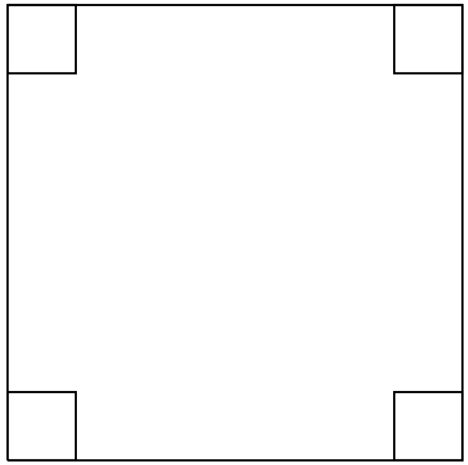
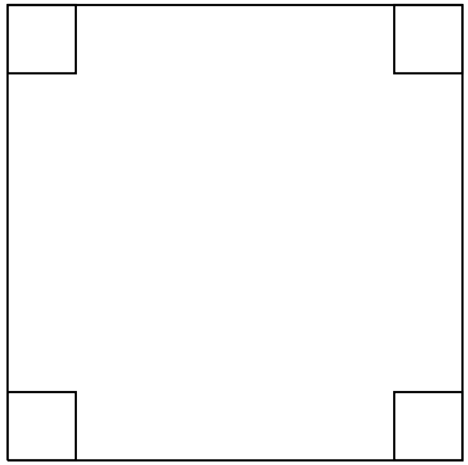
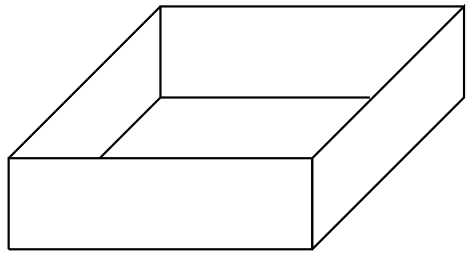
Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides.

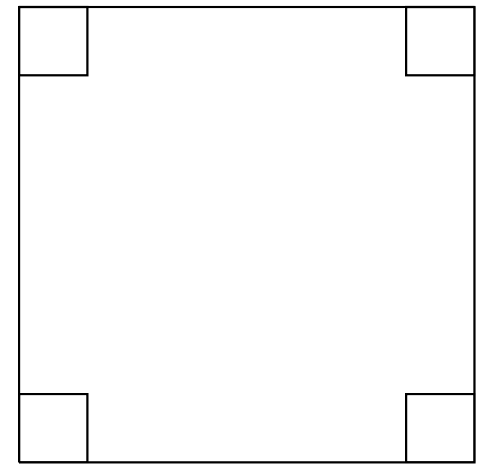


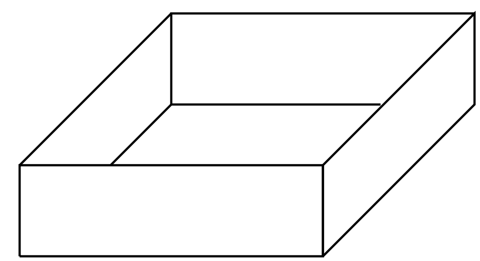
Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides.



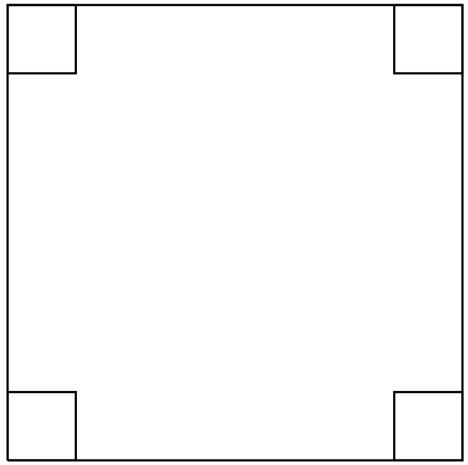


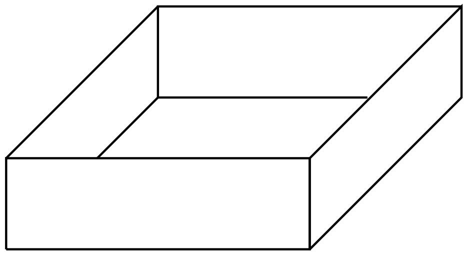
Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off



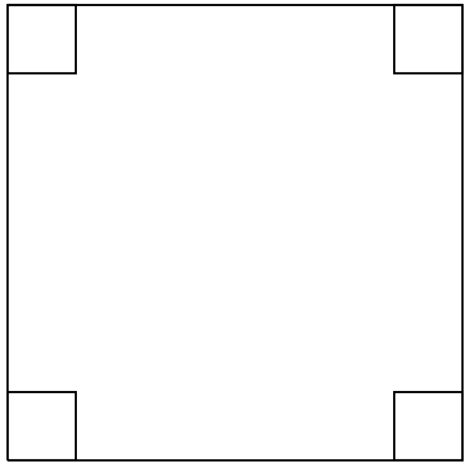


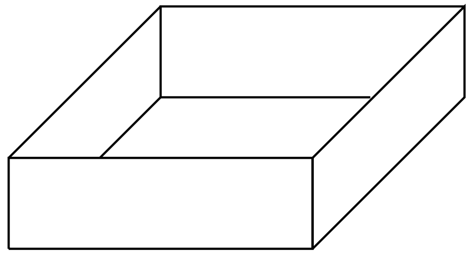
Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off if the volume of the box is to be maximized.



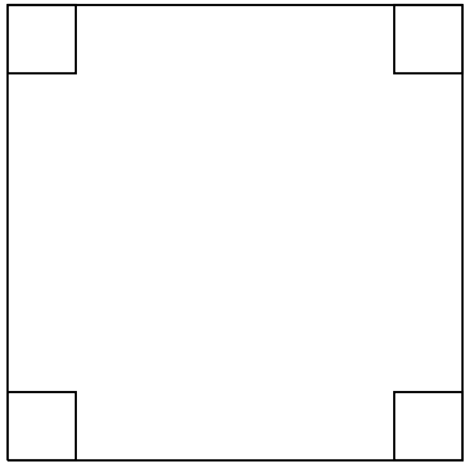


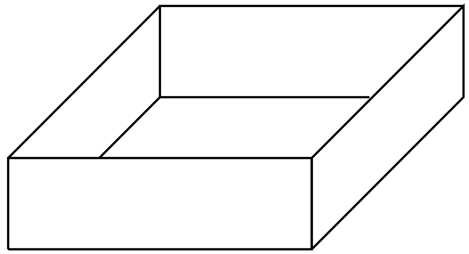
Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off if the volume of the box is to be maximized. What is the maximum volume?





Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off if the volume of the box is to be maximized. What is the maximum volume?

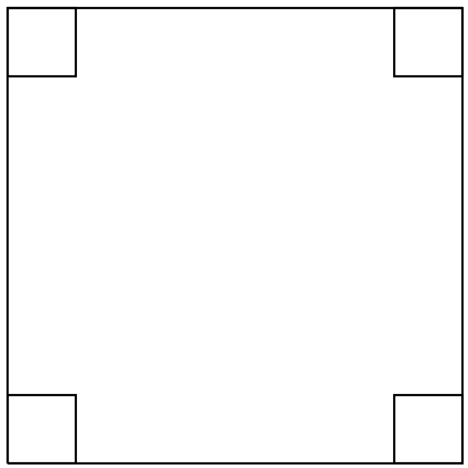


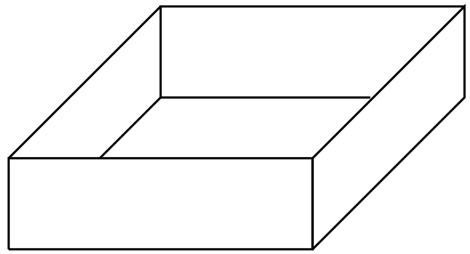


Analysis

- Suppose the side is x.

Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off if the volume of the box is to be maximized. What is the maximum volume?

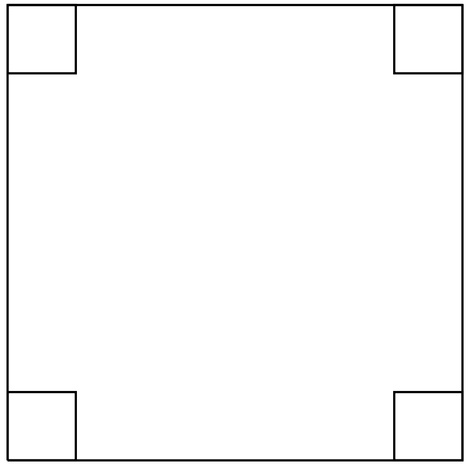


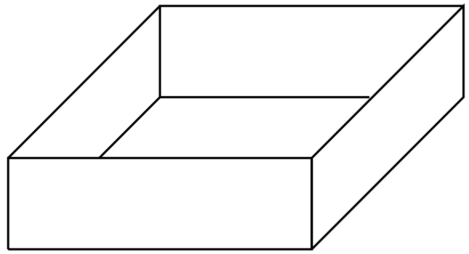


Analysis

- Suppose the side is x.
- Can express length, width and height in terms of x.

Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off if the volume of the box is to be maximized. What is the maximum volume?

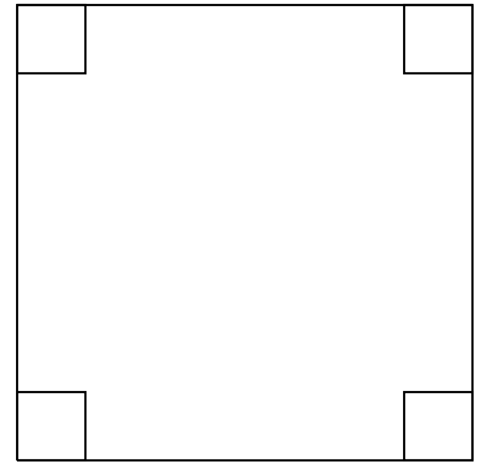


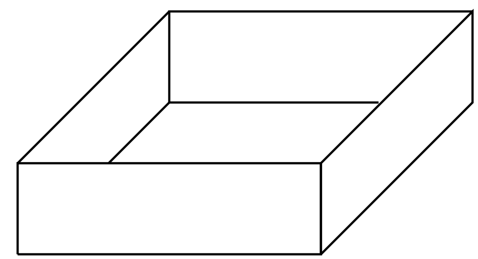


Analysis

- Suppose the side is x.
- Can express length, width and height in terms of x.
- Can express volume V in terms of x.

Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off if the volume of the box is to be maximized. What is the maximum volume?

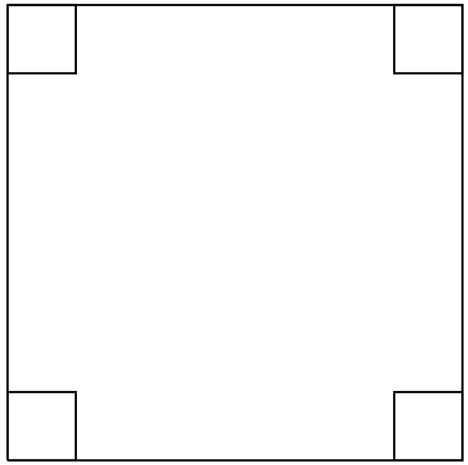




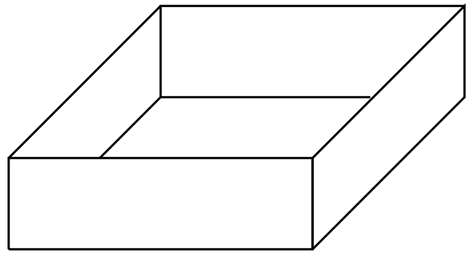
Analysis

- Suppose the side is x.
- Can express length, width and height in terms of x.
- Can express volume V in terms of x.
- There are restrictions on x *(domain of the function V)*

Example A rectangular box without lid is to be made from a square cardboard of sides $18\,\mathrm{cm}$ by cutting equal squares from each corner and then folding up the sides. Find the length of the side of the square that must be cut off if the volume of the box is to be maximized. What is the maximum volume?

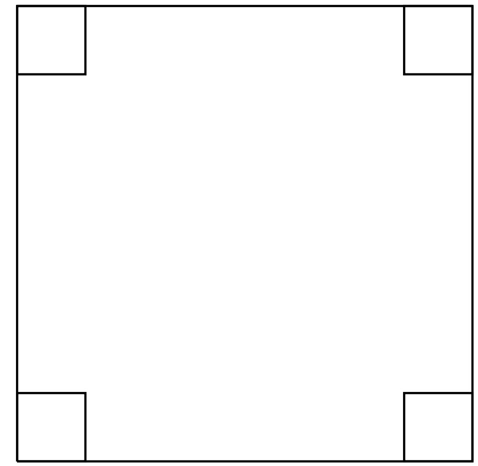


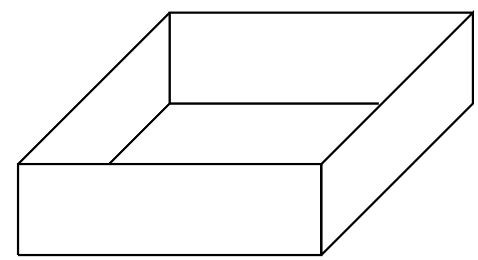
Analysis



- Suppose the side is x.
- Can express length, width and height in terms of x.
- Can express volume V in terms of x.
- There are restrictions on x *(domain of the function V)*
- *Maximize $V(x)$ for x belonging to the domain*.

