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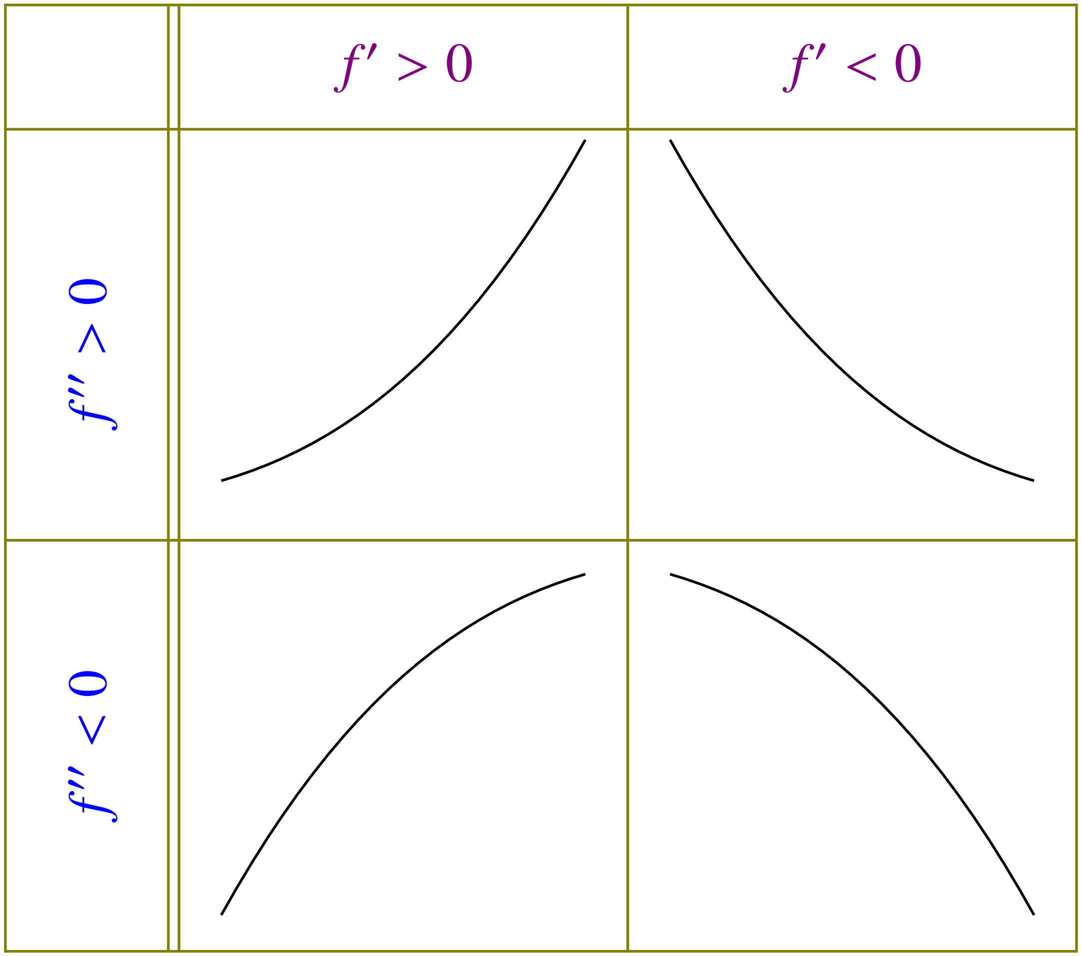
$f' > 0$
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Example Sketch the graph of the function f given by

$$f(x) = x^3 + 3x^2 - 45x$$

Indicate (a) relative extremum points and inflection points of the graph

(b) intervals on which f is increasing or decreasing; concave or convex.

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	$(-\infty, -5)$	-5	$(-5, -1)$	-1	$(-1, 3)$	3	$(3, \infty)$
f'	$+$	0	$-$	$-$	$-$	0	$+$
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					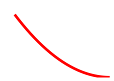		

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					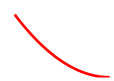		

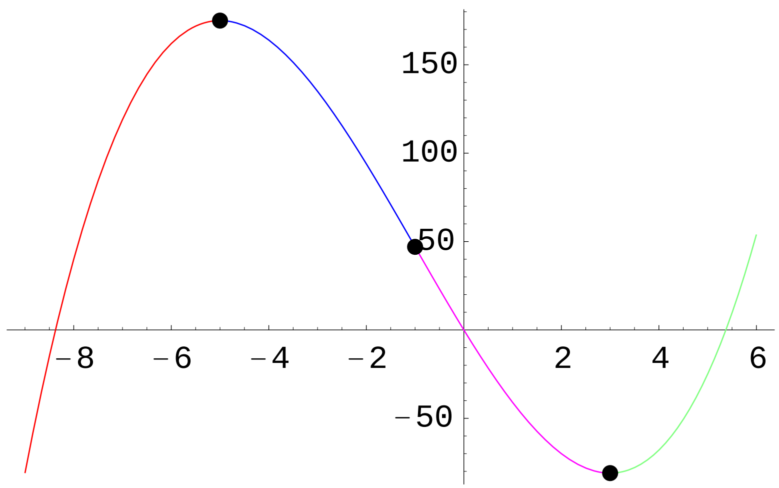

150
100
50
-8
-6
-4
-2
2
4
6
-50

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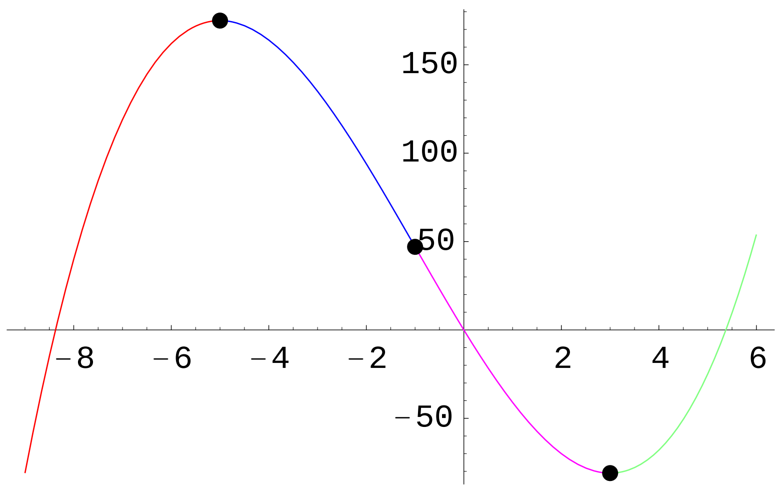
- y-intercept: $(0, 0)$



