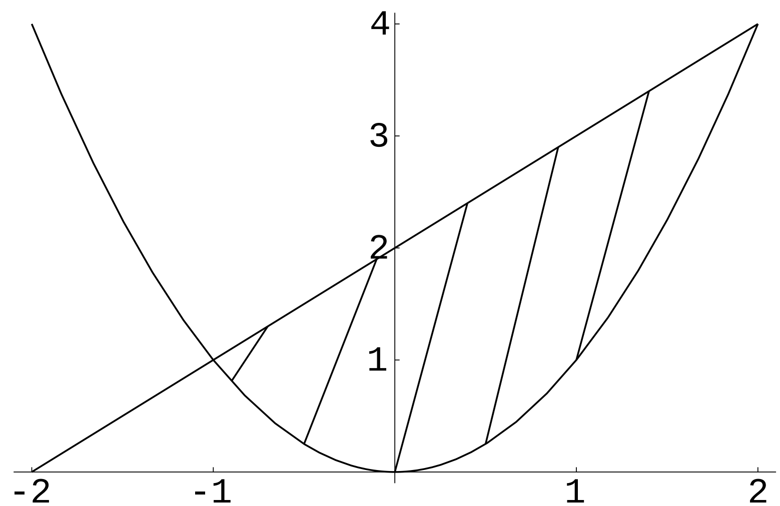

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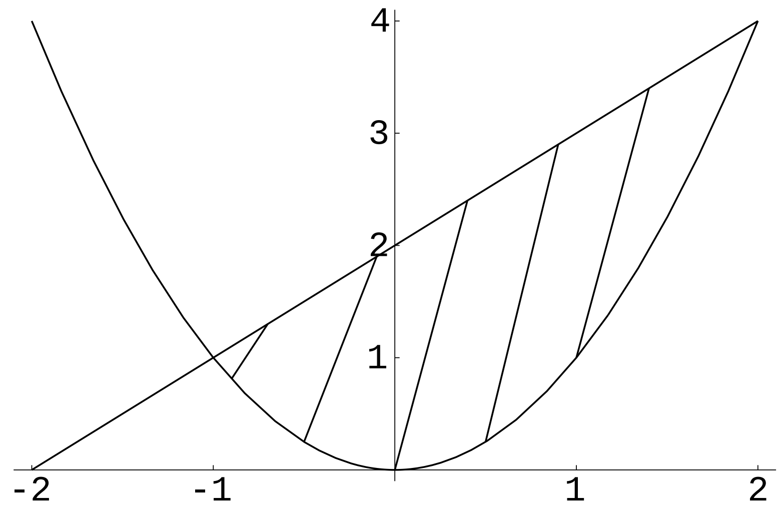

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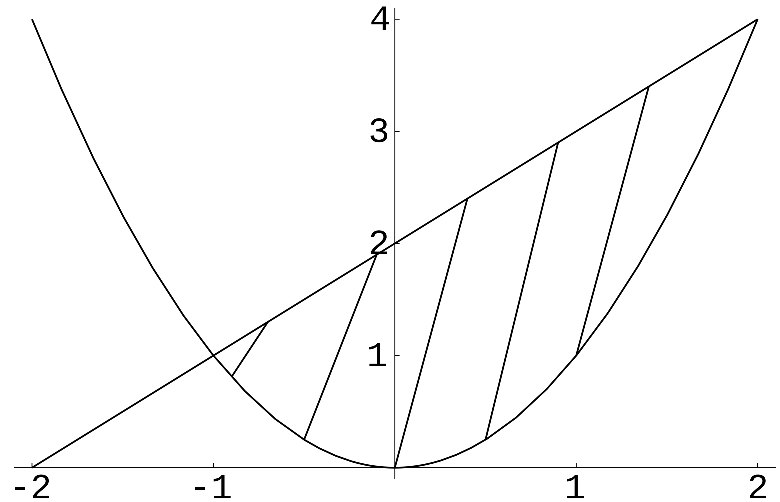

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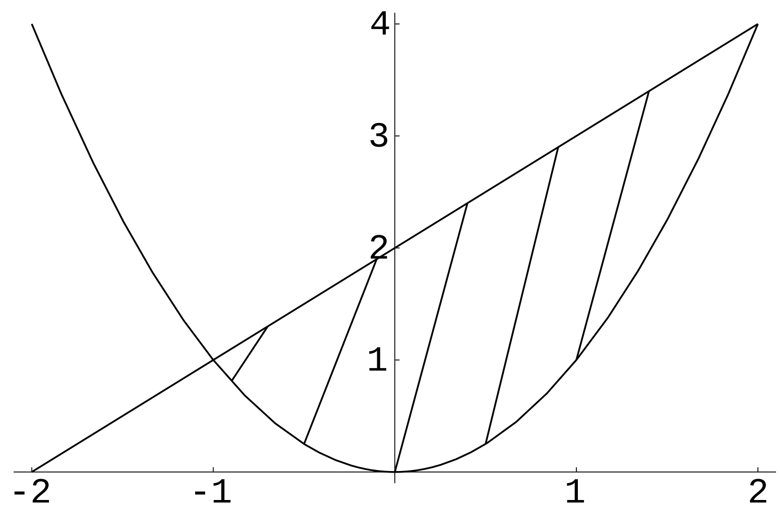

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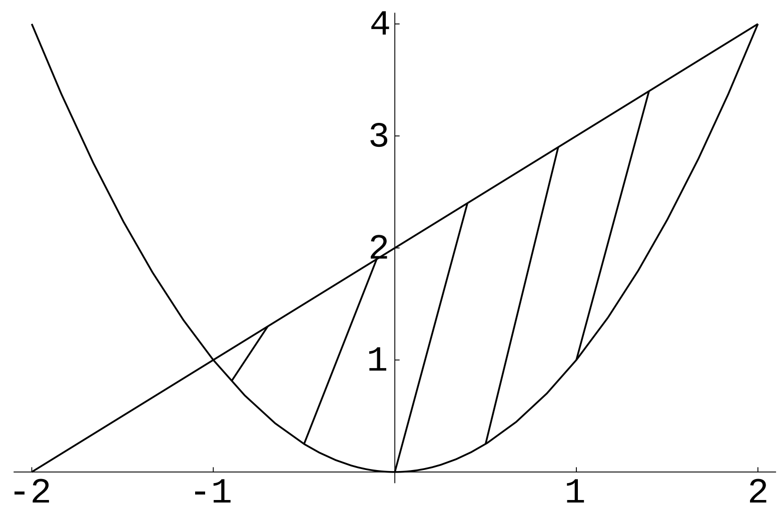

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$$= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^{2} = \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{9}{2}$$

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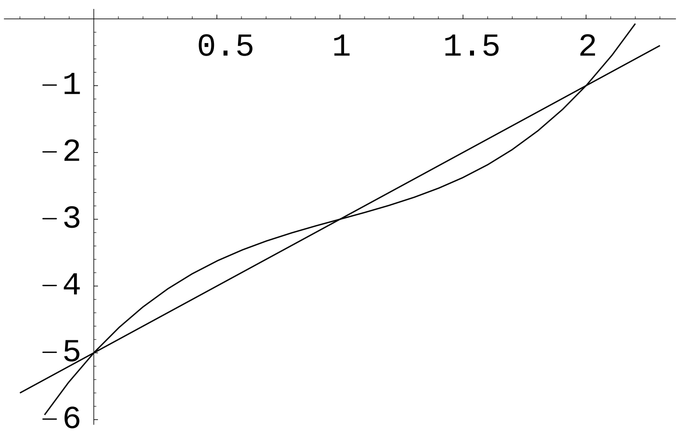