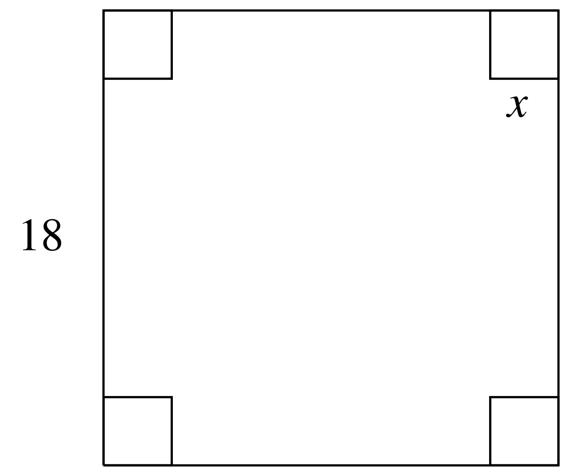
18

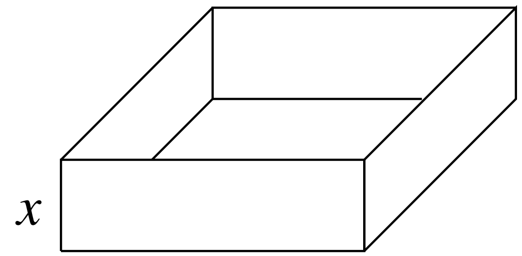




Solution

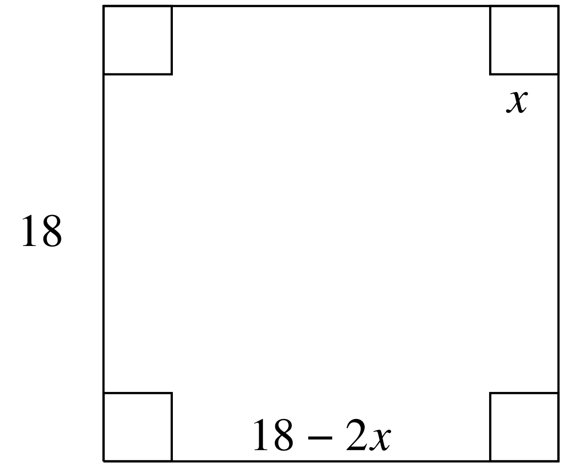
- Let length of side of the square to be cut off be x cm.

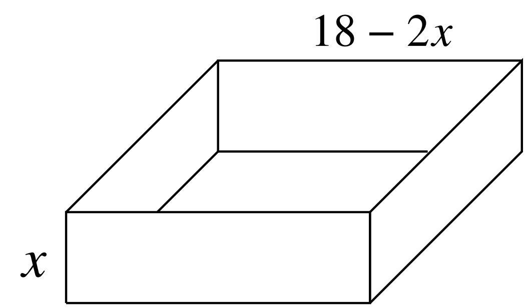




Solution

- Let length of side of the square to be cut off be x cm.

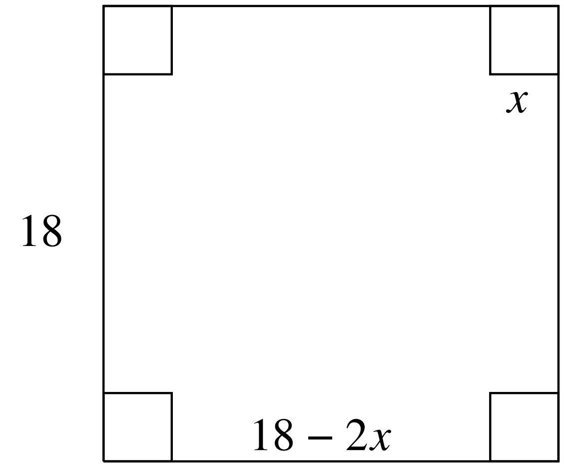


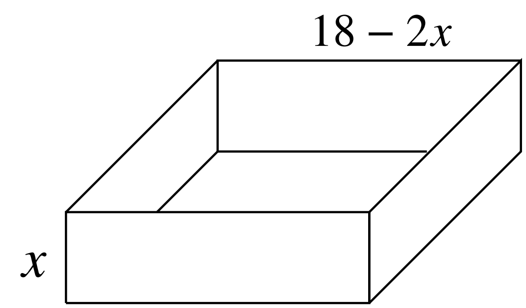


Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

$$V(x) \quad = \quad x(18-2x)^2, \qquad 0 < x < 9$$



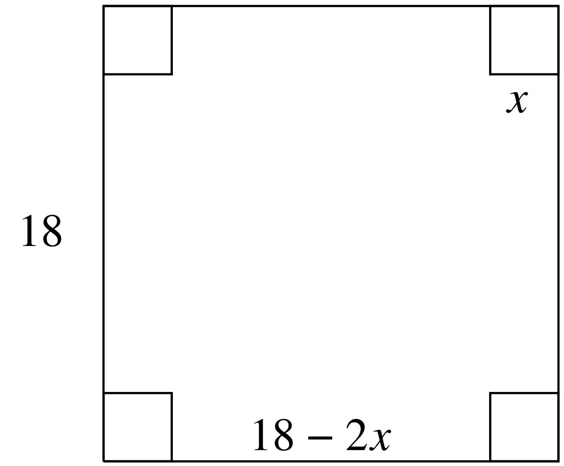


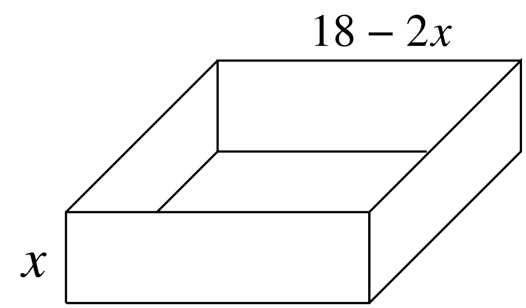
Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

$$V(x) = x(18 - 2x)^2, \qquad 0 < x < 9$$

$$= x(324 - 72x + 4x^2)$$

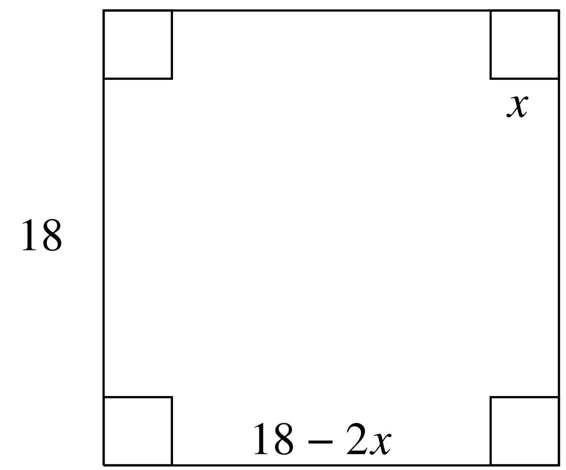


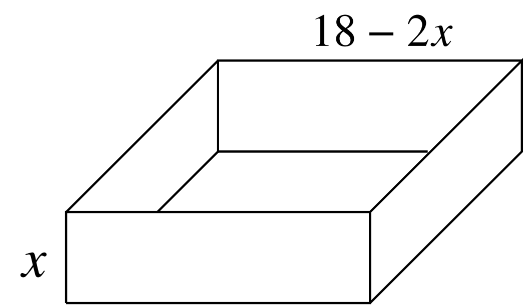


Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

$$V(x) = x(18 - 2x)^2, \qquad 0 < x < 9$$
$$= x(324 - 72x + 4x^2)$$
$$= 324x - 72x^2 + 4x^3$$





Solution

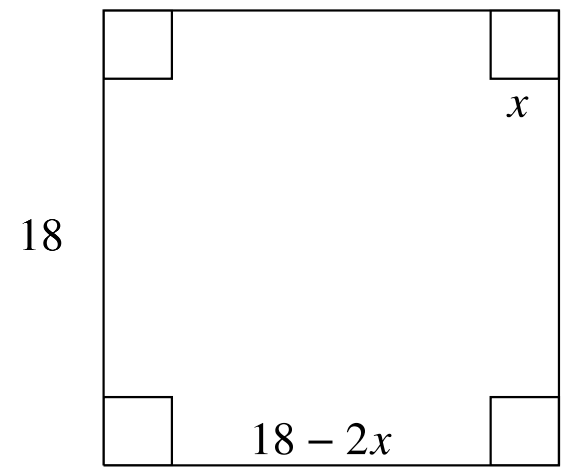
- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

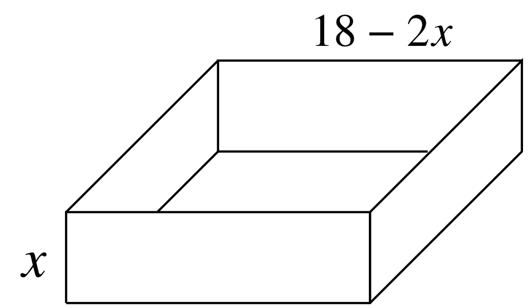
$$V(x) = x(18 - 2x)^2, \qquad 0 < x < 9$$

$$= x(324 - 72x + 4x^2)$$

$$= 324x - 72x^2 + 4x^3$$

- $V'(x) = 324 - 144x + 12x^2$





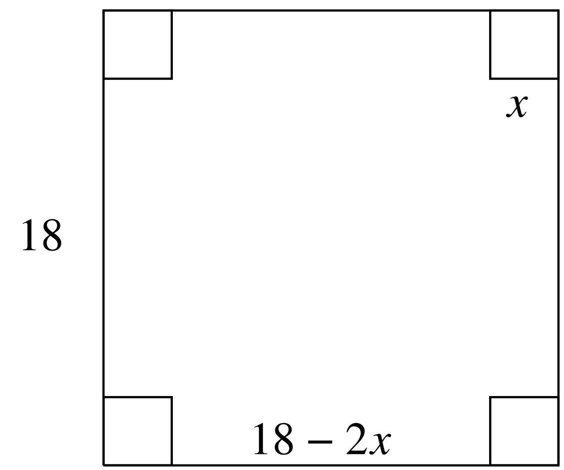
Solution

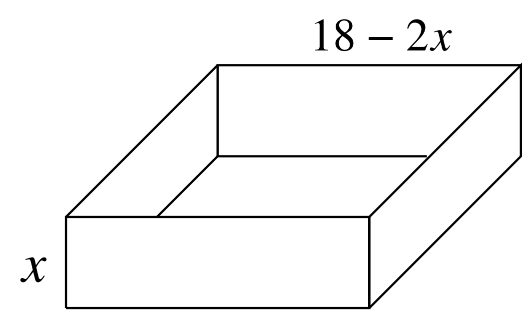
- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

$$V(x) = x(18-2x)^2, \qquad 0 < x < 9$$
$$= x(324 - 72x + 4x^2)$$
$$= 324x - 72x^2 + 4x^3$$

- $V'(x) = 324 - 144x + 12x^2$

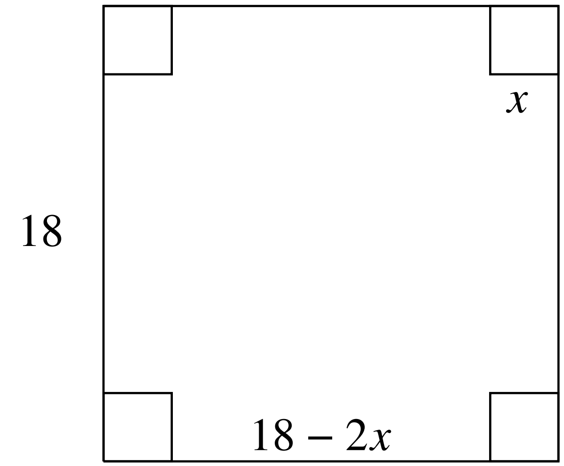
$$= 12(x^2 - 12x + 27) = 12(x-3)(x-9)$$



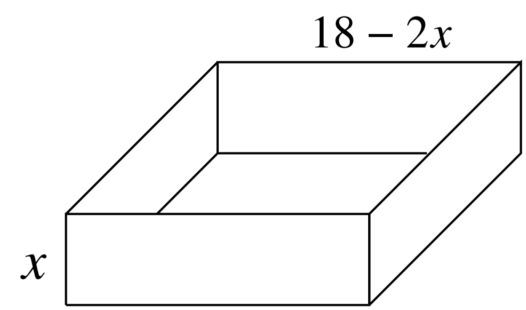


Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is



$$V(x) = x(18 - 2x)^2, \qquad 0 < x < 9$$
$$= x(324 - 72x + 4x^2)$$
$$= 324x - 72x^2 + 4x^3$$