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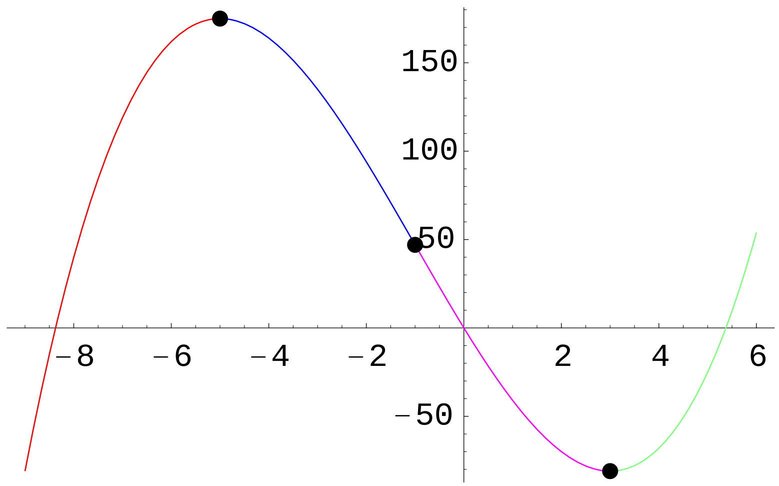

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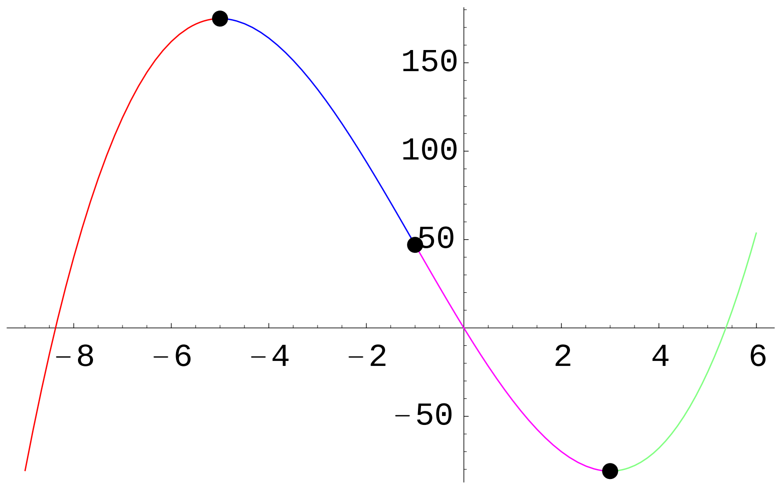

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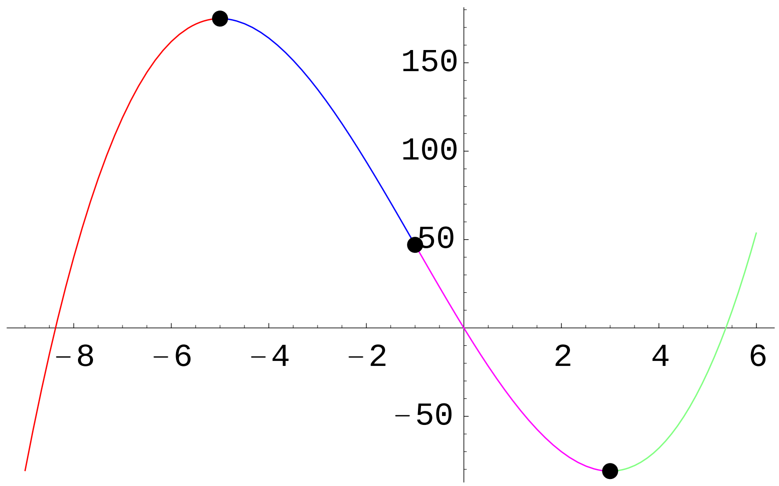

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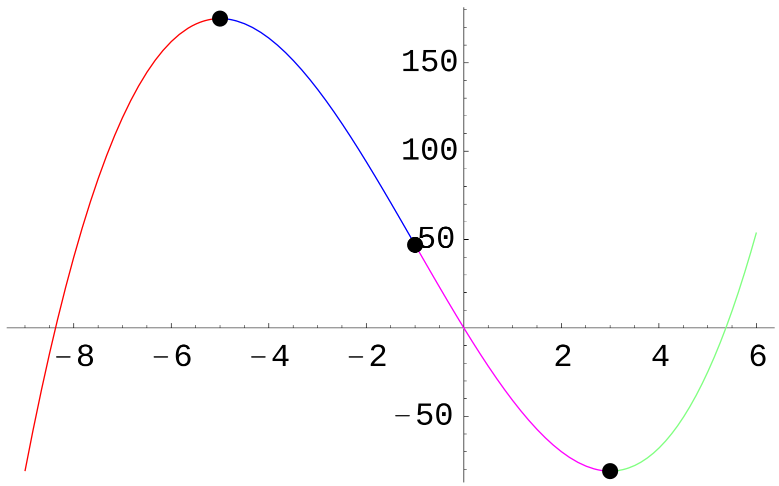
x-intercepts: $(0, 0)$, $(-8.37, 0)$, $(5.37, 0)$



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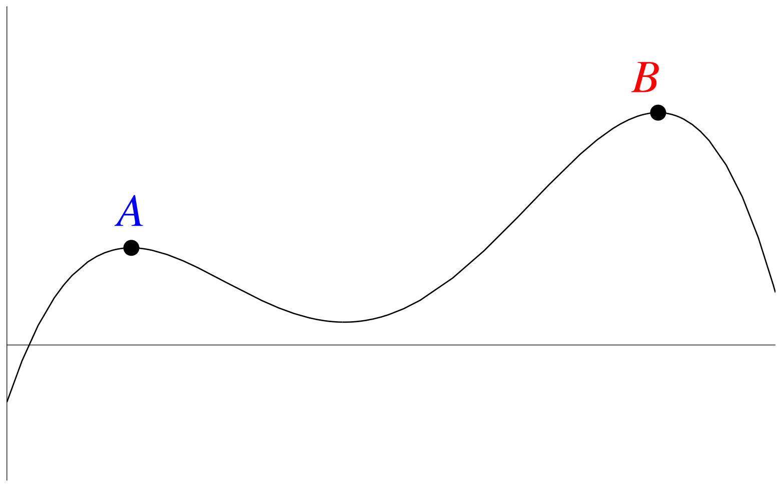