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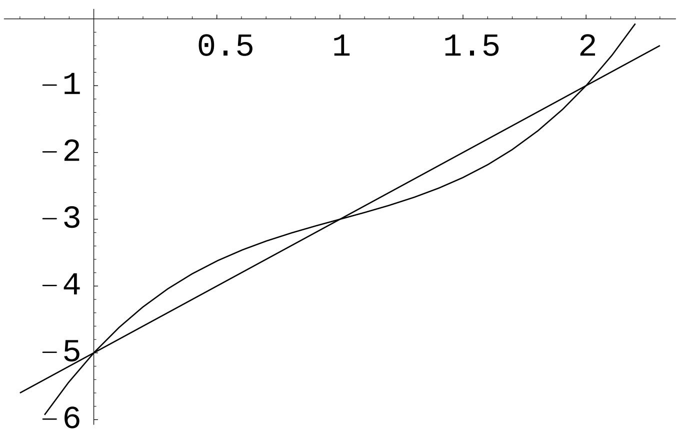

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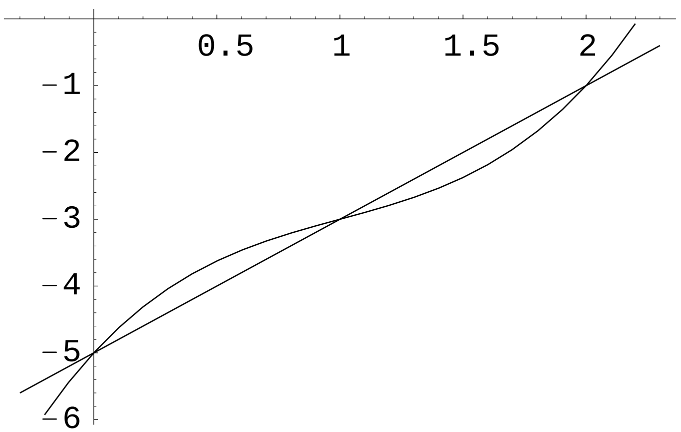
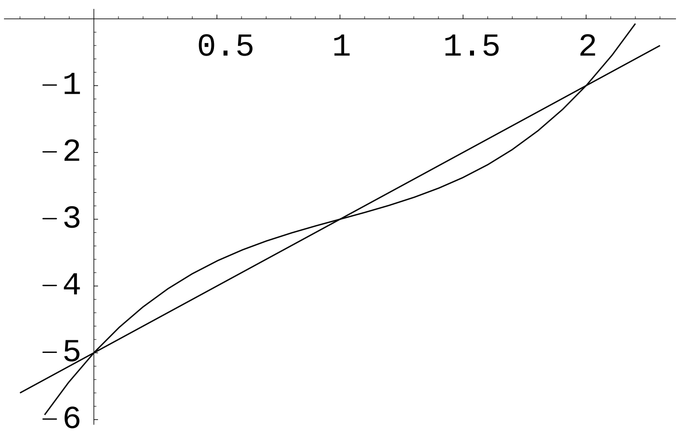
	$0 < x < 1$	$1 < x < 2$
$y_1 = x^3 - 3x^2 + 4x - 5$	$y_1(0.5) = -3.625$	$y_1(1.5) = -2.375$
$y_2 = 2x - 5$	$y_2(0.5) = -4$	$y_2(1.5) = -2$
	$y_1 > y_2$	$y_1 < y_2$



The required area is

$$A = \int_0^1 \left((x^3 - 3x^2 + 4x - 5) - (2x - 5)\right)\mathrm{d}x +$$



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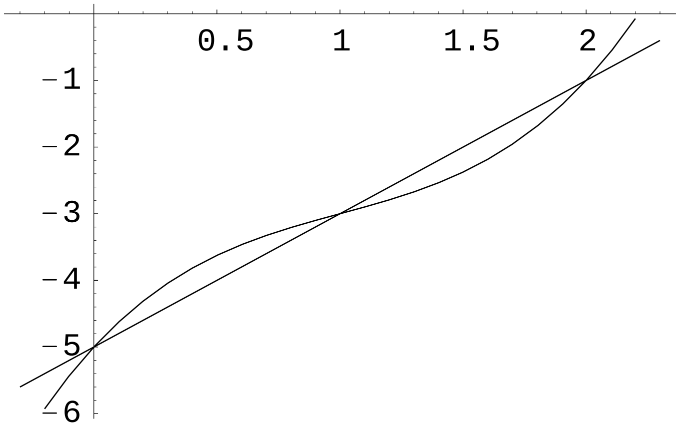

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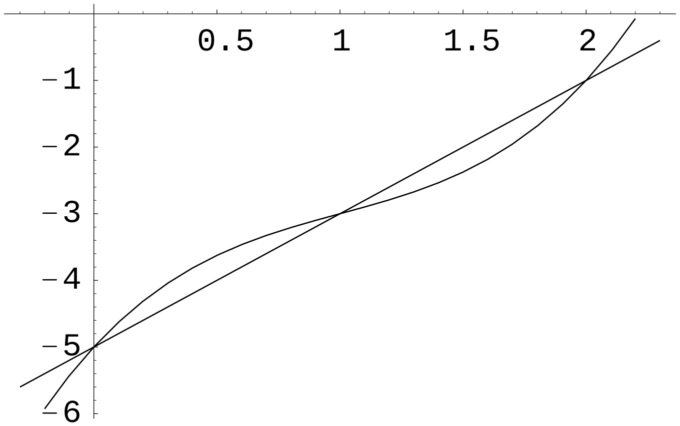

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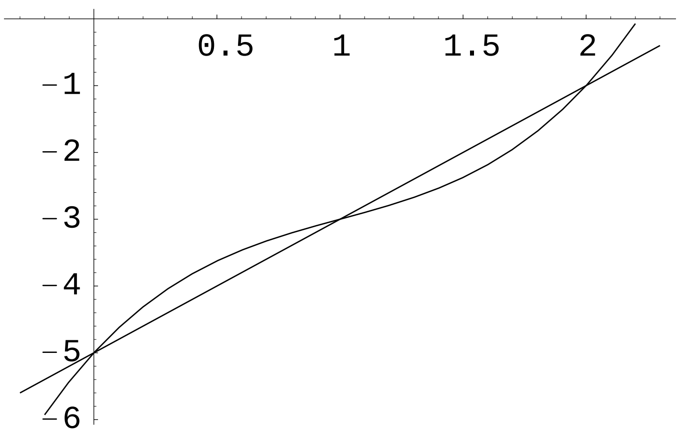

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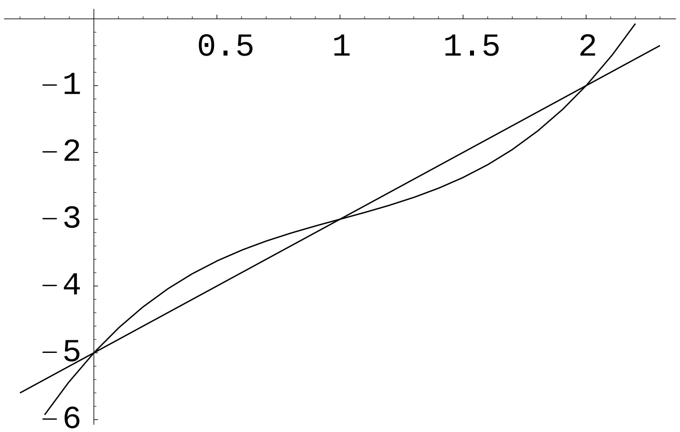

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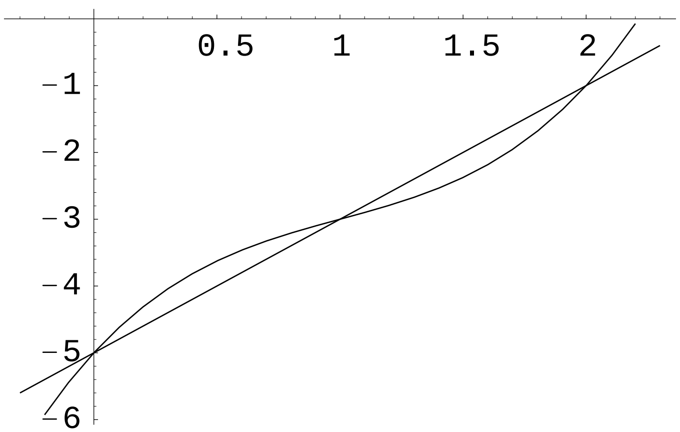