- $$V'(x) = 324 - 144x + 12x^2$$
$$= 12(x^2 - 12x + 27) = 12(x - 3)(x - 9)$$

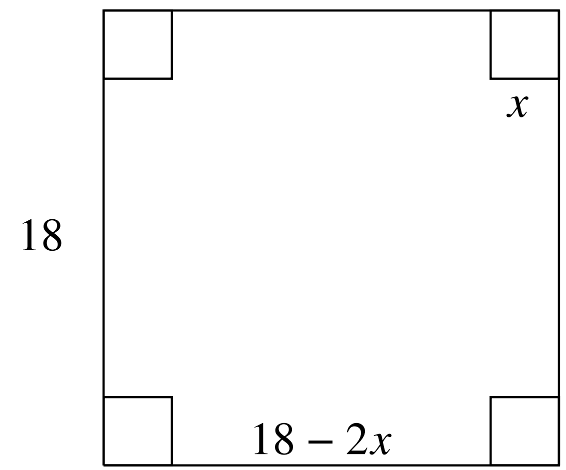
- *There is only one critical point in* $(0, 9)$ $x = 3$

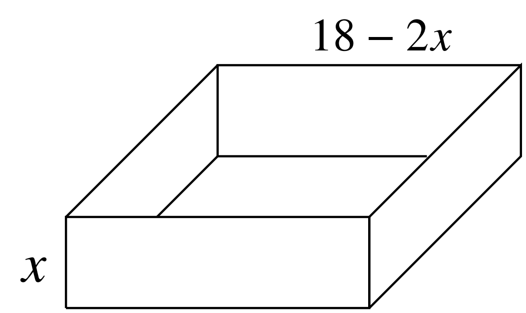
($x = 9$ is rejected; not belong to the domain of V)

Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

$$V(x) = x(18-2x)^2, \qquad 0 < x < 9$$
$$= x(324 - 72x + 4x^2)$$
$$= 324x - 72x^2 + 4x^3$$





- $V'(x) = 324 - 144x + 12x^2$

$$= 12(x^2 - 12x + 27) = 12(x-3)(x-9)$$

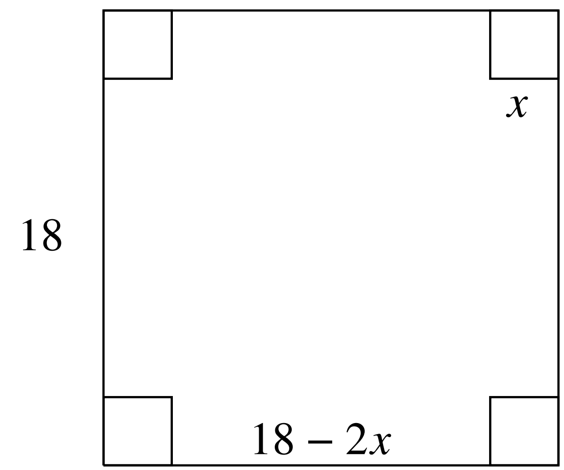
- *There is only one critical point in* $(0, 9)$ $x = 3$

  ($x = 9$ is rejected; not belong to the domain of V)

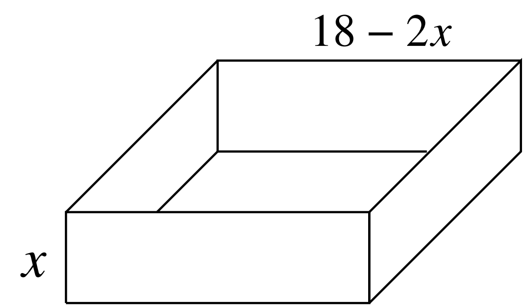
- $V''(x) = -144 + 24x$,

Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is



$$\begin{aligned} V(x) &= x(18-2x)^2, \qquad 0 < x < 9 \\ &= x(324 - 72x + 4x^2) \\ &= 324x - 72x^2 + 4x^3 \end{aligned}$$

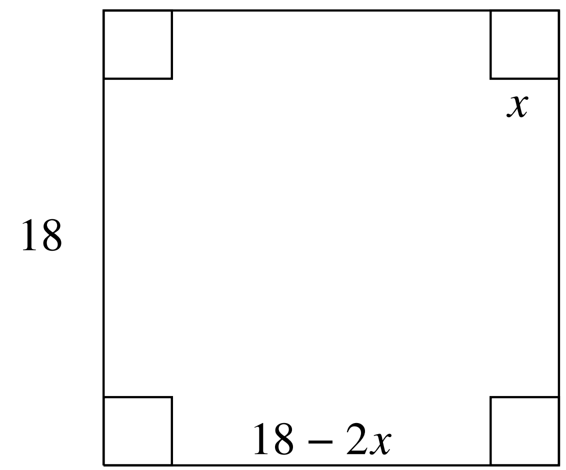


- $$\begin{aligned} V'(x) &= 324 - 144x + 12x^2 \\ &= 12(x^2 - 12x + 27) = 12(x-3)(x-9) \end{aligned}$$
- *There is only one critical point in* $(0, 9)$ $x = 3$

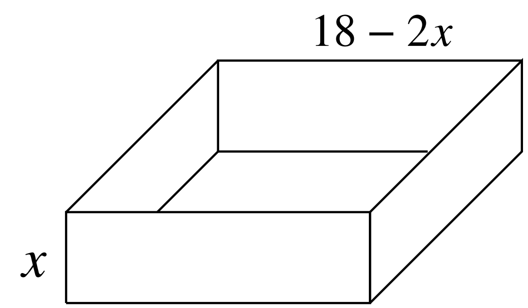
  ($x = 9$ is rejected; not belong to the domain of V)
- $V''(x) = -144 + 24x, \quad V''(3) = -72$

Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is



$$V(x) = x(18-2x)^2, \qquad 0 < x < 9$$
$$= x(324 - 72x + 4x^2)$$
$$= 324x - 72x^2 + 4x^3$$

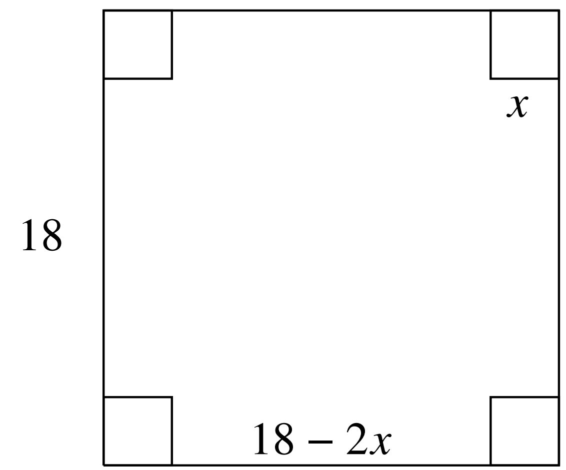


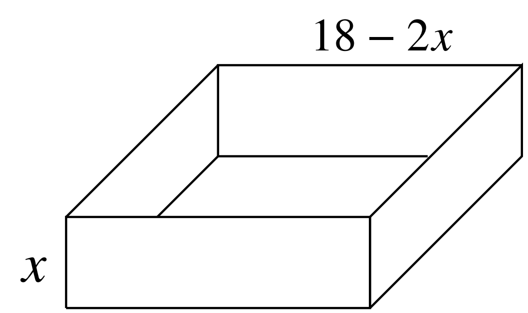
- $V'(x) = 324 - 144x + 12x^2$
  $= 12(x^2 - 12x + 27) = 12(x-3)(x-9)$
- *There is only one critical point in* $(0, 9)$ $x = 3$
  ($x = 9$ is rejected; not belong to the domain of V)
- $V''(x) = -144 + 24x$, $\quad V''(3) = -72 < 0$

Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

$$\begin{aligned} V(x) &= x(18-2x)^2, \qquad 0 < x < 9 \\ &= x(324 - 72x + 4x^2) \\ &= 324x - 72x^2 + 4x^3 \end{aligned}$$





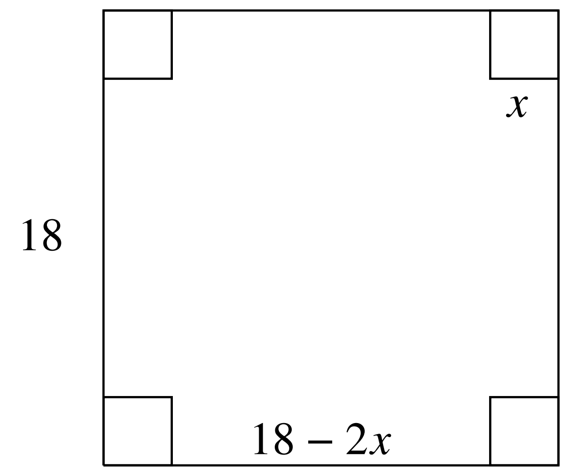
- $$\begin{aligned} V'(x) &= 324 - 144x + 12x^2 \\ &= 12(x^2 - 12x + 27) = 12(x-3)(x-9) \end{aligned}$$
- *There is only one critical point in* $(0, 9)$ $x = 3$

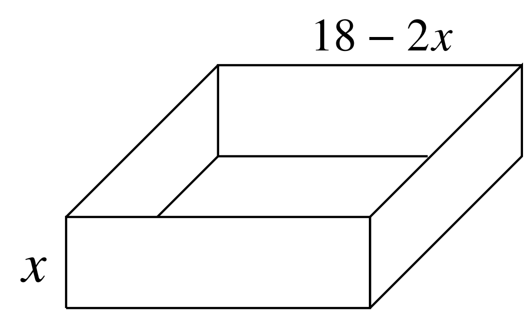
  ($x = 9$ is rejected; not belong to the domain of V)
- $V''(x) = -144 + 24x$, $\quad V''(3) = -72 < 0$
- By 2nd deriv. test (sp ver), V has max at $x = 3$.

Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

$$V(x) = x(18-2x)^2, \qquad 0 < x < 9$$
$$= x(324 - 72x + 4x^2)$$
$$= 324x - 72x^2 + 4x^3$$





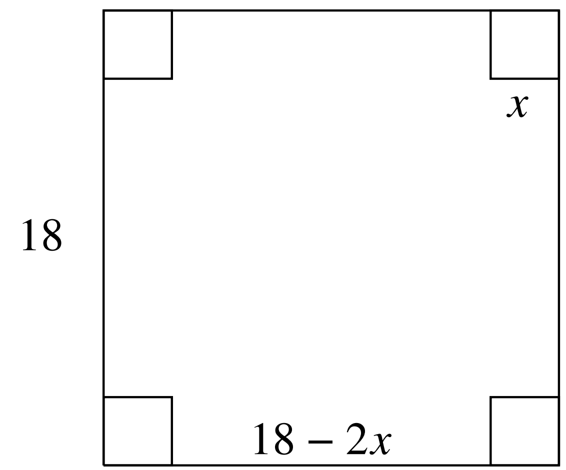
- $$V'(x) = 324 - 144x + 12x^2$$
$$= 12(x^2 - 12x + 27) = 12(x-3)(x-9)$$
- *There is only one critical point in* $(0,9)$ $x = 3$

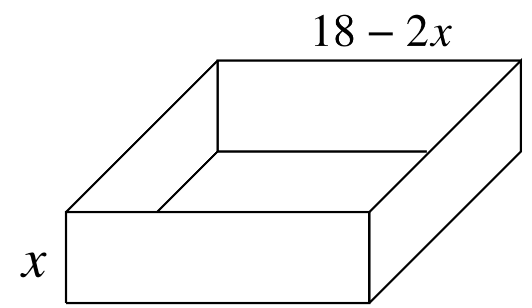
  ($x = 9$ is rejected; not belong to the domain of V)
- $V''(x) = -144 + 24x$, $V''(3) = -72 < 0$
- By 2nd deriv. test (sp ver), V has max at $x = 3$.
- Maximum volume is $V(3) = 432\ \text{cm}^3$.