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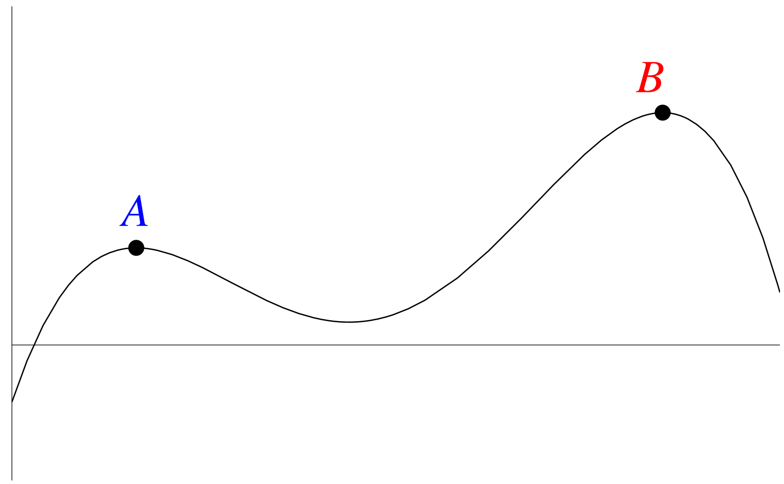

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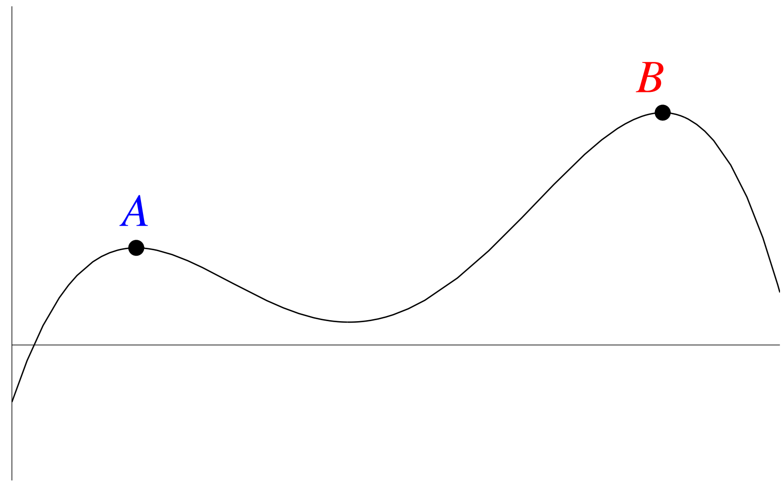

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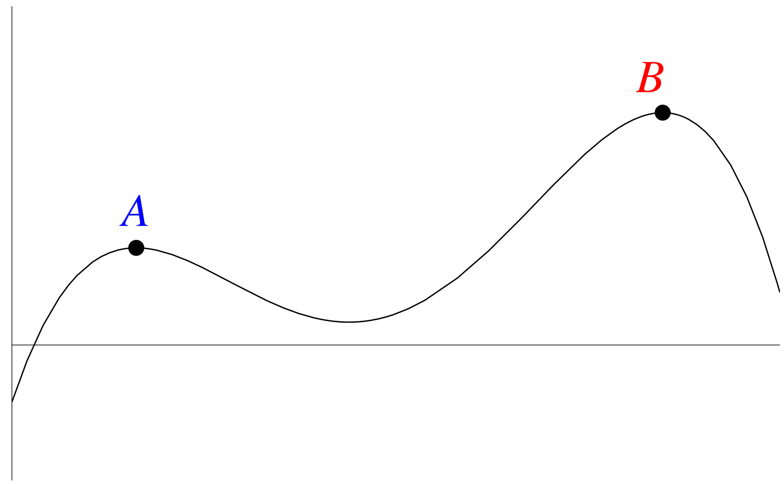

Question Does maximum always exists ?

Extreme Value Theorem *Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then f attains its (absolute) maximum and minimum. That is, there exist $x_1, x_2 \in [a, b]$ such that*

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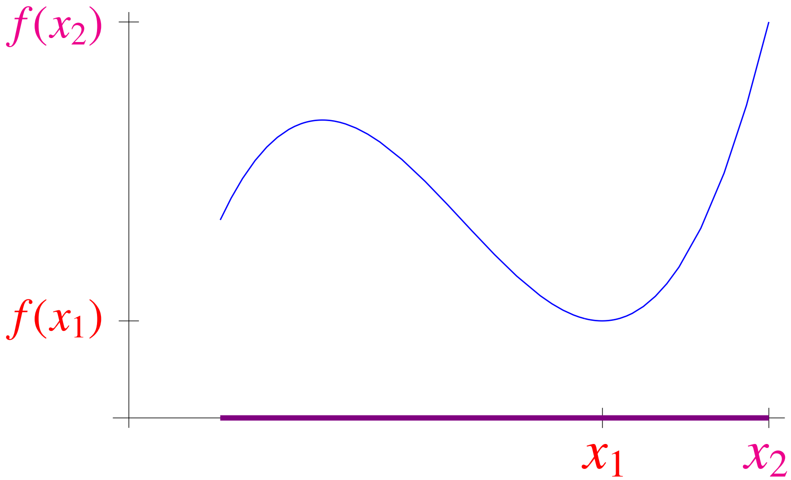

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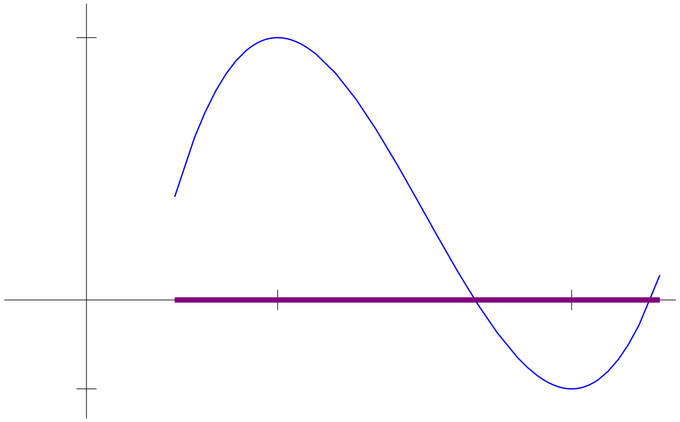