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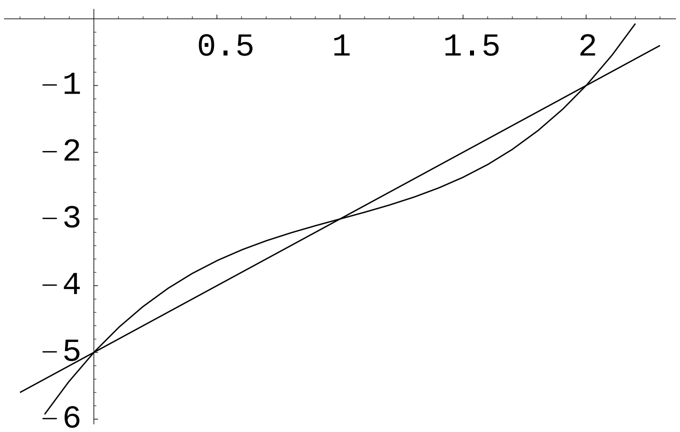
$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$



Chapter 7: Trigonometric Functions

- Angles
- Trigonometric Functions
- Differentiation of Trigonometric Functions

Objectives

- To differentiate trigonometric functions.

Angles

Definition (Idea)

An *angle* is formed by rotating a ray about its endpoint.

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An *angle* is formed by rotating a ray about its endpoint.

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- The endpoint of the ray is called the *vertex*.
- The final position is called the *terminal side*. See angle in motion

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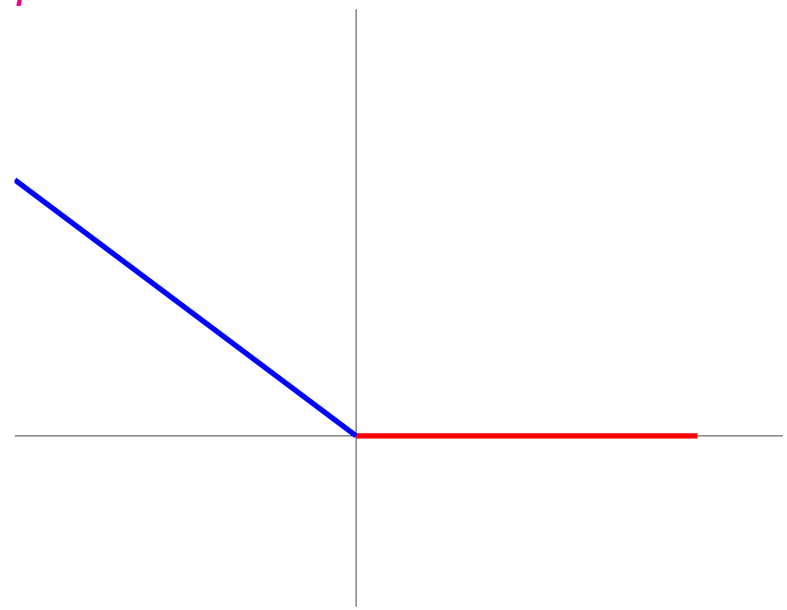
See angle in motion

Definition An angle is said to be in *standard position* if

- its vertex is at the origin and
- its initial side is along the positive x-axis.



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- Can use numbers to represent angles.
  - The direction of rotation may be
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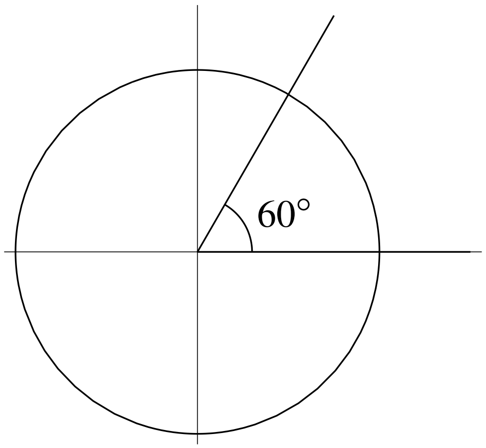
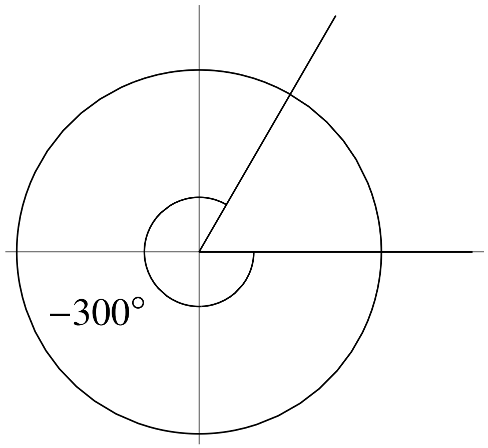
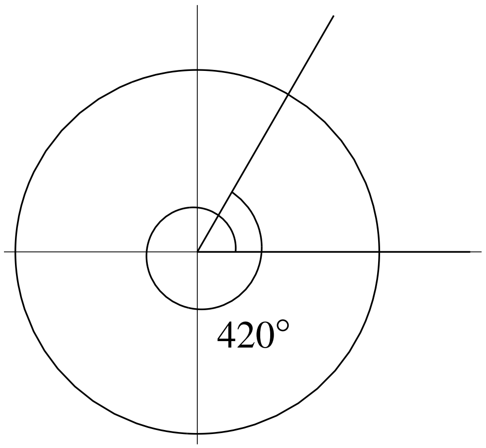
  - Magnitudes of rotation are traditionally measured in *degrees* where

$$\text{one revolution} = 360^\circ$$





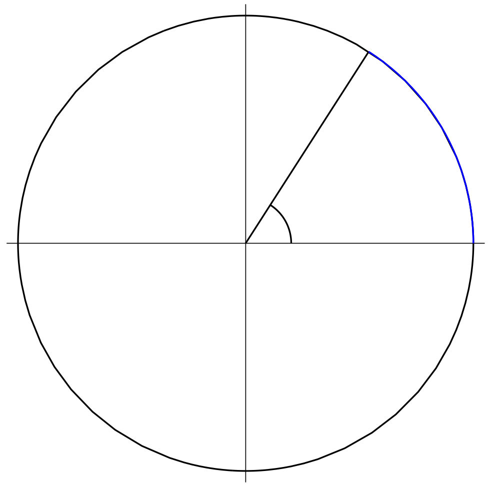


See 60° angle

See -300° angle

See 420° angle

Another unit for measuring angles is the radian.



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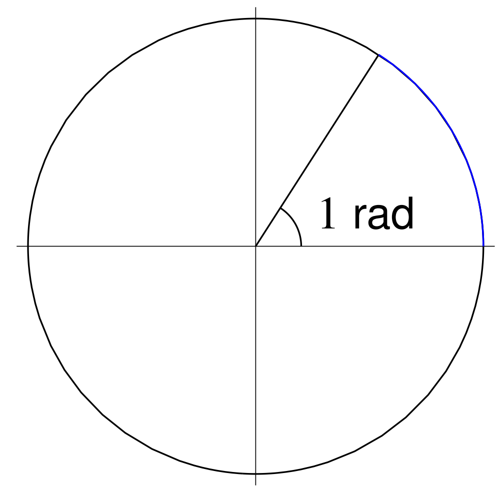