Solution

- Let length of side of the square to be cut off be x cm.
- The volume V, in cm^3, of the open box is

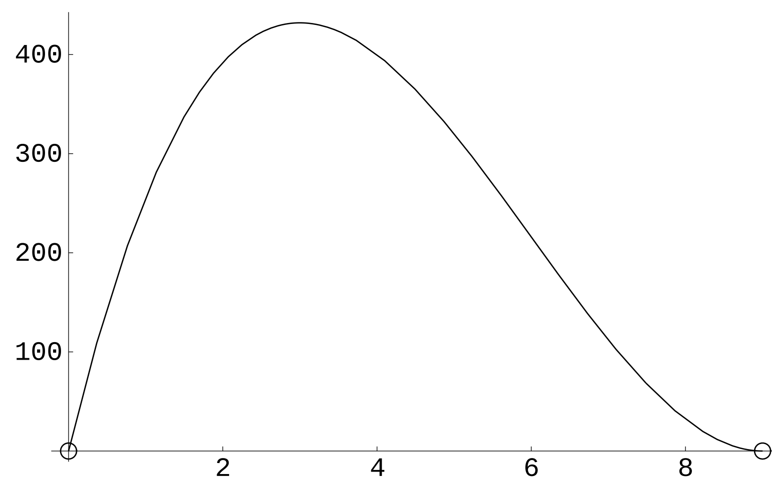
$$\begin{aligned} V(x) &= x(18-2x)^2, \qquad 0 < x < 9 \\ &= x(324 - 72x + 4x^2) \\ &= 324x - 72x^2 + 4x^3 \end{aligned}$$





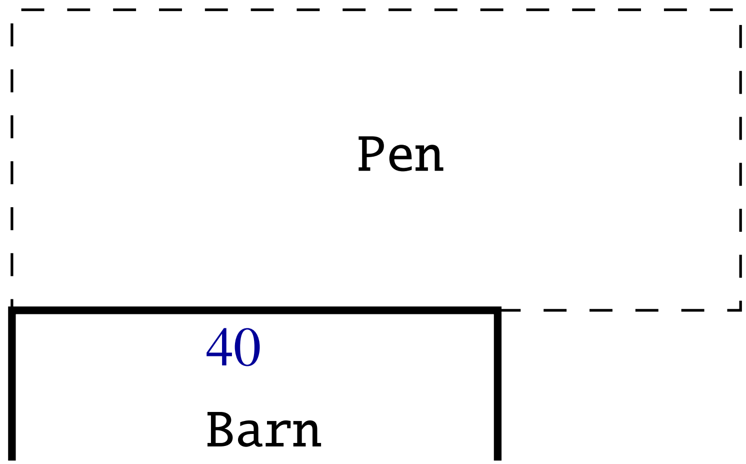
- $$\begin{aligned} V'(x) &= 324 - 144x + 12x^2 \\ &= 12(x^2 - 12x + 27) = 12(x-3)(x-9) \end{aligned}$$
- *There is only one critical point in* $(0, 9)$ $x = 3$

  ($x = 9$ is rejected; not belong to the domain of V)
- $V''(x) = -144 + 24x$, $V''(3) = -72 < 0$
- By 2nd deriv. test (sp ver), V has max at $x = 3$.
- Maximum volume is $V(3) = 432\,\text{cm}^3$.

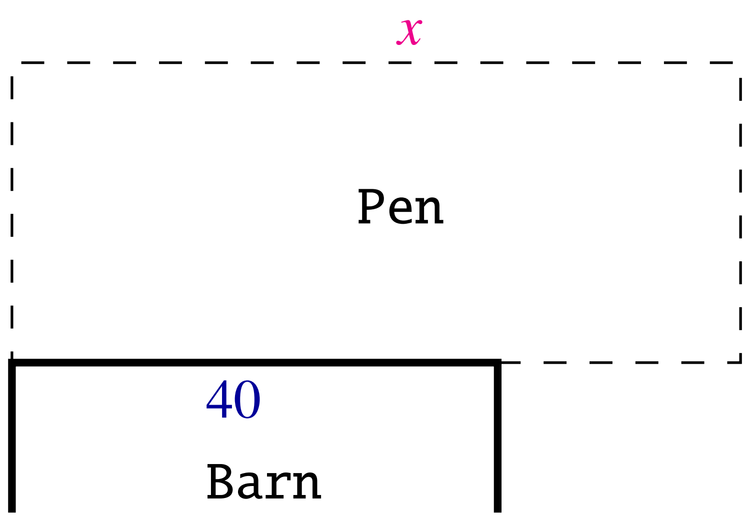


Example A farmer wants to construct a rectangular pen next to a barn $40\,m$ long, using all of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build if $200\,m$ of fencing material is available.

Example A farmer wants to construct a rectangular pen next to a barn $40\,m$ long, using all of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build if $200\,m$ of fencing material is available.

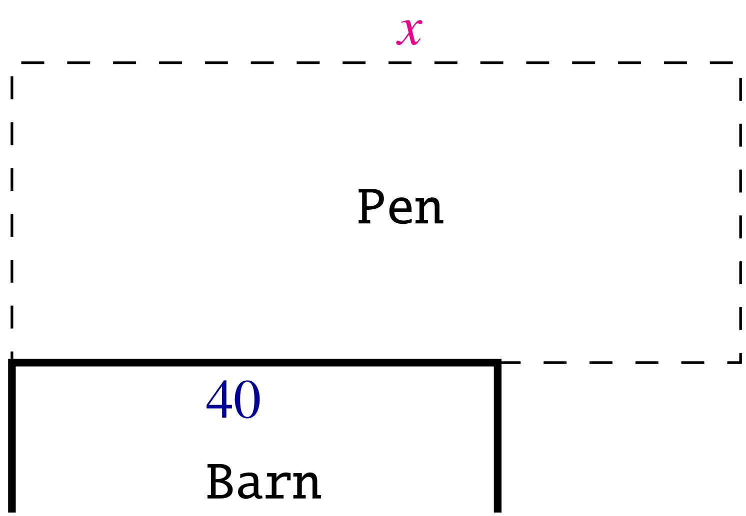


Example A farmer wants to construct a rectangular pen next to a barn $40\,m$ long, using all of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build if $200\,m$ of fencing material is available.



- Let x be length of side (in meters) of pen that is parallel to barn

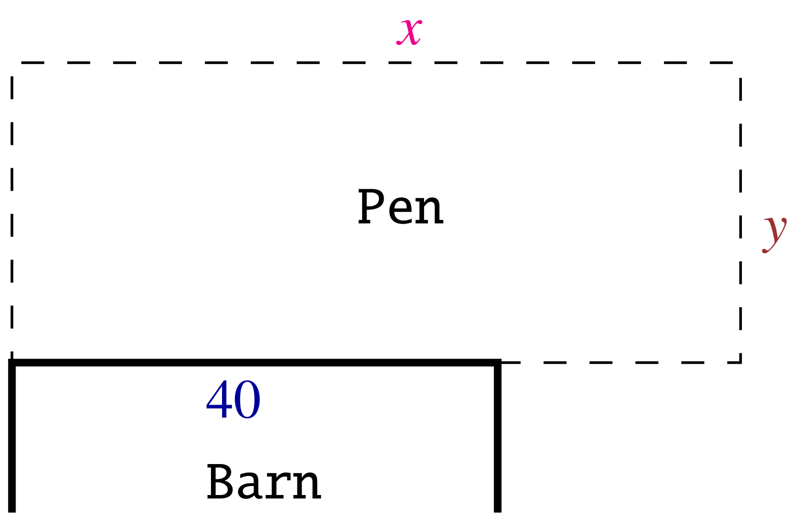
Example A farmer wants to construct a rectangular pen next to a barn $40\,m$ long, using all of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build if $200\,m$ of fencing material is available.



- Let x be length of side (in meters) of pen that is parallel to barn

  Note that $x \geq 40$

Example A farmer wants to construct a rectangular pen next to a barn $40\,m$ long, using all of the barn as part of one side of the pen. Find the dimensions of the pen with the largest area that the farmer can build if $200\,m$ of fencing material is available.



- Let x be length of side (in meters) of pen that is parallel to barn

  Note that $x \geq 40$

- Let y be length of side $\perp$ to barn.

Denote the area of the pen by A

Denote the area of the pen by A

Note that $A = xy$

Denote the area of the pen by A

Note that $A = xy$

Moreover, $x + (x - 40) + 2y \quad = \quad 200$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \end{aligned}$$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \\ y &= 120 - x \end{aligned}$$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \\ y &= 120 - x \end{aligned}$$

Therefore, $A = x(120 - x)$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \\ y &= 120 - x \end{aligned}$$

Therefore, $A = x(120 - x) \qquad 40 \le x < 120$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \\ y &= 120 - x \end{aligned}$$

Therefore,

$$\begin{aligned} A &= x(120 - x) \qquad 40 \le x < 120 \\ &= 120x - x^2 \end{aligned}$$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \\ y &= 120 - x \end{aligned}$$

Therefore,

$$\begin{aligned} A &= x(120 - x) \qquad 40 \leq x < 120 \\ &= 120x - x^2 \end{aligned}$$

Differentiating $A'(x) = 120 - 2x$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$
\begin{aligned}
x + (x - 40) + 2y &= 200 \\
2x + 2y &= 200 + 40 \\
y &= 120 - x
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A &= x(120 - x) \qquad 40 \leq x < 120 \\
&= 120x - x^2
\end{aligned}
$$

Differentiating

$$
\begin{aligned}
A'(x) &= 120 - 2x \\
&= 2(60 - x)
\end{aligned}
$$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \\ y &= 120 - x \end{aligned}$$

Therefore,

$$\begin{aligned} A &= x(120 - x) \qquad 40 \leq x < 120 \\ &= 120x - x^2 \end{aligned}$$

Differentiating

$$\begin{aligned} A'(x) &= 120 - 2x \\ &= 2(60 - x) \end{aligned}$$

Critical point of A is $x_1 = 60$

Denote the area of the pen by A

Note that $A = xy$

Moreover,

$$\begin{aligned} x + (x - 40) + 2y &= 200 \\ 2x + 2y &= 200 + 40 \\ y &= 120 - x \end{aligned}$$

Therefore,

$$\begin{aligned} A &= x(120 - x) \qquad 40 \leq x < 120 \\ &= 120x - x^2 \end{aligned}$$

Differentiating

$$\begin{aligned} A'(x) &= 120 - 2x \\ &= 2(60 - x) \end{aligned}$$

Critical point of A is $x_1 = 60$

Remark Have to check whether A has max at x_1

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$		0	
A			

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	
A			

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A			

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A	$\nearrow$		

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A	$\nearrow$		$\searrow$

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A	$\nearrow$		$\searrow$

Method 2

$$A''(x) = -2$$

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A	$\nearrow$		$\searrow$

Method 2 $A''(x) = -2$

$A''(60) < 0$ and 60 is the only critical pt

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A	$\nearrow$		$\searrow$

Method 2 $A''(x) = -2$

$A''(60) < 0$ and 60 is the only critical pt

Therefore, A *has global maximum when* $x = 60$

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A	$\nearrow$		$\searrow$

Method 2 $A''(x) = -2$

$A''(60) < 0$ and 60 is the only critical pt

Therefore, A *has global maximum when* $x = 60$

When $x = 60$, $y = 60$

Method 1

x	$(40, 60)$	60	$(60, 120)$
$A'(x)$	$+$	0	$-$
A	$\nearrow$		$\searrow$

Method 2 $A''(x) = -2$

$A''(60) < 0$ and 60 is the only critical pt

Therefore, A *has global maximum when* $x = 60$

When $x = 60$, $y = 60$

The dimensions of the largest pen are $60\,m \times 60\,m$

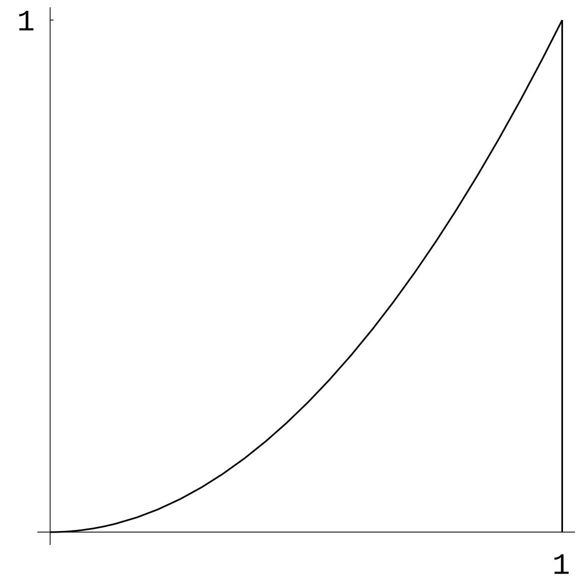
Chapter 6: Integration

- Definite Integrals
- Indefinite Integrals
- Applications of Definite Integrals

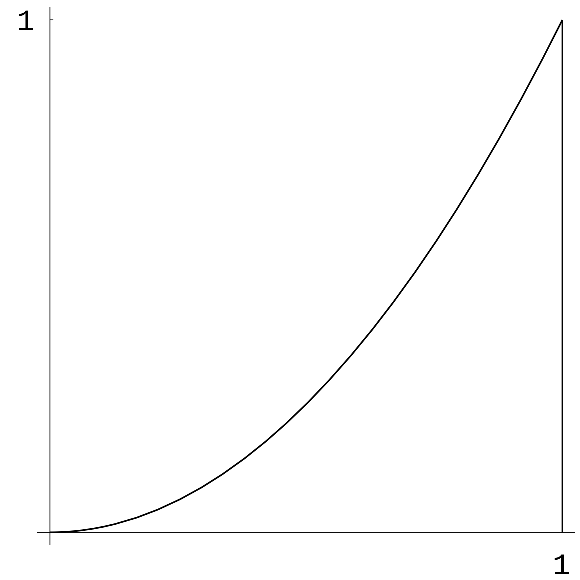
Objectives

- To evaluate definite integrals using Fundamental Theorem of Integral Calculus.
- To find indefinite integrals by formula.
- To find area of “simple” regions using definite integrals.