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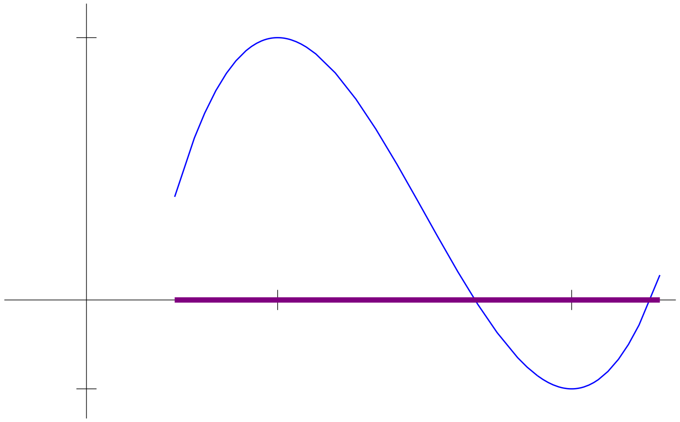

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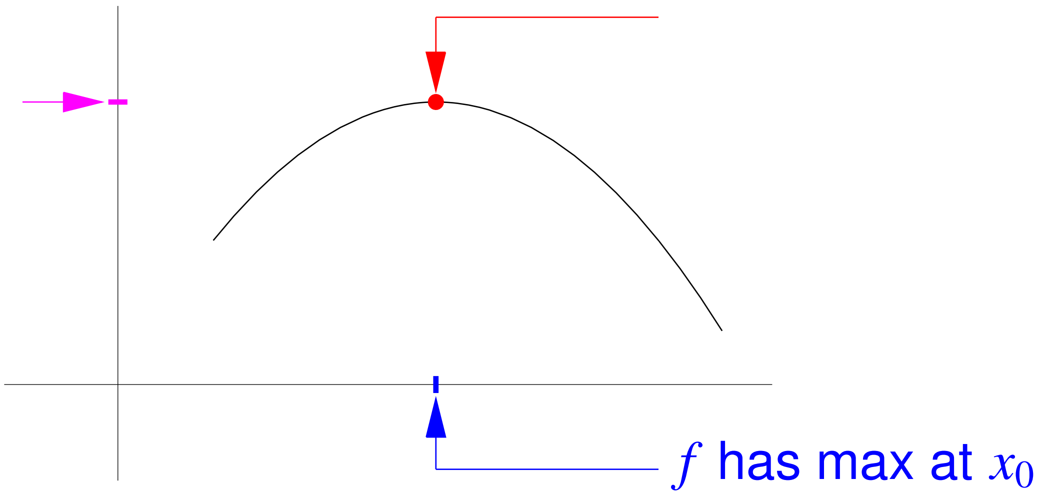

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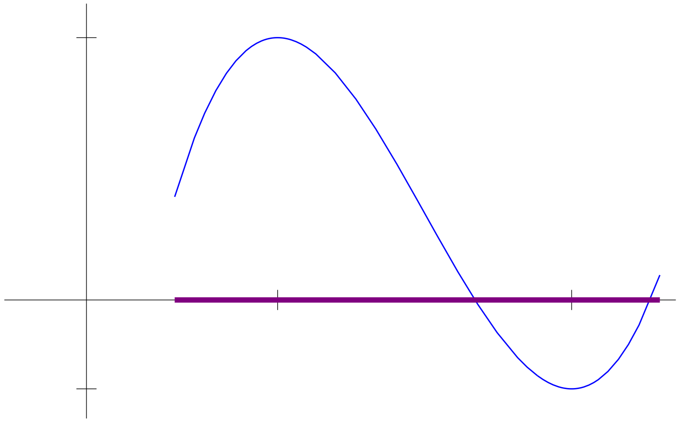

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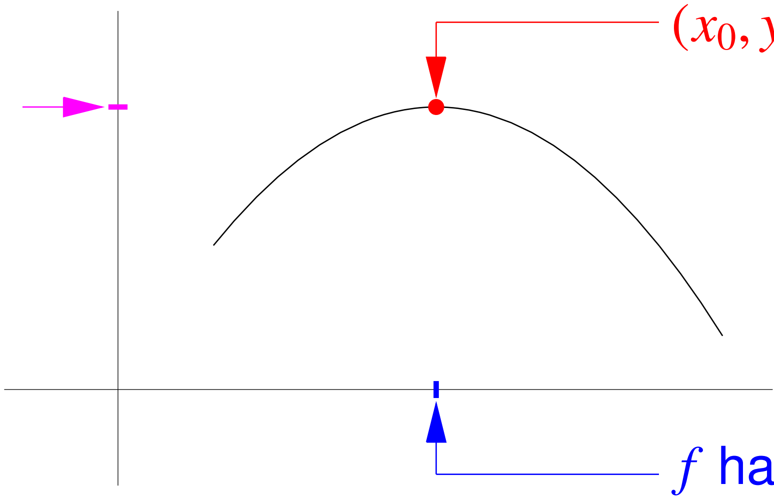

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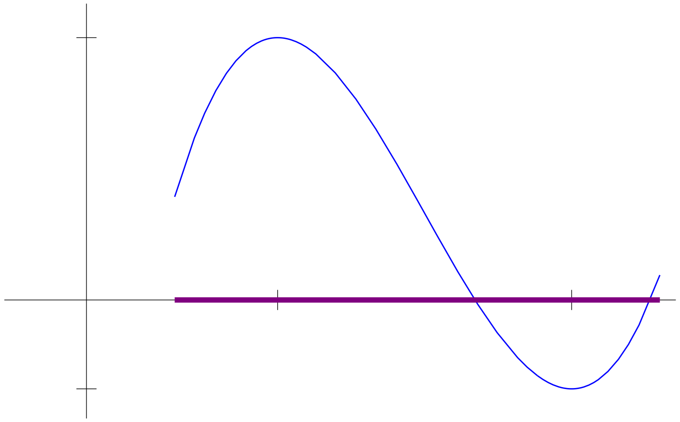