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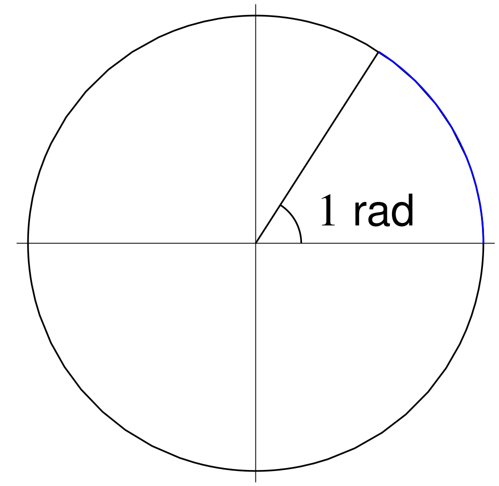

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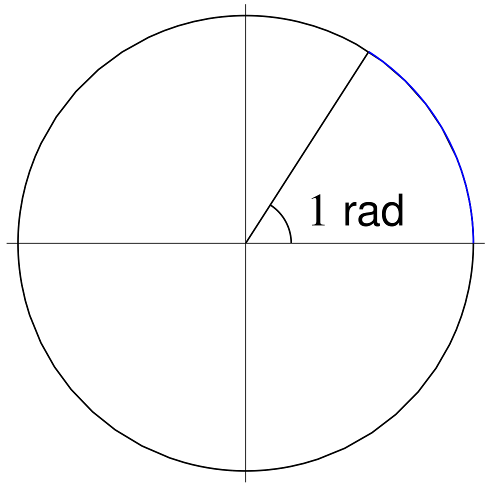

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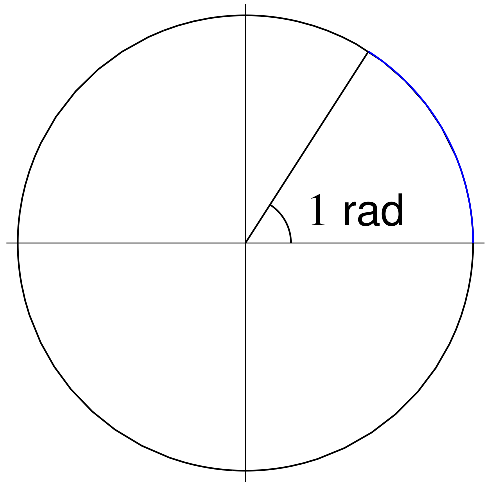

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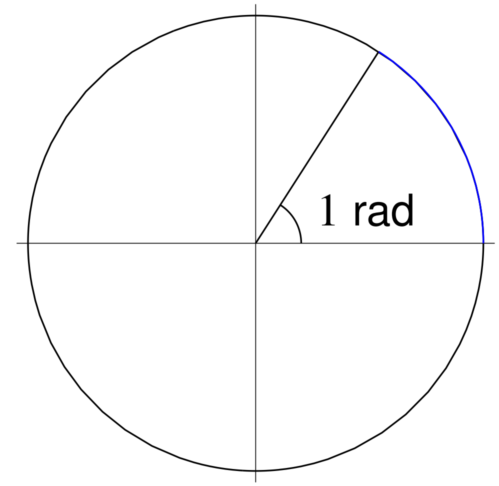

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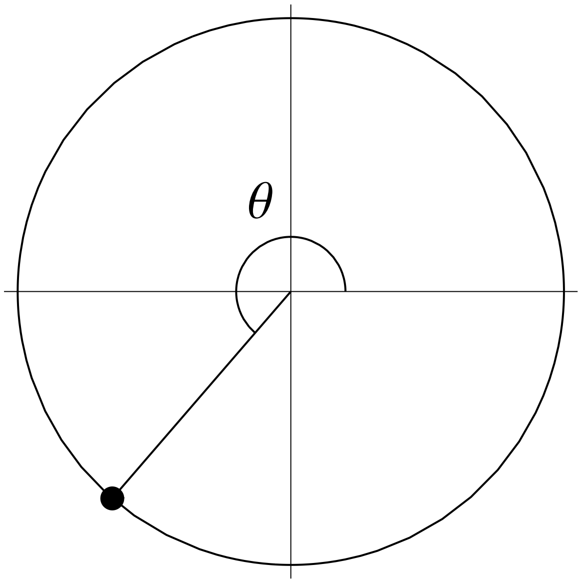
Using (1):

$$90^\circ = \frac{\pi}{2}, \quad 60^\circ = \frac{\pi}{3} \text{ etc}$$

Trigonometric Functions

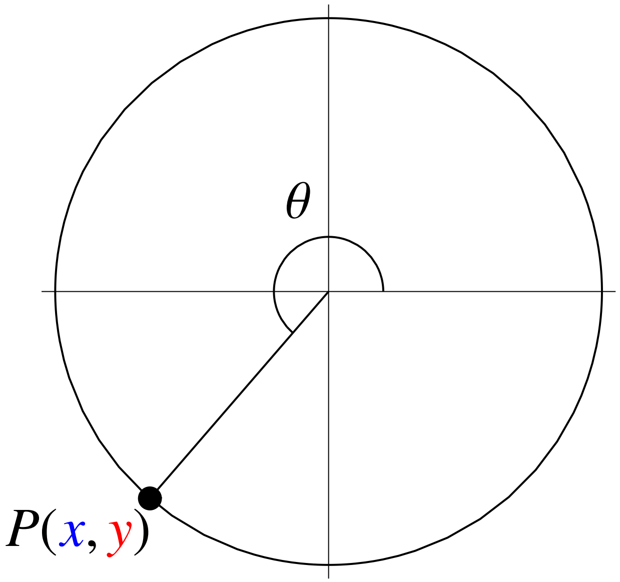
Consider an angle θ in *standard position*.



Trigonometric Functions

Consider an angle θ in *standard position*.

Let P be the point of intersection of the terminal side and *the unit circle*.



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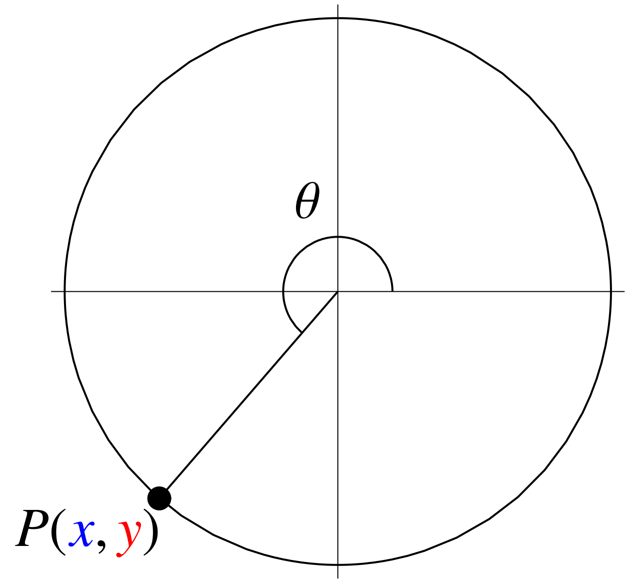

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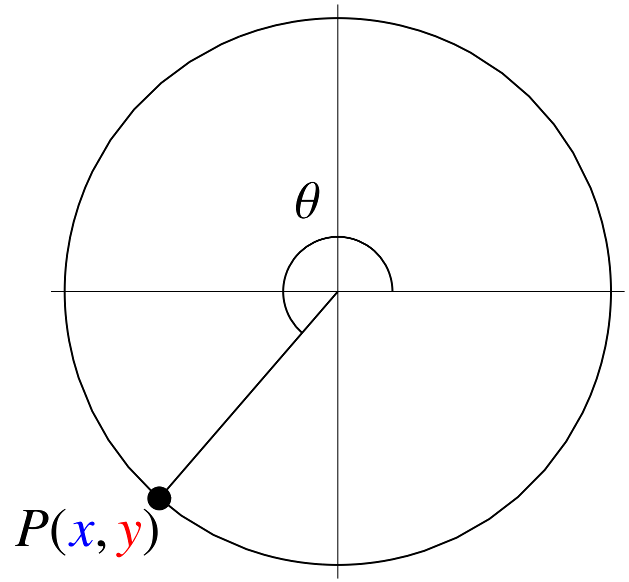