Problem Find the area of the region bounded by the curve $y = x^2$, the x-axis and the vertical line $x = 1$.



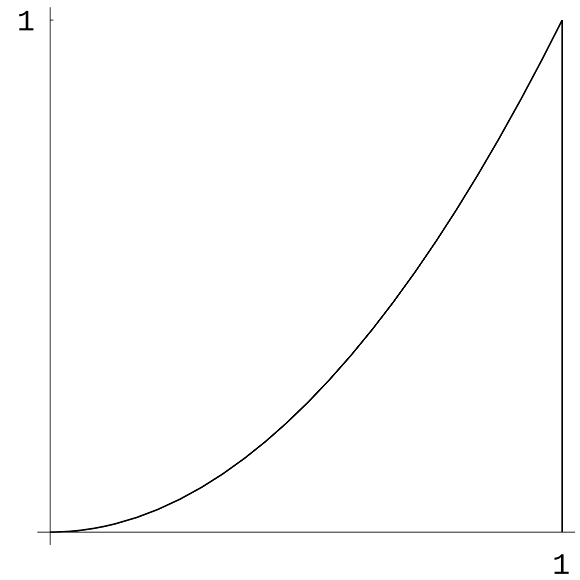
Problem Find the area of the region bounded by the curve $y = x^2$, the x-axis and the vertical line $x = 1$.



Method Divide $[0, 1]$ into n equal subintervals.

$$[0, \tfrac{1}{n}], \quad [\tfrac{1}{n}, \tfrac{2}{n}], \quad , \ldots , \quad [\tfrac{n-1}{n}, 1]$$

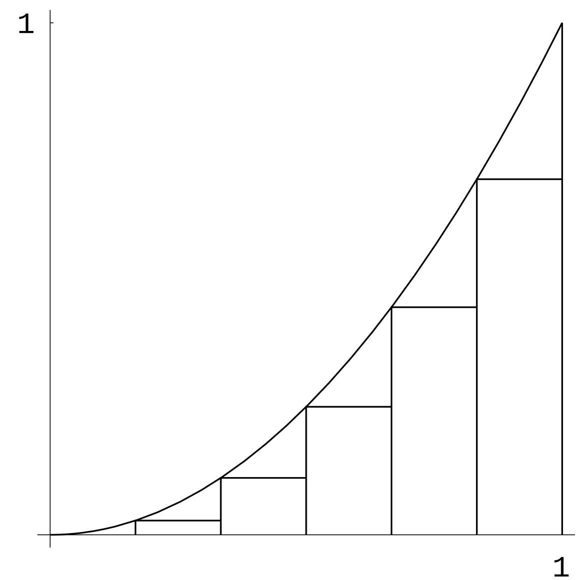
Problem Find the area of the region bounded by the curve $y = x^2$, the x-axis and the vertical line $x = 1$.



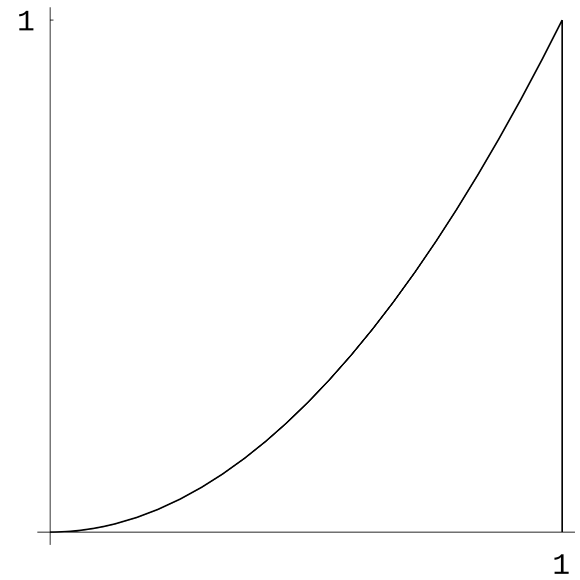
Method Divide $[0, 1]$ into n equal subintervals.

$$[0, \tfrac{1}{n}], \quad [\tfrac{1}{n}, \tfrac{2}{n}], \quad , \ldots , \quad [\tfrac{n-1}{n}, 1]$$

- On each subinterval, consider the largest rectangular regions inside the given region, *height of the i-th region is $f(\frac{i-1}{n})$*.



Problem Find the area of the region bounded by the curve $y = x^2$, the x-axis and the vertical line $x = 1$.



Method Divide $[0, 1]$ into n equal subintervals.

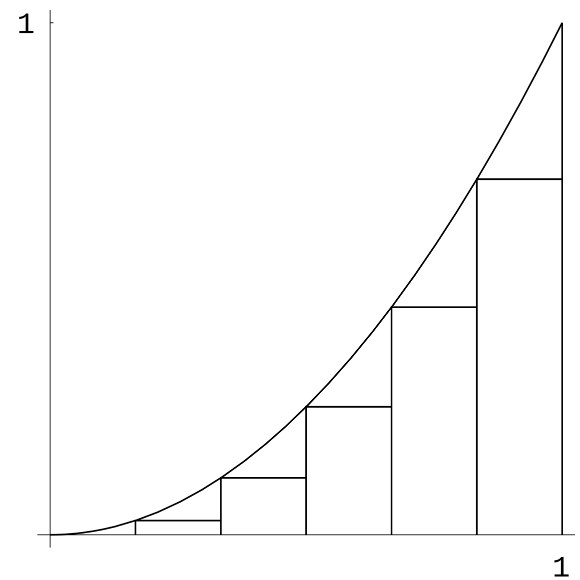
$$[0, \tfrac{1}{n}], \quad [\tfrac{1}{n}, \tfrac{2}{n}], \quad , \ldots , \quad [\tfrac{n-1}{n}, 1]$$

- On each subinterval, consider the largest rectangular regions inside the given region, *height of the i-th region is $f(\frac{i-1}{n})$*.

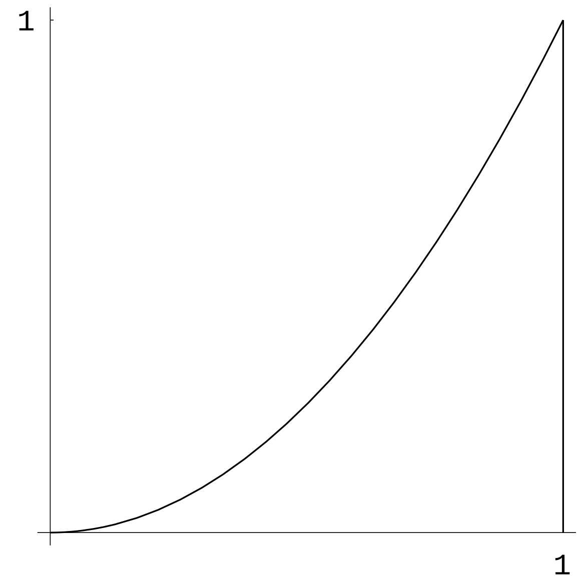
- Sum of areas of rectangular regions is

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

- Area of given region is $1/3$.



Problem Find the area of the region bounded by the curve $y = x^2$, the x-axis and the vertical line $x = 1$.



Method Divide $[0, 1]$ into n equal subintervals.

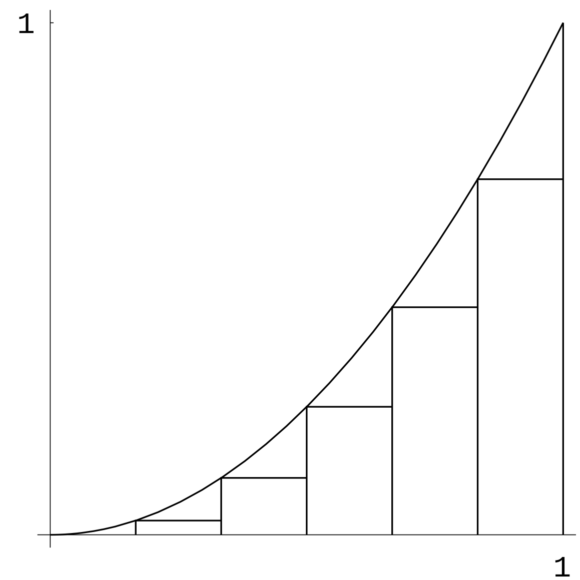
$$[0, \tfrac{1}{n}], \quad [\tfrac{1}{n}, \tfrac{2}{n}], \quad , \ldots , \quad [\tfrac{n-1}{n}, 1]$$

- On each subinterval, consider the largest rectangular regions inside the given region, *height of the i-th region is* $f(\frac{i-1}{n})$.
- Sum of areas of rectangular regions is

$$\frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}$$

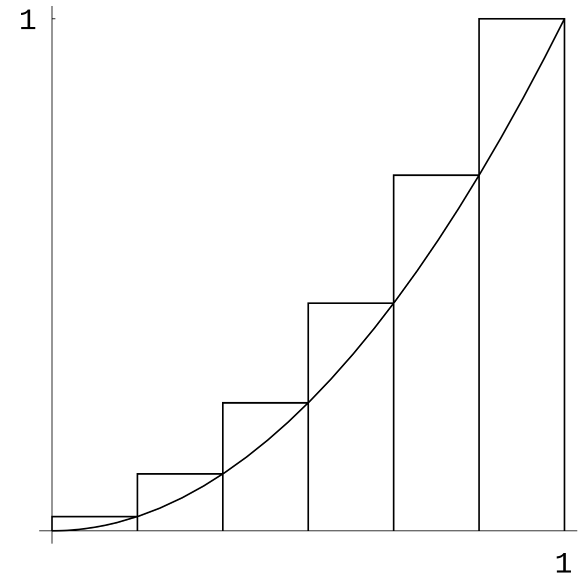
- Area of given region is $1/3$.

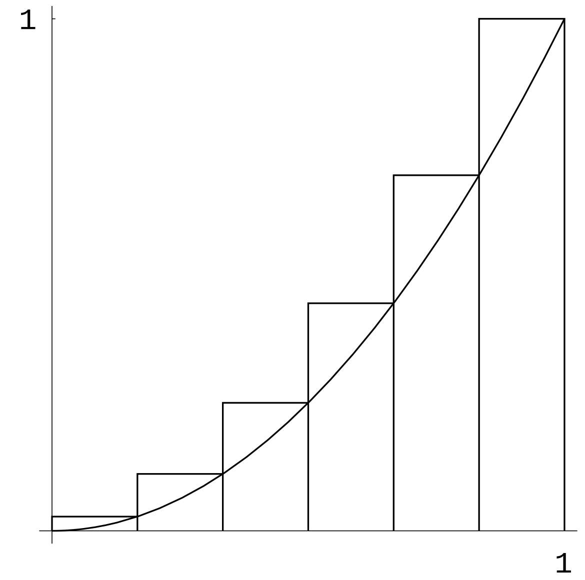
Approximation from below



Question Can we

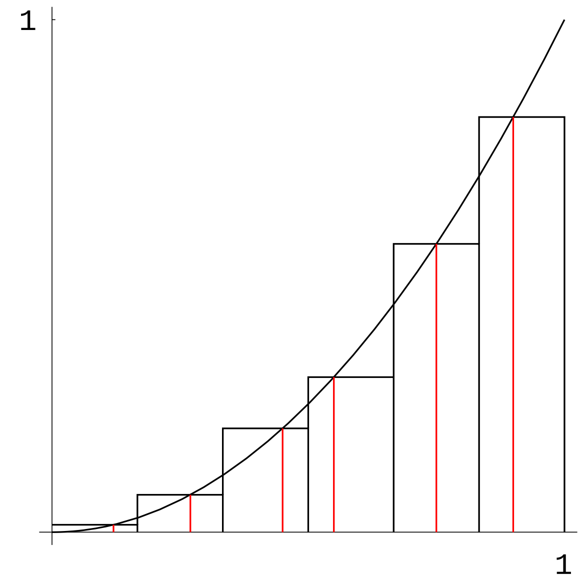
- use approximation from above ?





Question Can we

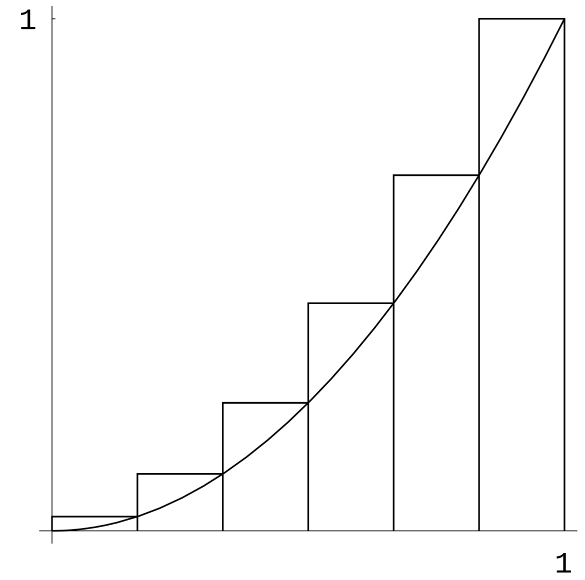
- use approximation from above ?