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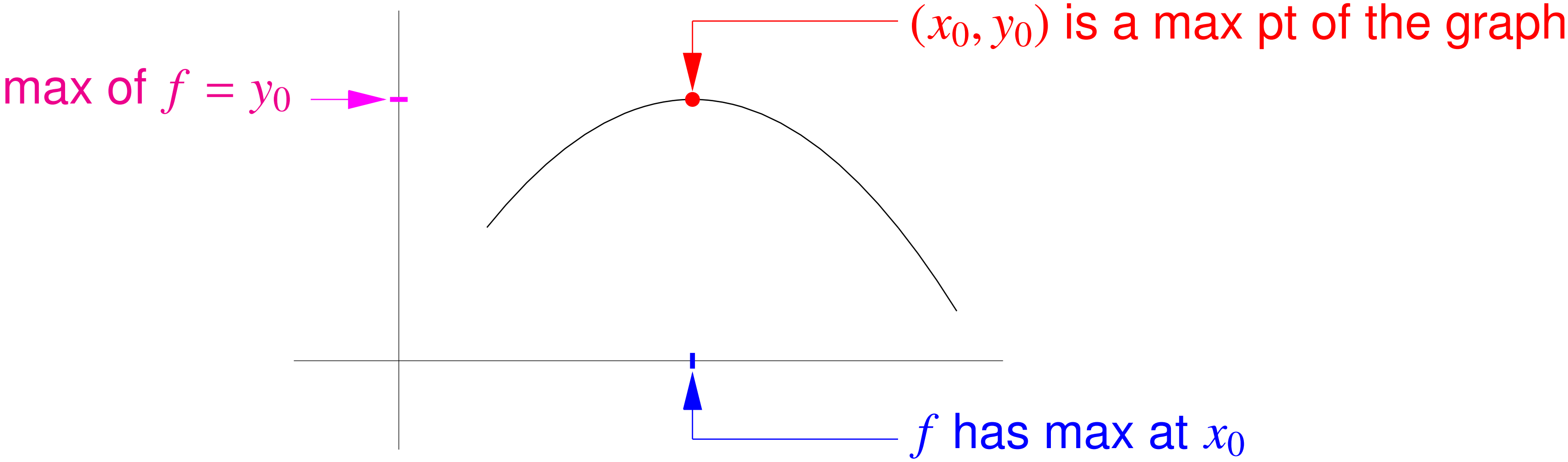

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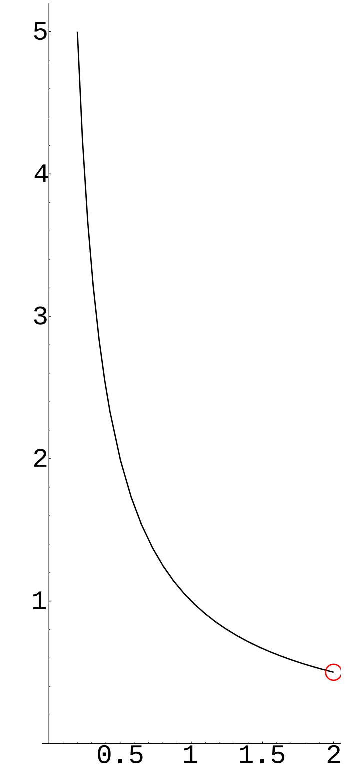

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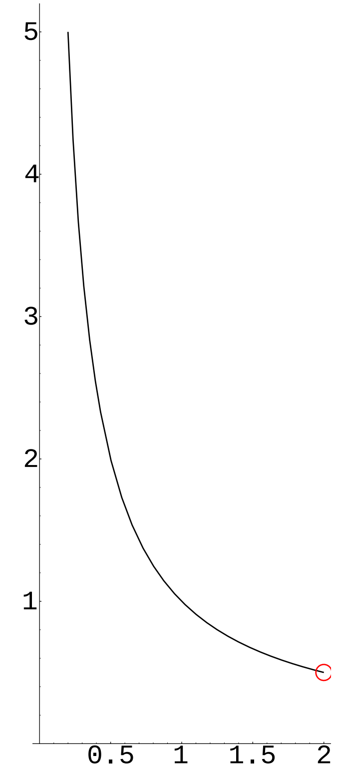

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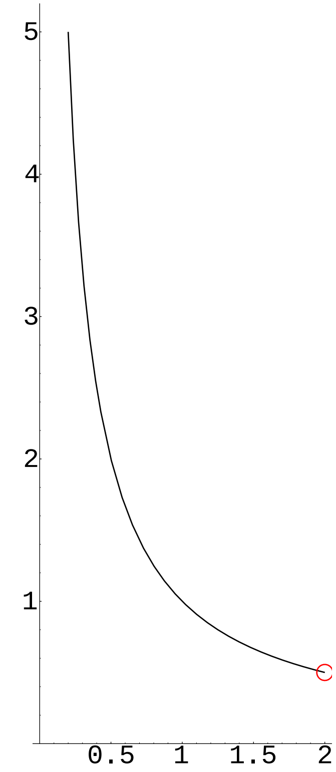

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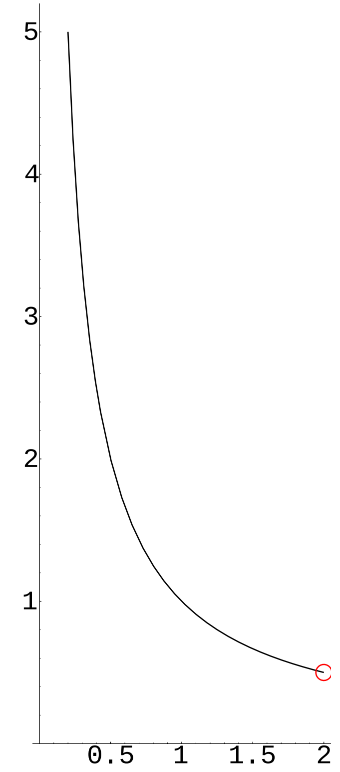