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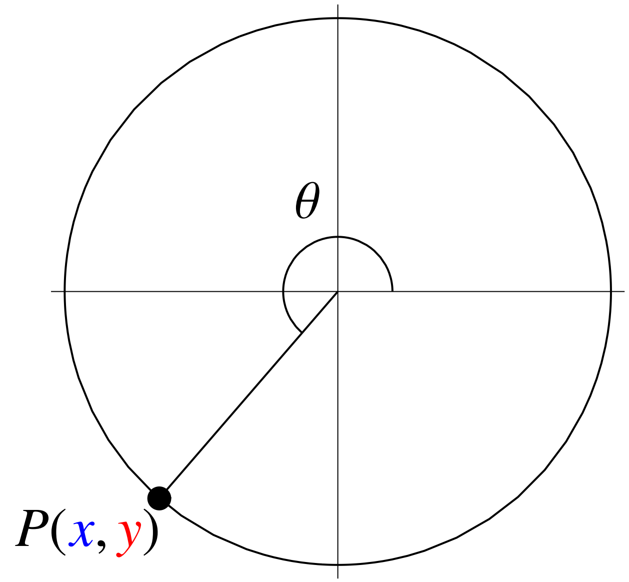

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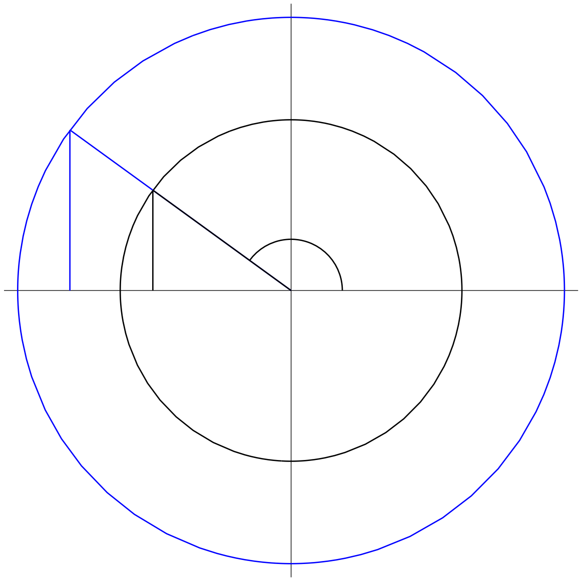
and we define $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ if $\cos\theta \neq 0$.

Remark

(1) Instead of using unit circle, can also use circle of radius r and define



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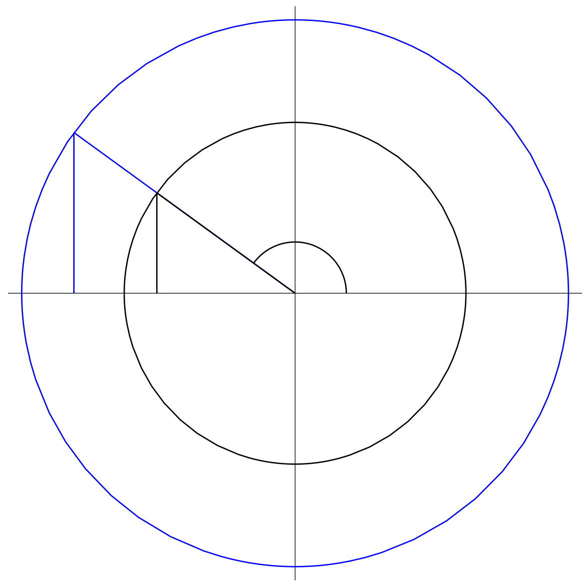

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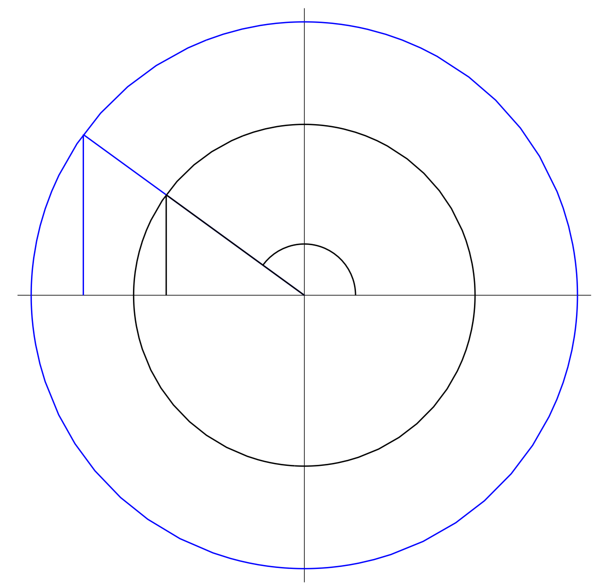

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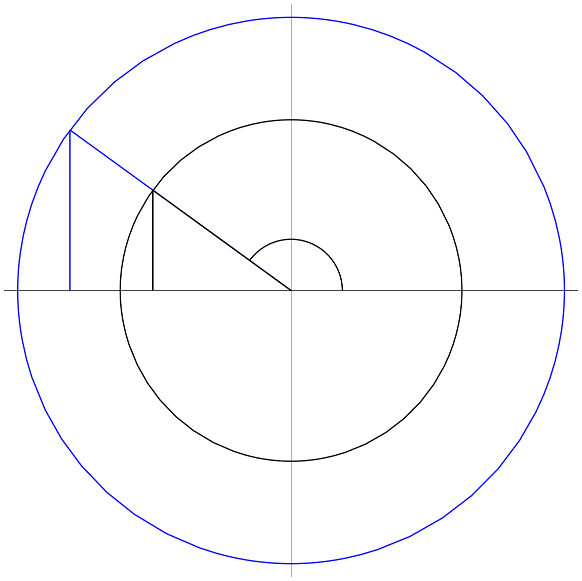

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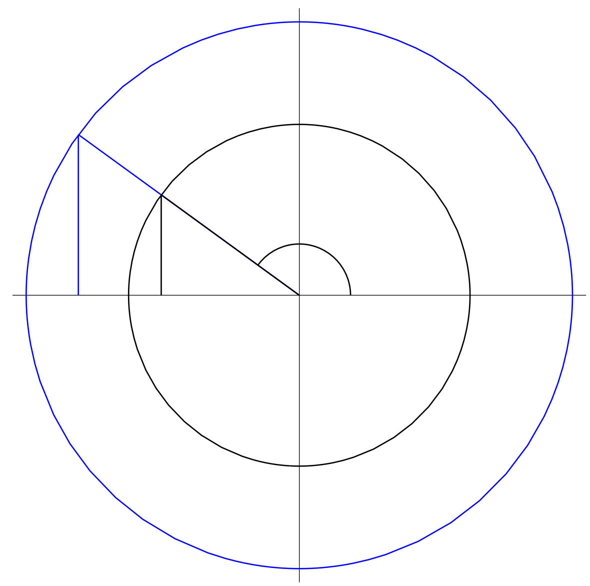

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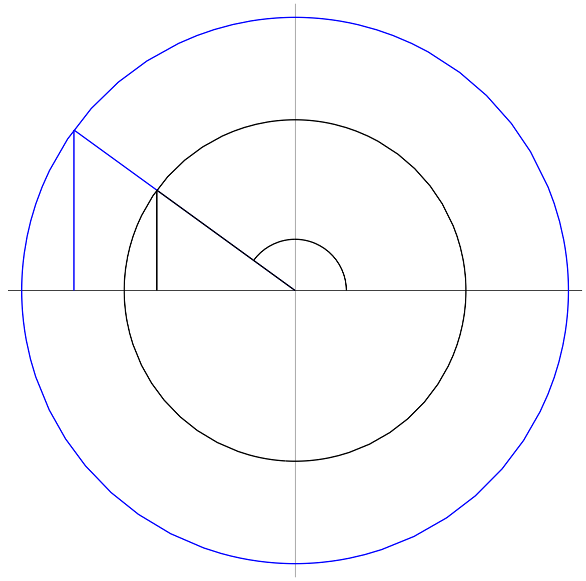