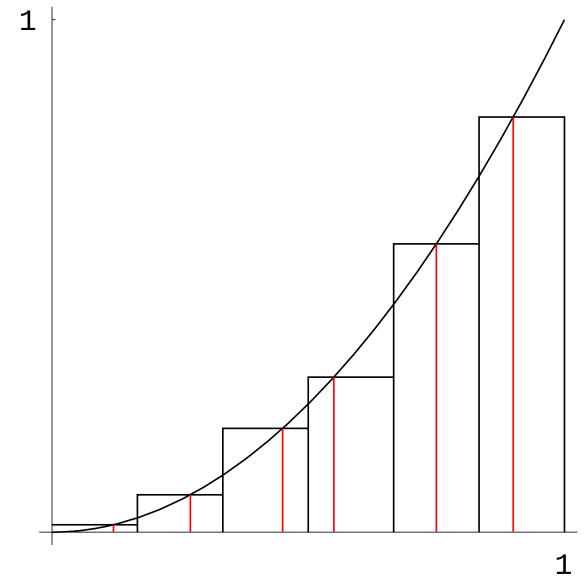
- take an arbitrary point t_i in each subinterval and use $f(t_i)$ as height ?

Question Can we

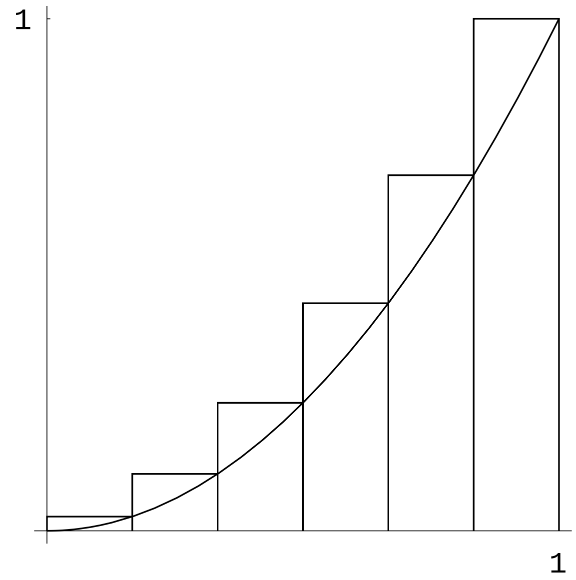
- use approximation from above ?



- take an arbitrary point t_i in each subinterval and use $f(t_i)$ as height ?

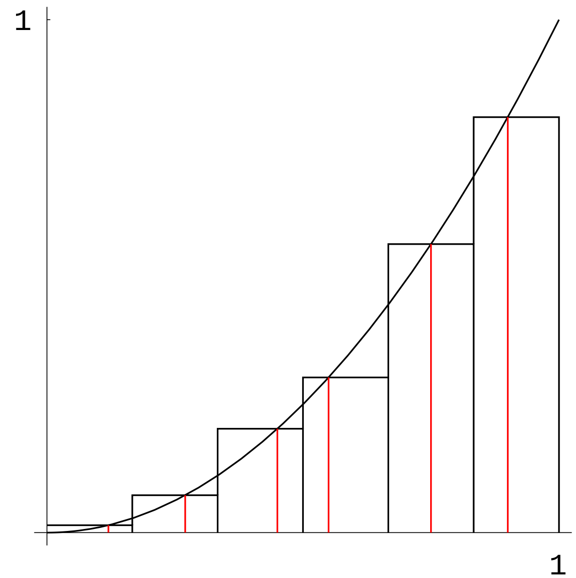


- use unequal subintervals ?



Question Can we

- use approximation from above ?



- take an arbitrary point t_i in each subinterval and use $f(t_i)$ as height ?

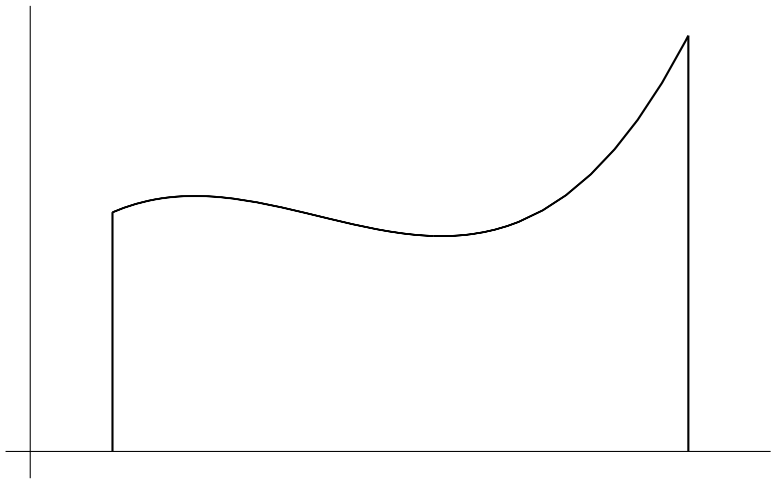
- use unequal subintervals ?

Remark Can replace

- $f(x) = x^2$ by any *continuous function*

- $[0, 1]$ by any *closed interval*.

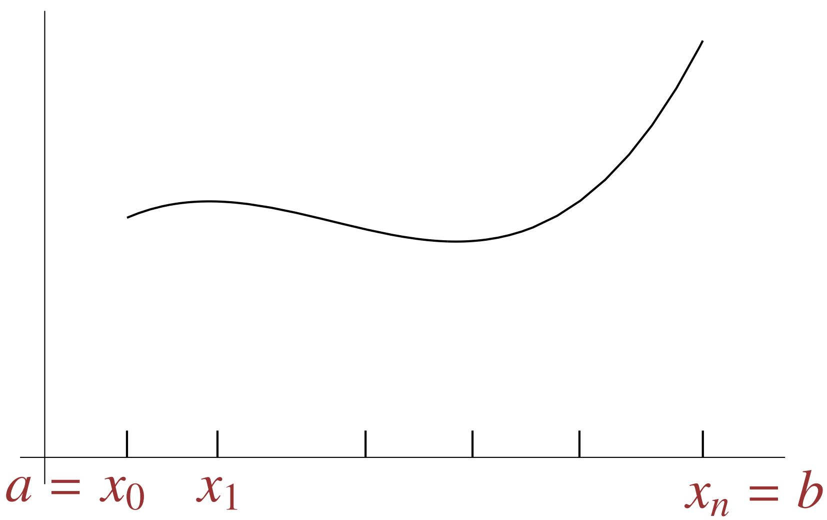
Definition Let f be defined and *continuous* on a closed interval $[a, b]$.



Definition Let f be defined and *continuous* on a closed interval $[a, b]$.

- Take finitely many points in $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$



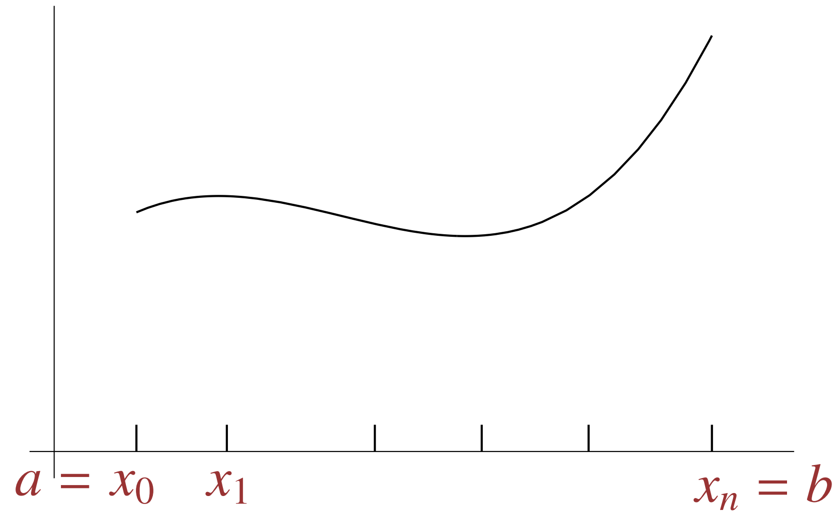
Definition Let f be defined and *continuous* on a closed interval $[a, b]$.

- Take finitely many points in $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

- These points divide $[a, b]$ into subintervals

$$[x_0, x_1]\ ,\ [x_1, x_2]\ ,\ \cdots\ ,\ [x_{n-1}, x_n]$$



Definition Let f be defined and *continuous* on a closed interval $[a, b]$.

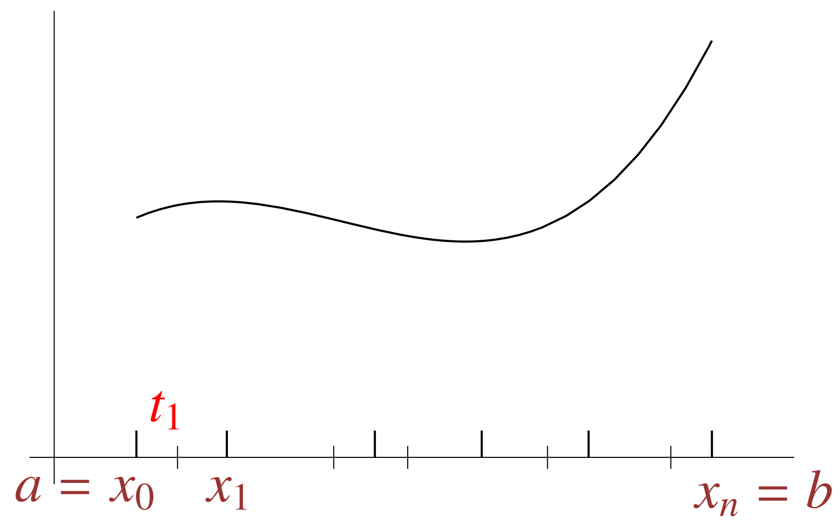
- Take finitely many points in $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

- These points divide $[a, b]$ into subintervals

$$[x_0, x_1] \; , \; [x_1, x_2] \; , \; \cdots \; , \; [x_{n-1}, x_n]$$

- For each $i = 1, \ldots, n$, take an arbitrary point t_i in the subinterval $[x_{i-1}, x_i]$.



Definition Let f be defined and *continuous* on a closed interval $[a, b]$.

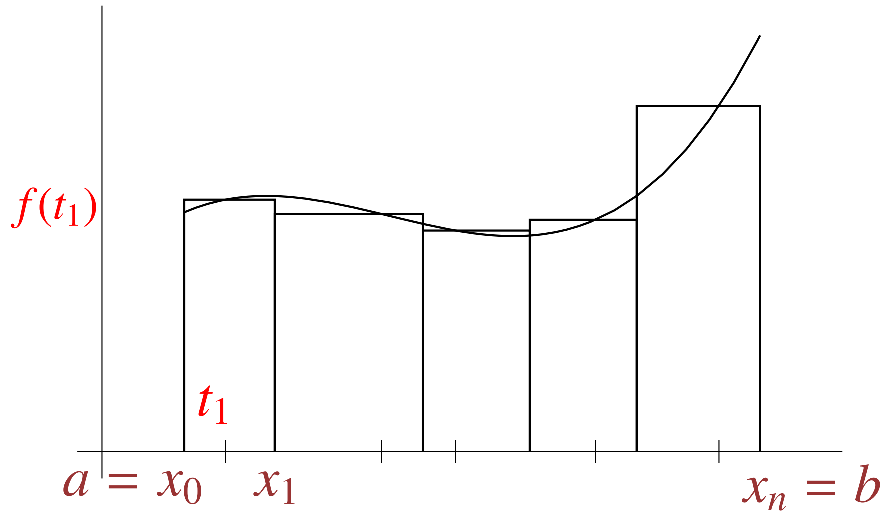
- Take finitely many points in $[a, b]$:

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

- These points divide $[a, b]$ into subintervals

$$[x_0, x_1]\ ,\ [x_1, x_2]\ ,\ \cdots\ ,\ [x_{n-1}, x_n]$$

- For each $i = 1, \ldots, n$, take an arbitrary point t_i in the subinterval $[x_{i-1}, x_i]$.



- Consider the sum

$$\sum_{i=1}^{n} f(t_i)\Delta x_i := f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \cdots + f(t_n)\Delta x_n$$

where $\Delta x_i = x_i - x_{i-1}$ is the length of the i-th subinterval.

- Can show that there exists a real number I *(which is unique)* such that

$$(*) \qquad \sum_{i=1}^{n} f(t_i)\Delta x_i \text{ is close to } I \text{ whenever } \Delta x_1, \ldots, \Delta x_n \text{ are small enough.}$$

- Can show that there exists a real number I *(which is unique)* such that

$$(*) \qquad \sum_{i=1}^{n} f(t_i)\Delta x_i \text{ is close to } I \text{ whenever } \Delta x_1, \ldots, \Delta x_n \text{ are small enough.}$$

The number I is called the *definite integral* of f from a to b, denoted by

$$\int_a^b f(x)\,\mathrm{d}x$$

- Can show that there exists a real number I *(which is unique)* such that

  $(*)\quad \sum_{i=1}^{n} f(t_i)\Delta x_i$ is close to I whenever $\Delta x_1, \ldots, \Delta x_n$ are small enough.