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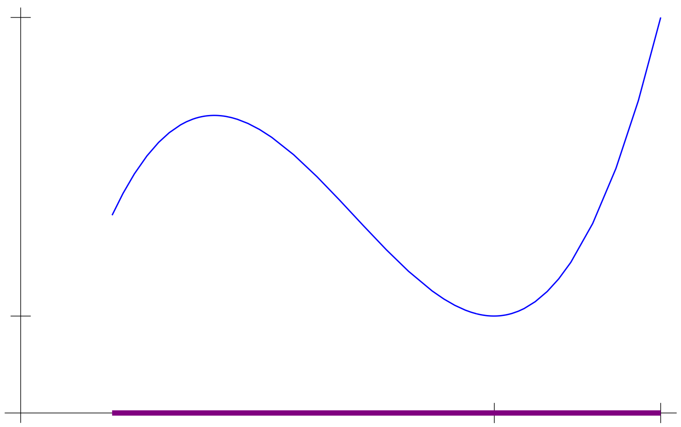

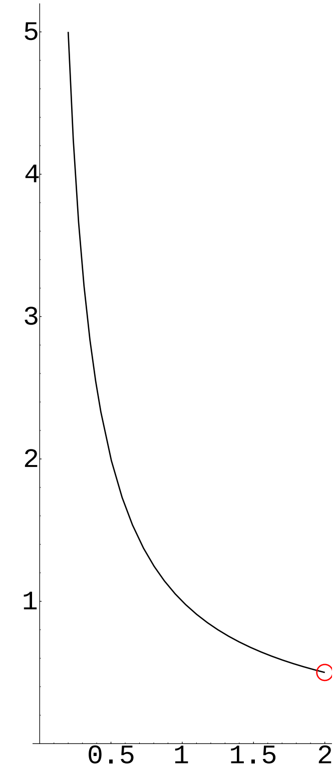

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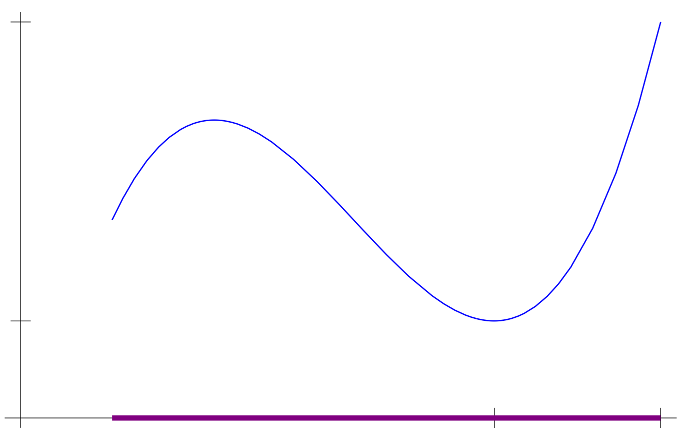

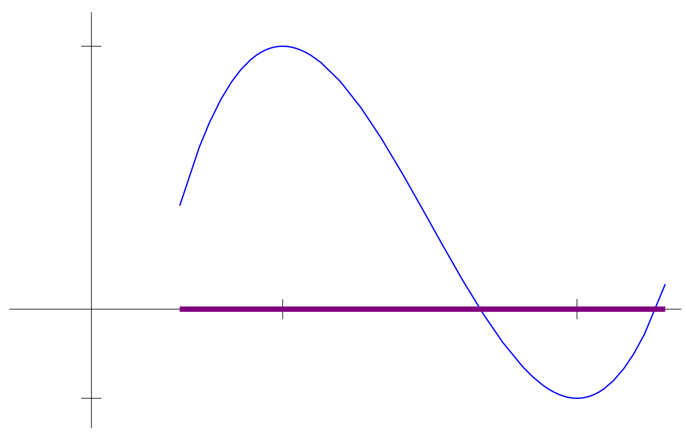

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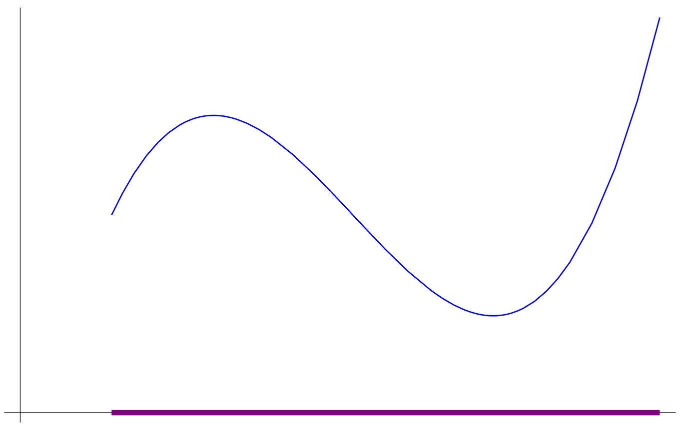

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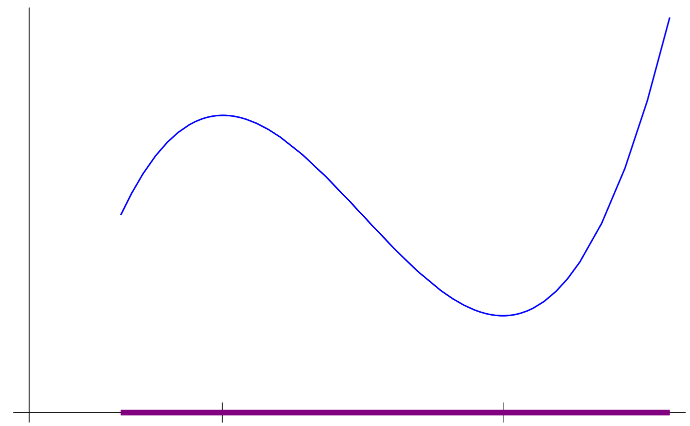

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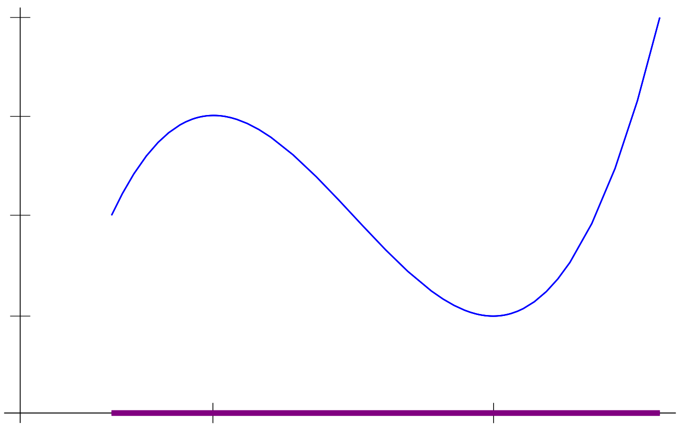