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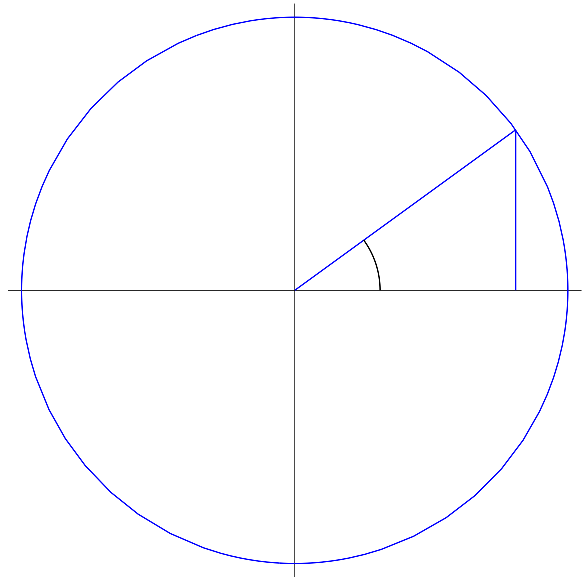

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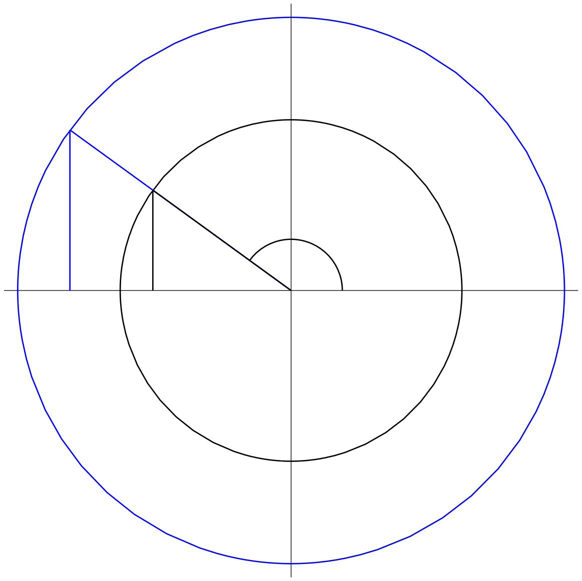

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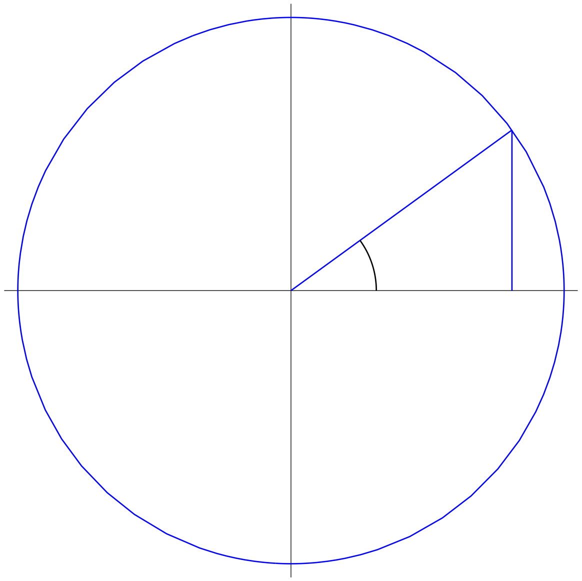

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- The sine and cosine functions are periodic with period 2π:

$$\begin{aligned} \sin(x + 2\pi) &= \sin x \\ \cos(x + 2\pi) &= \cos x \end{aligned}$$

for all $x \in \mathbb{R}$.

- ◇ The domain of the sine (and also cosine) function is $\mathbb{R}$.

  ◇ The range is $[-1, 1]$.

- The sine and cosine functions are periodic with period 2π:

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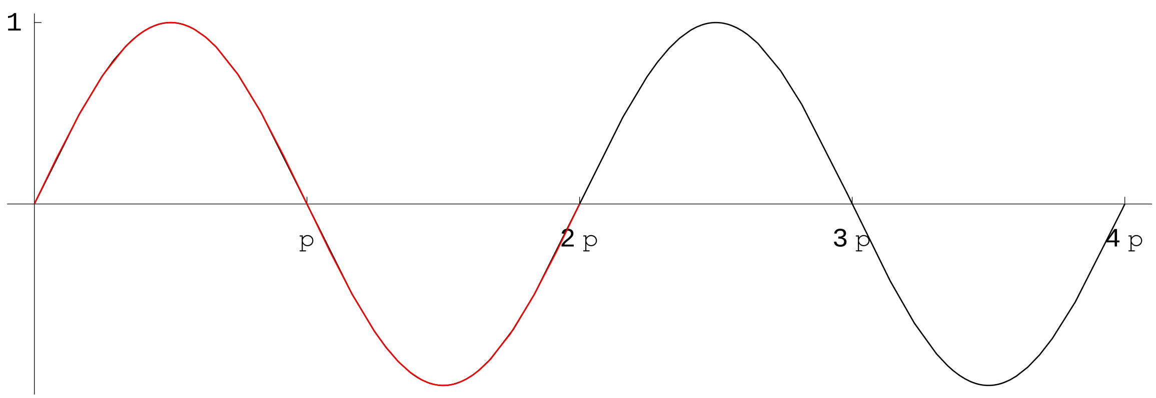
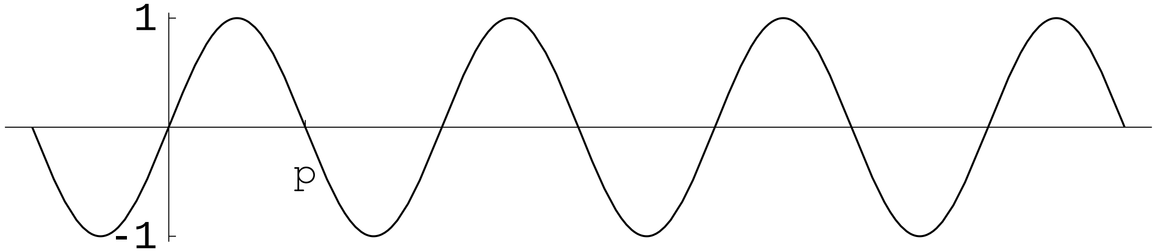
for all $x \in \mathbb{R}$.



◇ **Graph of** $y = \sin x$



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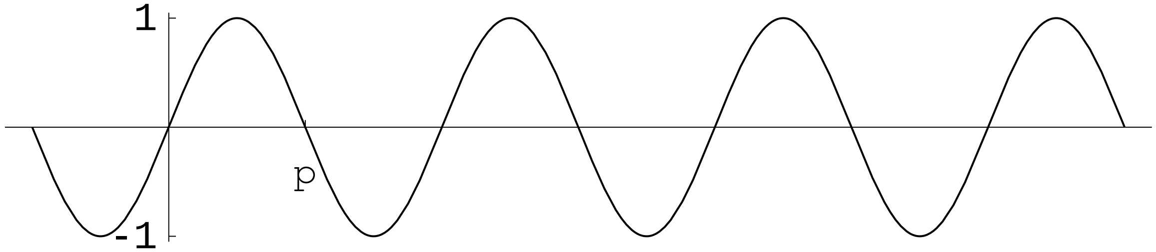

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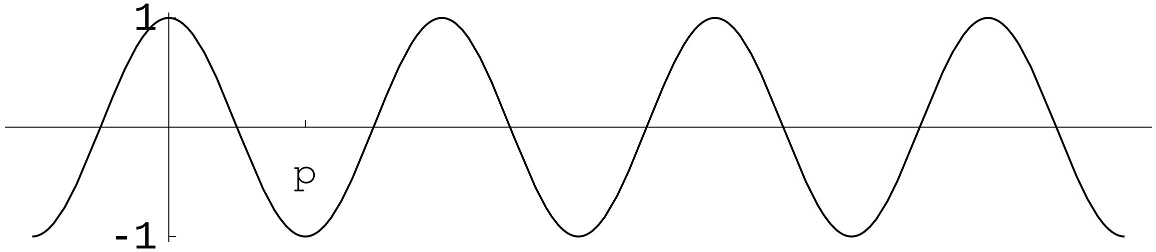