  The number I is called the *definite integral* of f from a to b, denoted by

$$\int_a^b f(x)\,\mathrm{d}x$$

b = upper limit of integration

a = lower limit

- Can show that there exists a real number I *(which is unique)* such that

$$(*) \quad \sum_{i=1}^{n} f(t_i)\Delta x_i \text{ is close to } I \text{ whenever } \Delta x_1, \ldots, \Delta x_n \text{ are small enough.}$$

The number I is called the *definite integral* of f from a to b, denoted by

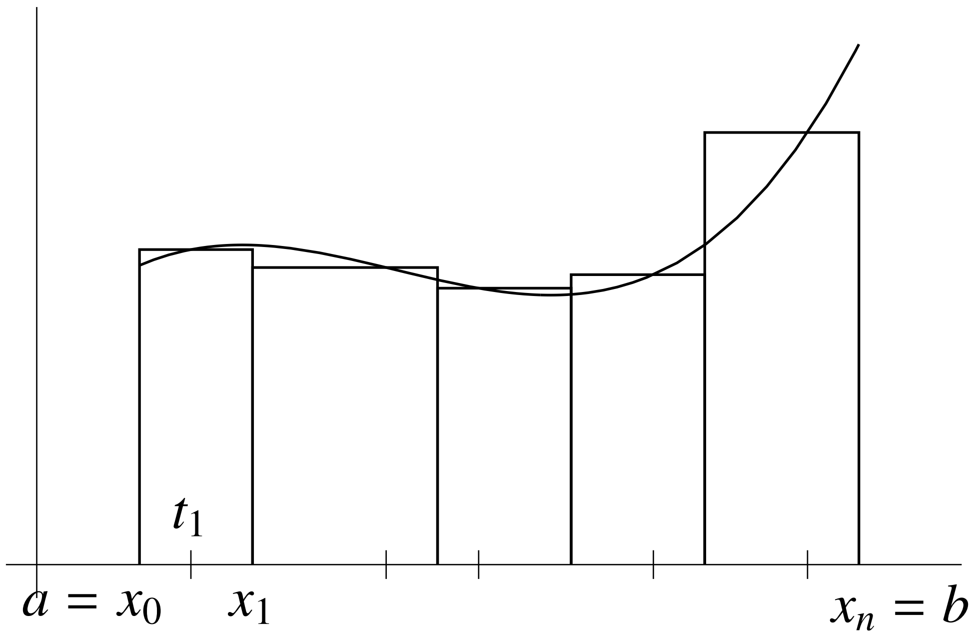
$$\int_a^b f(x)\,\mathrm{d}x$$

b = upper limit of integration

a = lower limit

Remark

- If f is non-negative, $\int_a^b f(x)\,\mathrm{d}x$ is the *area under graph of* f *for* $a \le x \le b$.



- Can show that there exists a real number I *(which is unique)* such that

$$(*) \qquad \sum_{i=1}^{n} f(t_i)\Delta x_i \text{ is close to } I \text{ whenever } \Delta x_1, \ldots, \Delta x_n \text{ are small enough.}$$

The number I is called the *definite integral* of f from a to b, denoted by

$$\int_a^b f(x)\,\mathrm{d}x$$

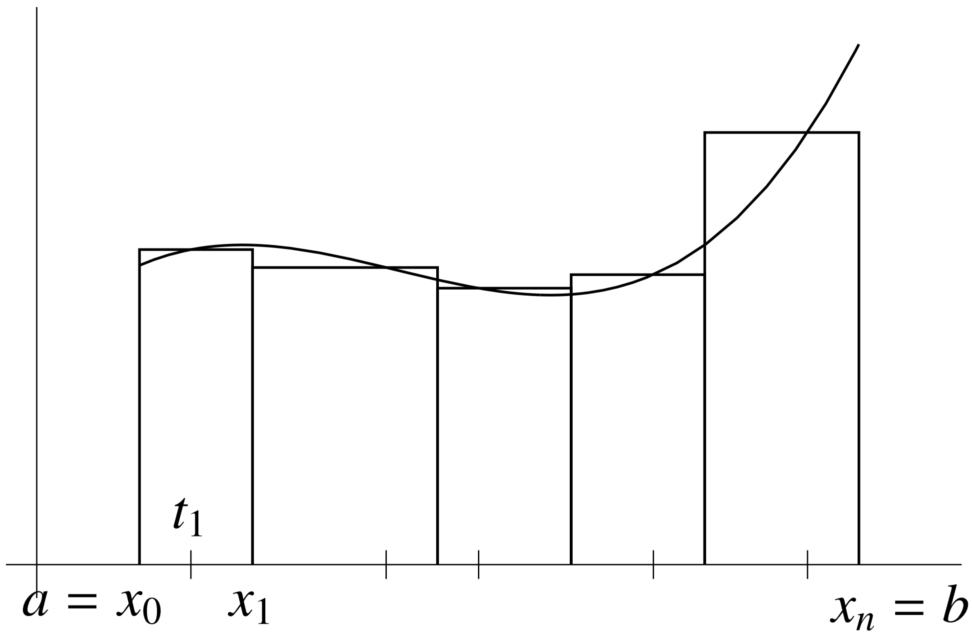
b = upper limit of integration

a = lower limit

Remark

- If f is non-negative, $\int_a^b f(x)\,\mathrm{d}x$ is the *area under graph of f for $a \le x \le b$*.

- If f is somewhere negative, definite integral does not mean area.



Example $\int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}$ *(can be found easily using Fundamental Theorem)*

Example $\int_0^1 x^2 \, dx = \frac{1}{3}$ *(can be found easily using Fundamental Theorem)*

Notation

- Can use other symbols, say t, instead of x, *eg.* $\int_a^b f(t) \, dt$

Example $\displaystyle\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$ *(can be found easily using Fundamental Theorem)*

Notation

- Can use other symbols, say t, instead of x, *eg.* $\displaystyle\int_a^b f(t)\,\mathrm{d}t$

For convenience, we define

- $\displaystyle\int_a^a f(x)\,\mathrm{d}x = 0$

Example $\int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}$ *(can be found easily using Fundamental Theorem)*

Notation

- Can use other symbols, say t, instead of x, *eg.* $\int_a^b f(t) \, \mathrm{d}t$

For convenience, we define

- $\int_a^a f(x) \, \mathrm{d}x = 0$
- $\int_b^a f(x) \, \mathrm{d}x = -\int_a^b f(x) \, \mathrm{d}x$

Example $\int_0^1 x^2 \, \mathrm{d}x = \frac{1}{3}$ *(can be found easily using Fundamental Theorem)*

Notation

- Can use other symbols, say t, instead of x, *eg.* $\int_a^b f(t) \, \mathrm{d}t$

For convenience, we define

- $\int_a^a f(x) \, \mathrm{d}x = 0$
- $\int_b^a f(x) \, \mathrm{d}x = -\int_a^b f(x) \, \mathrm{d}x$

Example $\int_1^0 x^2 \, \mathrm{d}x = -\int_0^1 x^2 \, \mathrm{d}x$

Example $\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}$ *(can be found easily using Fundamental Theorem)*

Notation

- Can use other symbols, say t, instead of x, *eg.* $\int_a^b f(t)\,\mathrm{d}t$

For convenience, we define

- $\int_a^a f(x)\,\mathrm{d}x = 0$
- $\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x$

Example
$$\begin{aligned}\int_1^0 x^2\,\mathrm{d}x &= -\int_0^1 x^2\,\mathrm{d}x \\ &= -\frac{1}{3}\end{aligned}$$

Properties Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions such that both $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ exist. Let $\alpha \in \mathbb{R}$ and $a < c < b$. Then

(1) $\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$

Properties Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions such that both $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ exist. Let $\alpha \in \mathbb{R}$ and $a < c < b$. Then

(1) $$\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

Reason Follows from the corresponding result for limits.

Properties Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions such that both $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ exist. Let $\alpha \in \mathbb{R}$ and $a < c < b$. Then

(1) $\displaystyle\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$

Reason Follows from the corresponding result for limits.

(2) $\displaystyle\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$

Properties Let $f, g : [a,b] \longrightarrow \mathbb{R}$ be functions such that both $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ exist. Let $\alpha \in \mathbb{R}$ and $a < c < b$. Then

(1) $$\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

Reason Follows from the corresponding result for limits.

(2) $$\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$$

(3) $$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

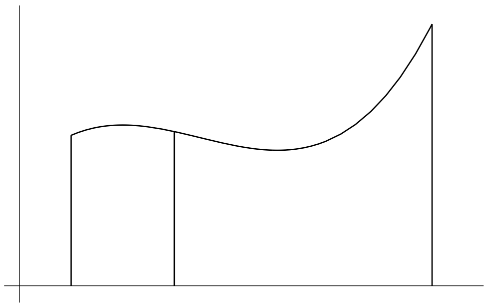
Properties Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be functions such that both $\int_a^b f(x)\,\mathrm{d}x$ and $\int_a^b g(x)\,\mathrm{d}x$ exist. Let $\alpha \in \mathbb{R}$ and $a < c < b$. Then

(1) $$\int_a^b [f(x) + g(x)]\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

Reason Follows from the corresponding result for limits.

(2) $$\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$$

(3) $$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$



Reason Use areas.

To find definite integrals using definition is very tedious.

To find definite integrals using definition is very tedious.

- **Fundamental Theorem of Integral Calculus**

  A convenient way to evaluate definite integrals *(provided that primitives / antiderivatives can be found).*

To find definite integrals using definition is very tedious.

- **Fundamental Theorem of Integral Calculus**

  A convenient way to evaluate definite integrals *(provided that primitives / antiderivatives can be found).*

  It is quite **surprising** that *integration is related to differentiation*.

To find definite integrals using definition is very tedious.

- **Fundamental Theorem of Integral Calculus**

  A convenient way to evaluate definite integrals *(provided that primitives / antiderivatives can be found).*

  It is quite **surprising** that *integration is related to differentiation*.

To tackle the problem, to find $\int_a^b f(x)\,\mathrm{d}x$

introduce an auxiliary function $F : [a,b] \longrightarrow \mathbb{R}$

defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

To find definite integrals using definition is very tedious.

- **Fundamental Theorem of Integral Calculus**

  A convenient way to evaluate definite integrals *(provided that primitives / antiderivatives can be found).*

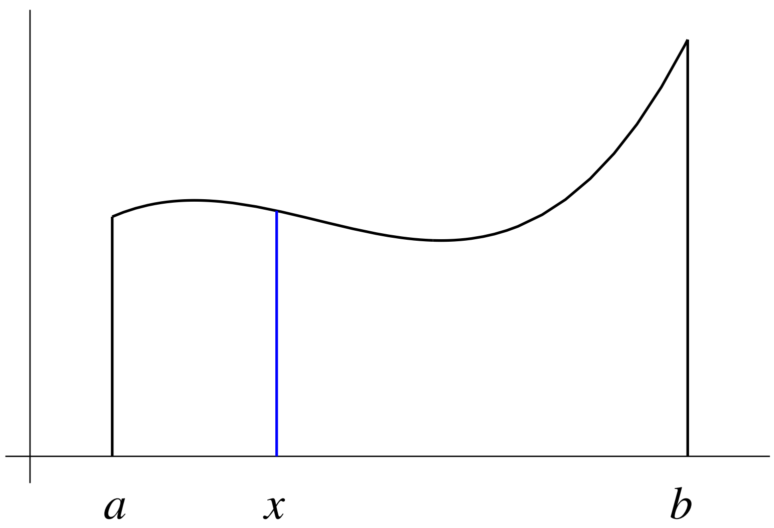
  It is quite **surprising** that *integration is related to differentiation*.

To tackle the problem, to find $\displaystyle\int_a^b f(x)\,\mathrm{d}x$

introduce an auxiliary function $F : [a, b] \longrightarrow \mathbb{R}$

defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$



To find definite integrals using definition is very tedious.

- **Fundamental Theorem of Integral Calculus**

  A convenient way to evaluate definite integrals *(provided that primitives / antiderivatives can be found).*

  It is quite **surprising** that *integration is related to differentiation*.