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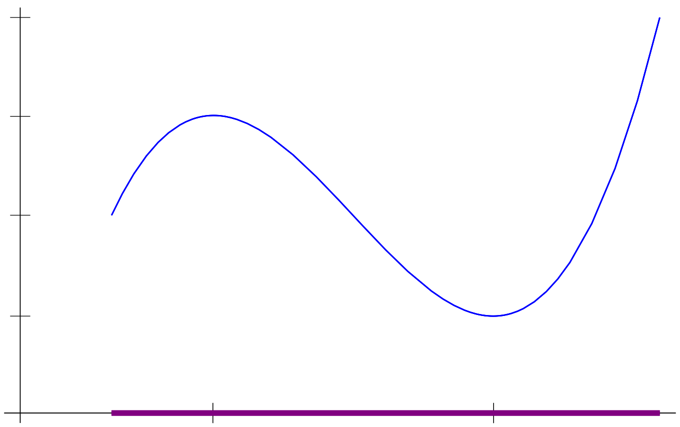

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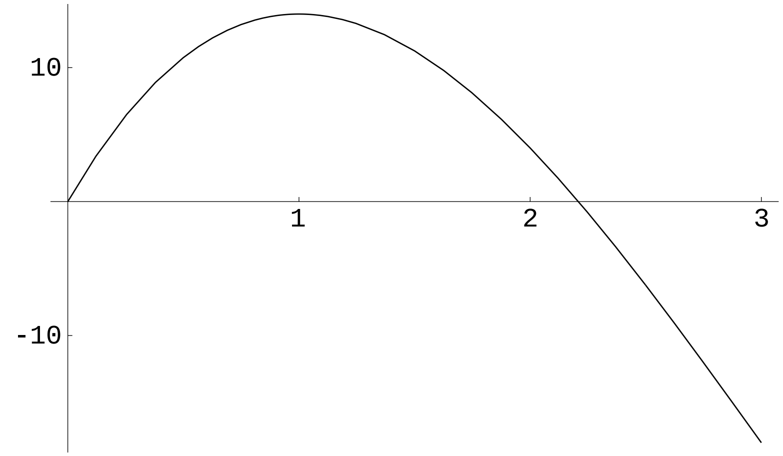

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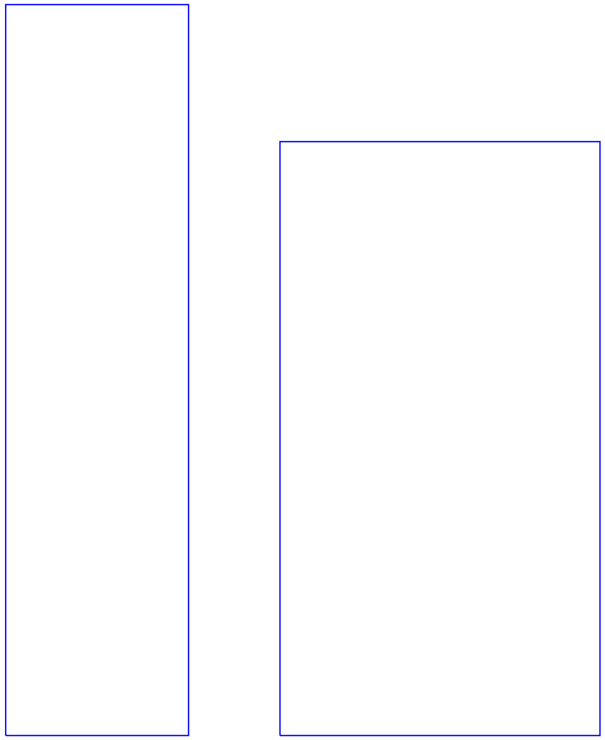

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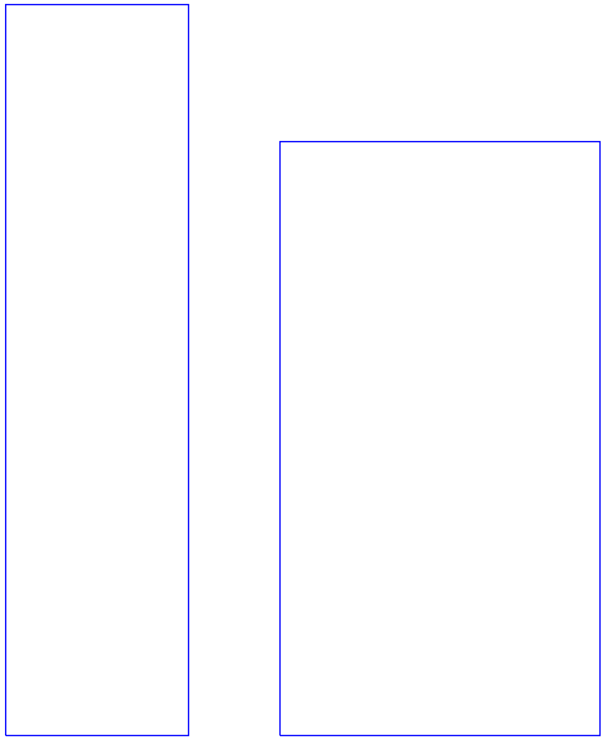

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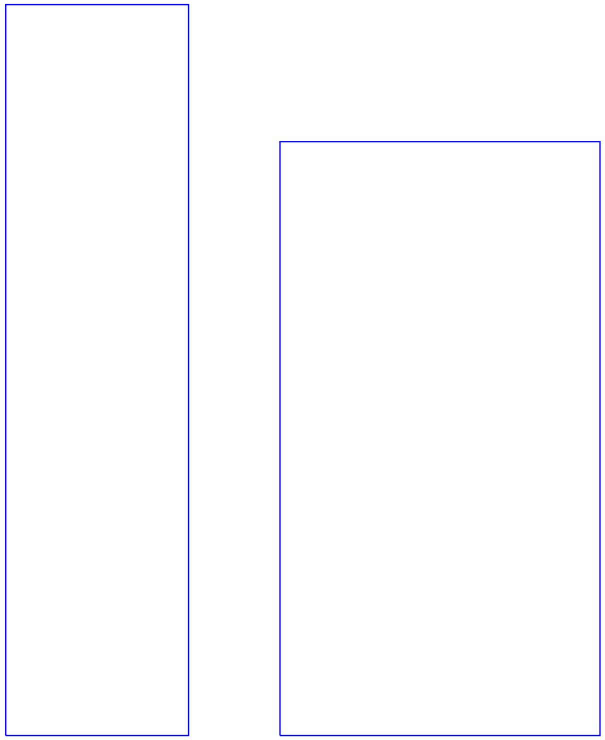