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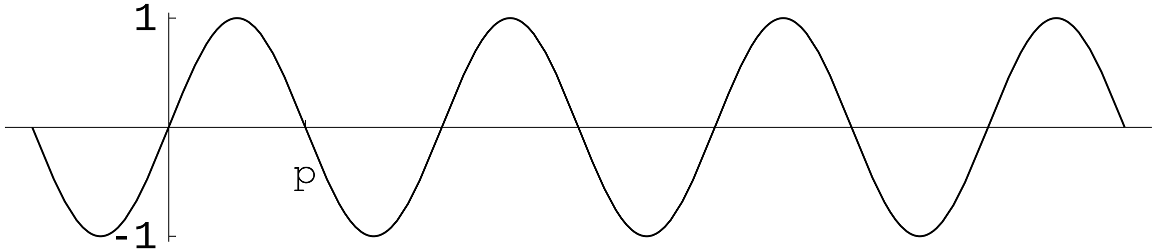

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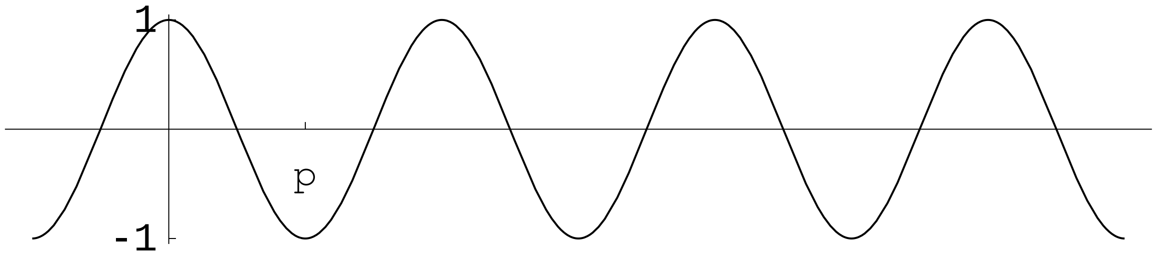

Remark Graph of cosine function can be obtained by *shifting* graph of sine function $\frac{\pi}{2}$ *units to the left*.

- Since $\cos x = 0$ for $x = \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ \ldots,$

  $\tan x = \frac{\sin x}{\cos x}$ is defined for $x \neq \pm\frac{\pi}{2},\ \pm\frac{3\pi}{2},\ \pm\frac{5\pi}{2},\ \ldots$.

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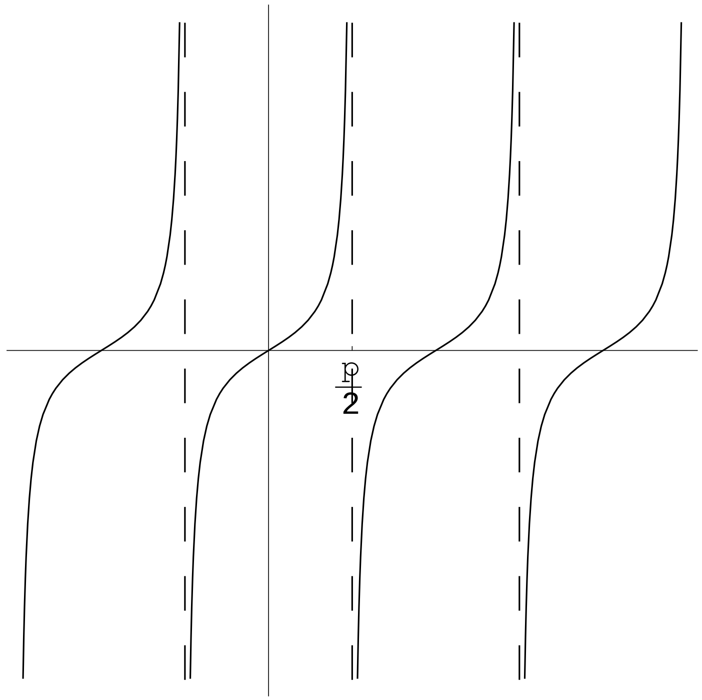

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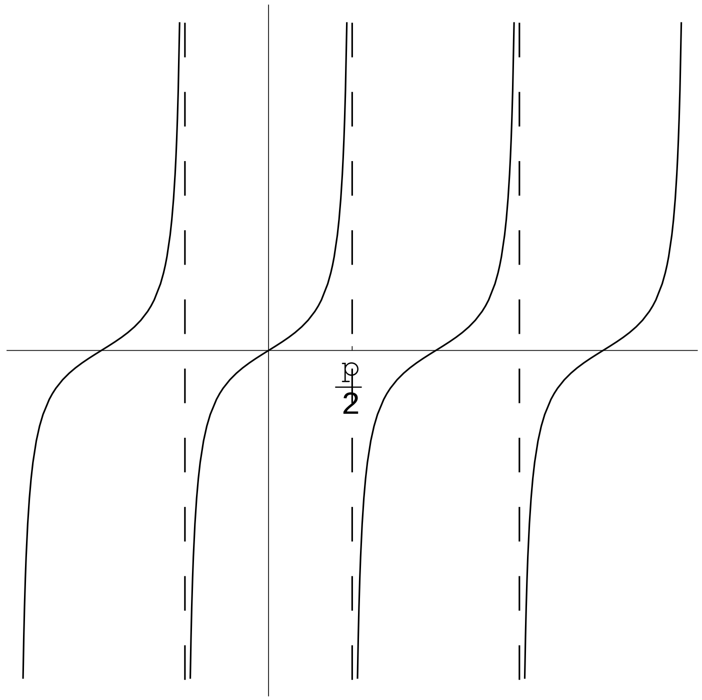

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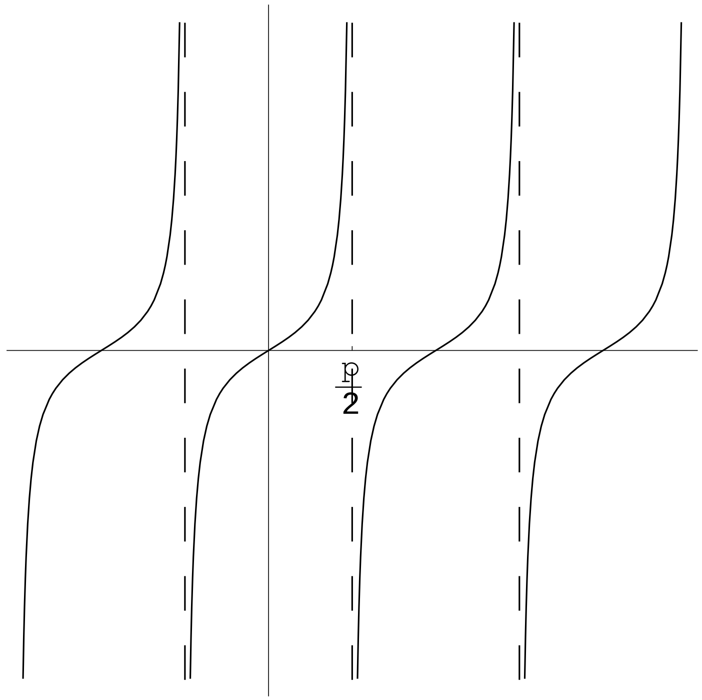


  ◇ The domain is $\mathbb{R} - \{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots\}$.