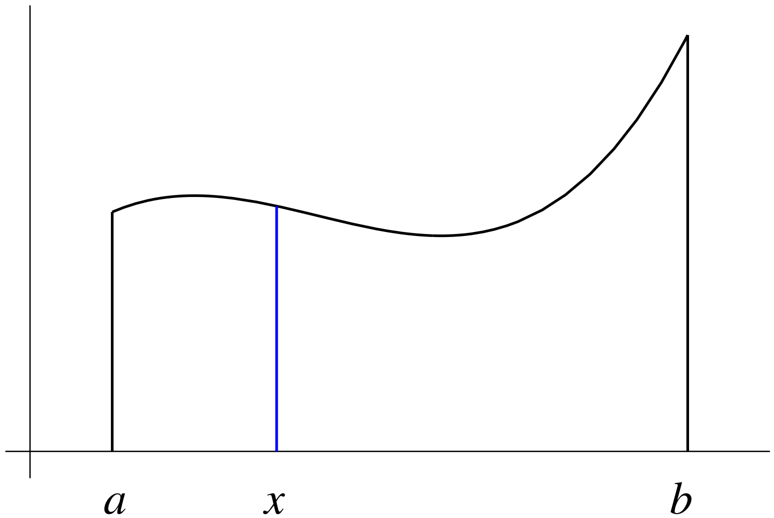
To tackle the problem, to find $\int_a^b f(x)\,\mathrm{d}x$

introduce an auxiliary function $F : [a, b] \longrightarrow \mathbb{R}$

defined by

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

The required integral $\int_a^b f(x)\,\mathrm{d}x$ is $F(b)$.



Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

- **Recall** that $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$

Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

- **Recall** that $F'(x) = \lim_{h \to 0} \dfrac{F(x+h) - F(x)}{h}$

- By construction

$$F(x+h) - F(x) \quad = \quad \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^{x} f(t)\,\mathrm{d}t$$

Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

- **Recall** that $F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h}$
- By construction

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t$$

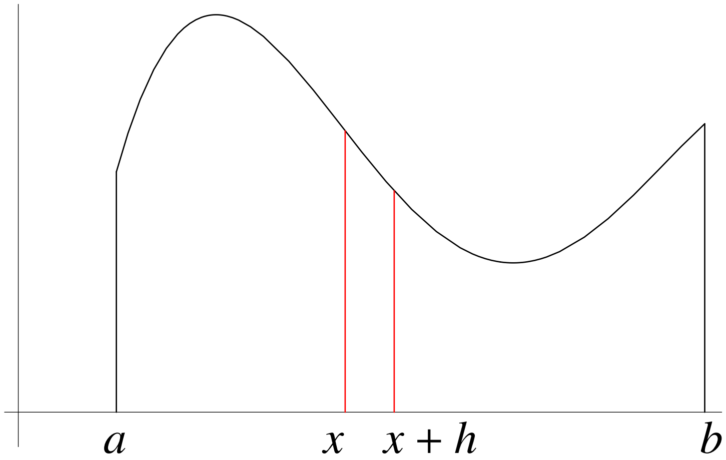
$$= \int_x^{x+h} f(t)\,\mathrm{d}t$$

Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

- **Recall** that $F'(x) = \lim_{h\to 0} \dfrac{F(x+h) - F(x)}{h}$
- By construction

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t$$

$$= \int_x^{x+h} f(t)\,\mathrm{d}t$$



Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

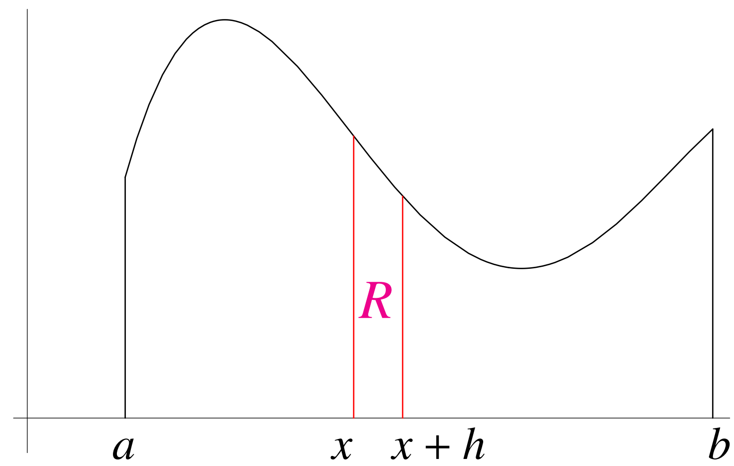
- **Recall** that $F'(x) = \lim_{h \to 0} \dfrac{F(x+h) - F(x)}{h}$

- By construction

$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^{x} f(t)\,\mathrm{d}t$$

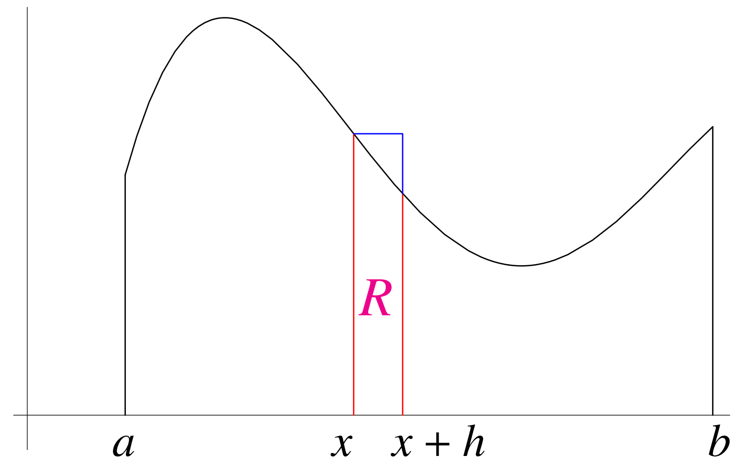
$$= \int_x^{x+h} f(t)\,\mathrm{d}t$$

$$= \text{area of } R \quad \textit{(assume } f \geq 0\textit{)}$$



Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

- **Recall** that $F'(x) = \lim_{h\to 0} \dfrac{F(x+h) - F(x)}{h}$
- By construction



$$F(x+h) - F(x) = \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^{x} f(t)\,\mathrm{d}t$$

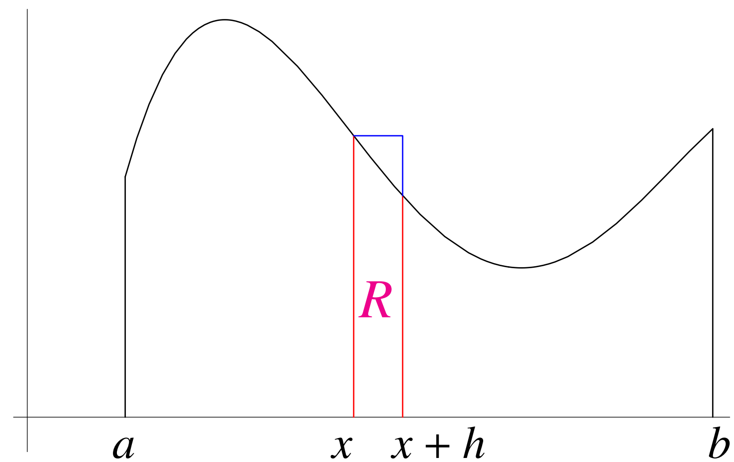
$$= \int_x^{x+h} f(t)\,\mathrm{d}t$$

$=$ area of R *(assume $f \geq 0$)*

$\approx f(x) \cdot h$ *when h is small positive*

Show that $\boxed{F'(x) = f(x)}$ *(idea only)*

- **Recall** that $F'(x) = \lim_{h \to 0} \dfrac{F(x+h) - F(x)}{h}$
- By construction



$$
\begin{aligned}
F(x+h) - F(x) &= \int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \\
&= \int_x^{x+h} f(t)\,\mathrm{d}t \\
&= \text{area of } R \qquad (\text{assume } f \geq 0) \\
&\approx f(x) \cdot h \qquad \text{when } h \text{ is small positive}
\end{aligned}
$$

Therefore $\dfrac{F(x+h) - F(x)}{h} \approx f(x)$ if h is small.

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is <u>*a*</u> *primitive* (or *antiderivative*) for f (on I).

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is a *primitive* (or *antiderivative*) for f (on I).

Example Let $f(x) = 3x^2$.

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is a *primitive* (or *antiderivative*) for f (on I).

Example Let $f(x) = 3x^2$.

- By inspection $G(x) = x^3$ is a primitive for f *(on $\mathbb{R}$)*.

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is <u>*a*</u> *primitive* (or *antiderivative*) for f (on I).

Example Let $f(x) = 3x^2$.

- By inspection $G(x) = x^3$ is a primitive for f *(on* $\mathbb{R}$*)*.

Remark Primitive is not unique.

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is <u>a</u> *primitive* (or *antiderivative*) for f (on I).

Example Let $f(x) = 3x^2$.

- By inspection $G(x) = x^3$ is a primitive for f *(on* $\mathbb{R}$*)*.

Remark Primitive is not unique.

- Above example, $G_1(x) = x^3 + 1$ also a primitive.

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is a *primitive* (or *antiderivative*) for f (on I).

Example Let $f(x) = 3x^2$.

- By inspection $G(x) = x^3$ is a primitive for f *(on* $\mathbb{R}$*)*.

Remark Primitive is not unique.

- Above example, $G_1(x) = x^3 + 1$ also a primitive.
- In fact, infinitely many primitives: $G(x) +$ any constant *that is,* $x^3 + C$

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is a *primitive* (or *antiderivative*) for f (on I).

Example Let $f(x) = 3x^2$.

- By inspection $G(x) = x^3$ is a primitive for f *(on $\mathbb{R}$)*.

Remark Primitive is not unique.

- Above example, $G_1(x) = x^3 + 1$ also a primitive.
- In fact, infinitely many primitives: $G(x)$ + any constant *that is,* $x^3 + C$
- Any more ??? *If $F' = f$, must F be in the above form ?*

Summary To find $\int_a^b f(x)\,\mathrm{d}x$ Introduce $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Required integral is $F(b)$.

It is shown that $F'(x) = f(x)$.

Definition Let f be a (continuous) function. Suppose G is a function such that $G' = f$ (on an open interval I), we say that G is a *primitive* (or *antiderivative*) for f (on I).

Example Let $f(x) = 3x^2$.

- By inspection $G(x) = x^3$ is a primitive for f *(on $\mathbb{R}$)*.

Remark Primitive is not unique.

- Above example, $G_1(x) = x^3 + 1$ also a primitive.
- In fact, infinitely many primitives: $G(x) +$ any constant *that is,* $x^3 + C$
- Any more ??? *If $F' = f$, must F be in the above form ?*
- NO MORE If $F' = f$, then $F(x) \equiv x^3 + C$ for some constant C.

Theorem *Let* F, G *be functions on an open interval* (a, b) *such that*

$$F'(x) = G'(x) \qquad \text{for all } x \in (a, b).$$

Then F *and* G *differ by a constant,*

Theorem *Let F, G be functions on an open interval (a,b) such that*

$$F'(x) = G'(x) \qquad \textit{for all } x \in (a,b).$$

Then F and G differ by a constant, that is, there exists a constant C such that

$$F(x) - G(x) = C \text{ for all } x \in (a,b)$$

Theorem *Let F, G be functions on an open interval (a, b) such that*

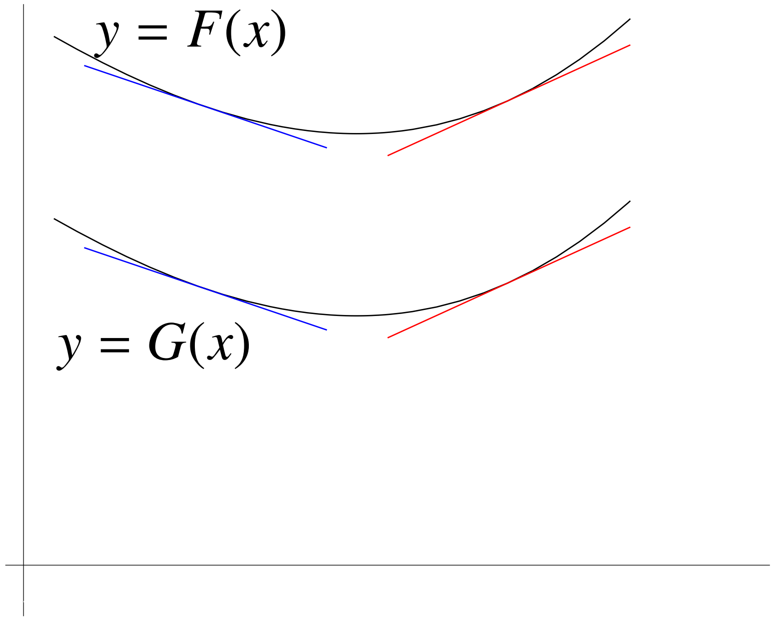
$$F'(x) = G'(x) \qquad \text{for all } x \in (a, b).$$

Then F and G differ by a constant, that is, there exists a constant C such that

$$F(x) - G(x) = C \text{ for all } x \in (a, b)$$

Geometry Meaning

- **Hypothesis:** at corresponding points, tangents to graphs of F and G are parallel.



Theorem *Let F, G be functions on an open interval (a, b) such that*

$$F'(x) = G'(x) \qquad \text{for all } x \in (a, b).$$

Then F and G differ by a constant, that is, there exists a constant C such that

$$F(x) - G(x) = C \text{ for all } x \in (a, b)$$

Geometry Meaning

- **Hypothesis:** at corresponding points, tangents to graphs of F and G are parallel.

- **Conclusion:** graph of F can be obtained by moving that of G upward ($C > 0$) or downward ($C < 0$).