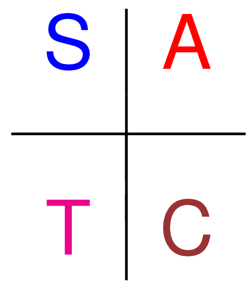
  ◇ The range is $\mathbb{R}$.

CAST rule

S	A
T	C

where

CAST rule



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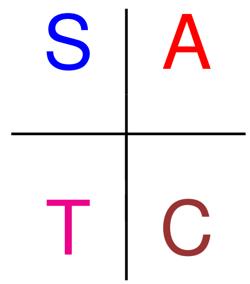
CAST rule



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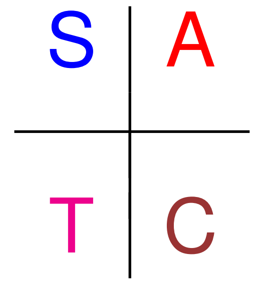

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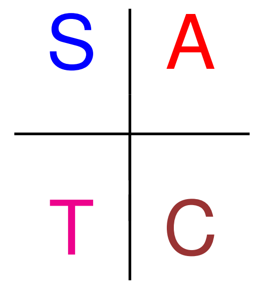

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For example, C in the 4th quadrant means that *cosine is positive* there whereas *the other two trigonometric functions are negative*.

Special Angles

$$
\begin{aligned}
\sin 0 &= 0 & \cos 0 &= 1 \\
\sin\frac{\pi}{6} &= \frac{1}{2} & \cos\frac{\pi}{6} &= \frac{\sqrt{3}}{2} \\
\sin\frac{\pi}{4} &= \frac{1}{\sqrt{2}} & \cos\frac{\pi}{4} &= \frac{1}{\sqrt{2}} \\
\sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2} & \cos\frac{\pi}{3} &= \frac{1}{2} \\
\sin\frac{\pi}{2} &= 1 & \cos\frac{\pi}{2} &= 0
\end{aligned}
$$

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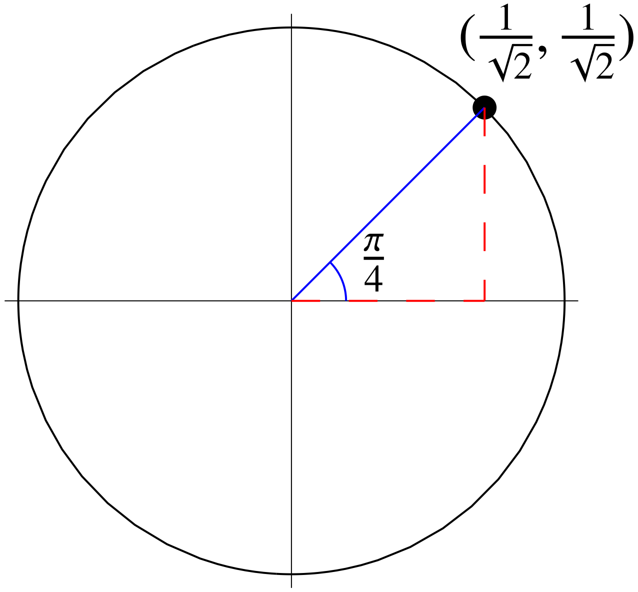
$$\sin\frac{\pi}{2} = 1 \qquad \cos\frac{\pi}{2} = 0$$



An important identity

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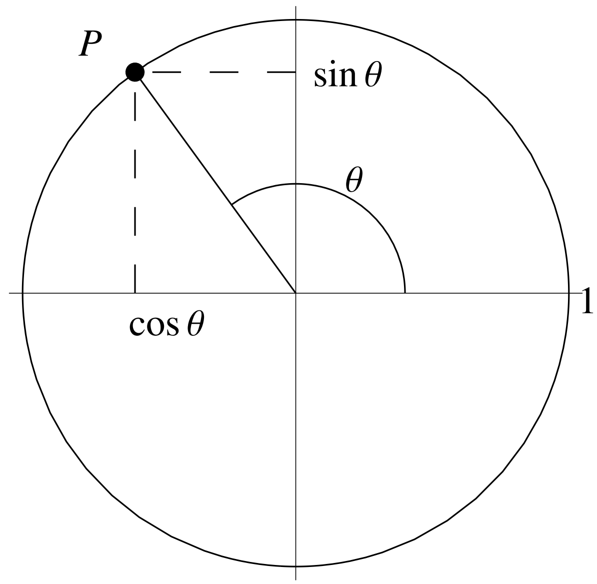

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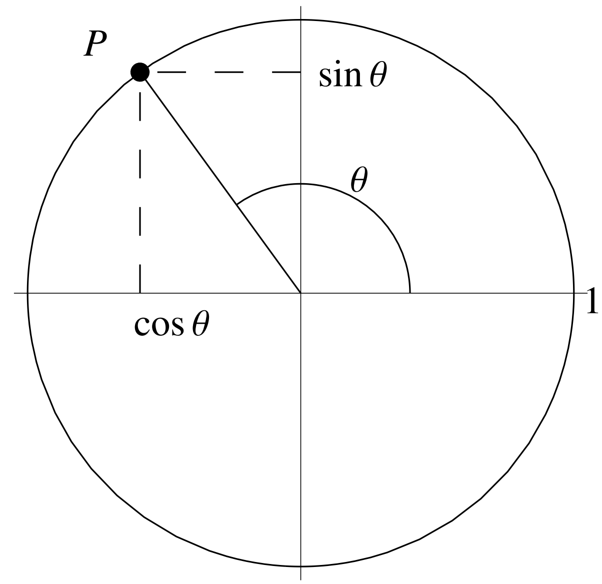
Proof $\sin\theta$ and $\cos\theta$ are the y- and x-coordinates of P where P lies on the unit circle $x^2 + y^2 = 1$.



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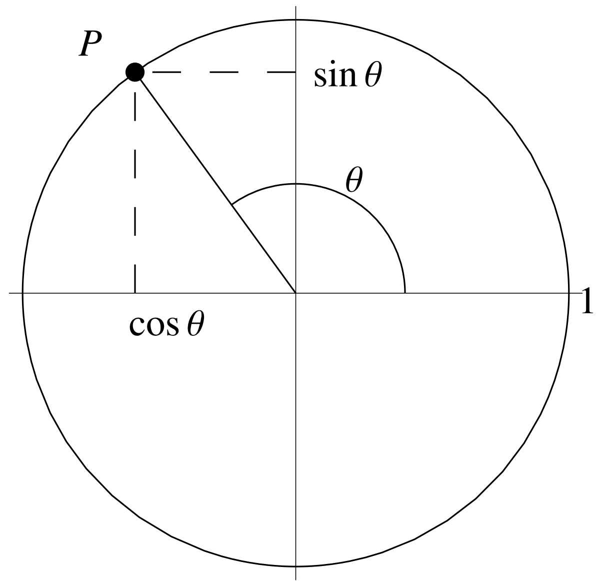

Remark (∗) is called an *identity* because the equality holds for all $\theta \in \mathbb{R}$.

More Identities

$$
\begin{aligned}
\sin(\tfrac{\pi}{2} - x) &= \cos x & \cos(\tfrac{\pi}{2} - x) &= \sin x \\
\sin(\tfrac{\pi}{2} + x) &= \cos x & \cos(\tfrac{\pi}{2} + x) &= -\sin x \\
\sin(\pi - x) &= \sin x & \cos(\pi - x) &= -\cos x \\
\sin(\pi + x) &= -\sin x & \cos(\pi + x) &= -\cos x \\
\sin(\tfrac{3\pi}{2} - x) &= -\cos x & \cos(\tfrac{3\pi}{2} - x) &= -\sin x \\
\sin(\tfrac{3\pi}{2} + x) &= -\cos x & \cos(\tfrac{3\pi}{2} + x) &= \sin x \\
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\end{aligned}
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- Can be derived using appropriate figures (see lecture notes).

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$$

- Can be derived using appropriate figures (see lecture notes).
- Can be memorized using some rules (see lecture notes).