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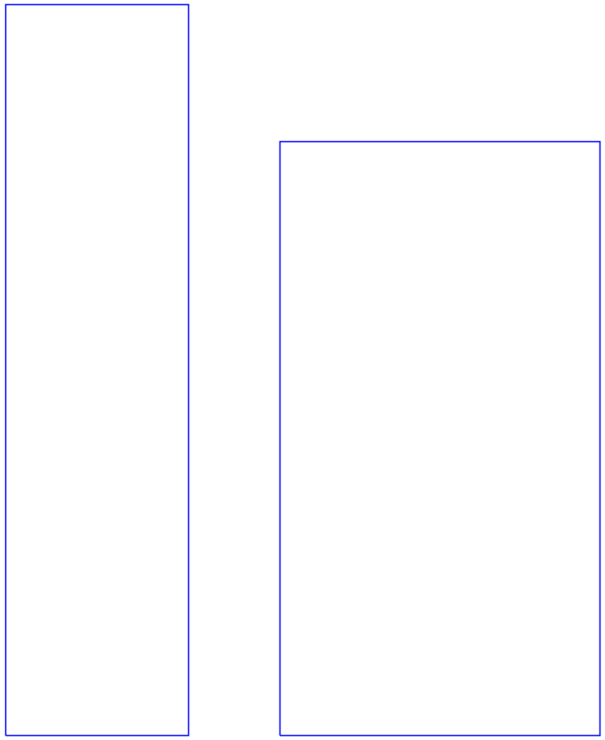

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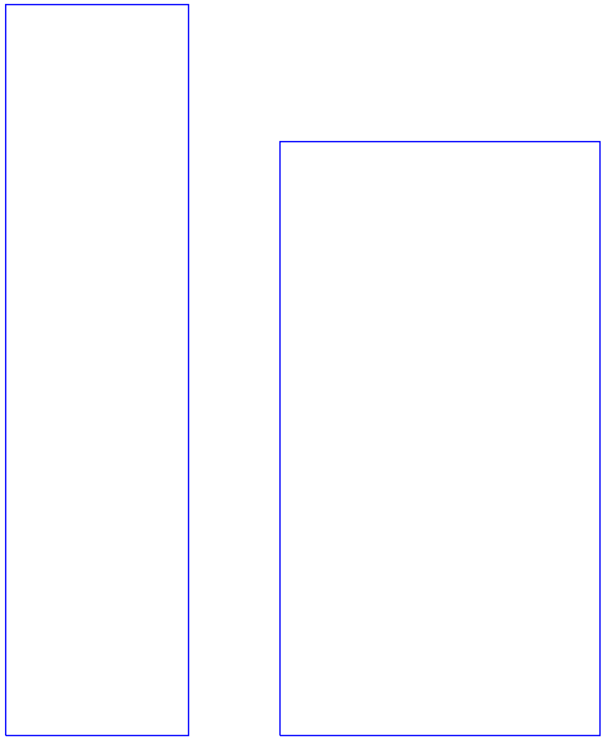

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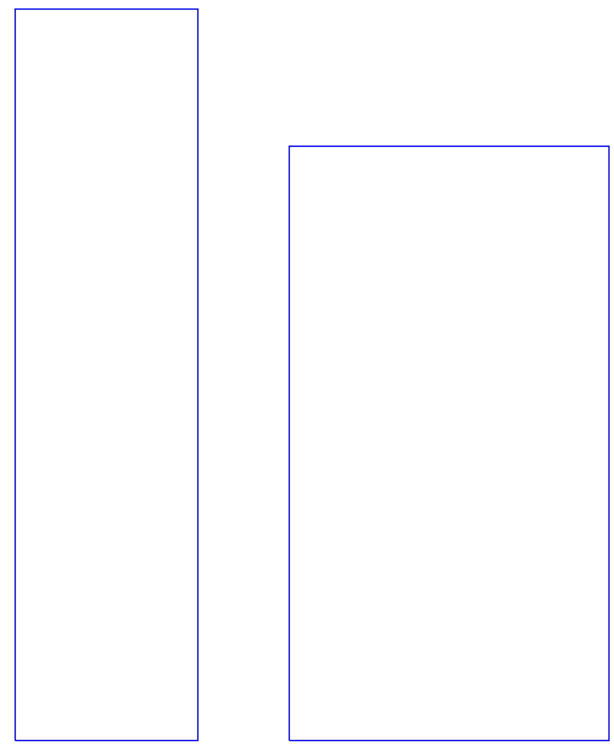

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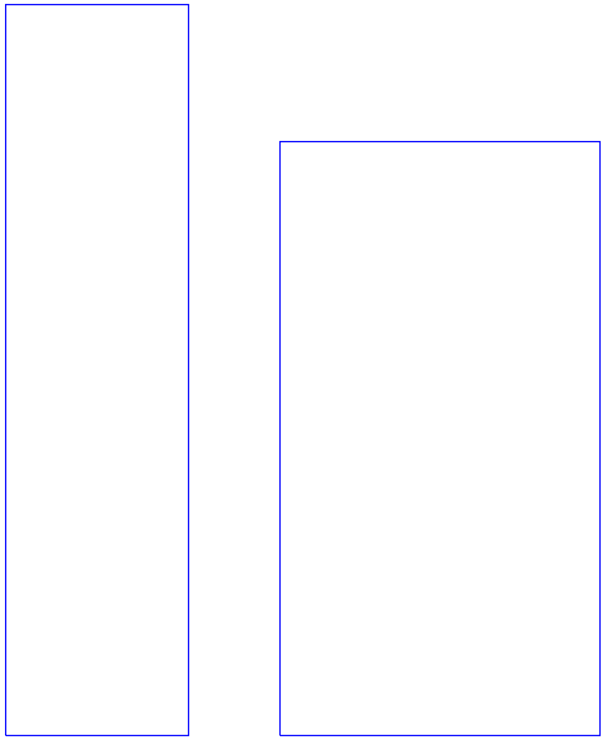

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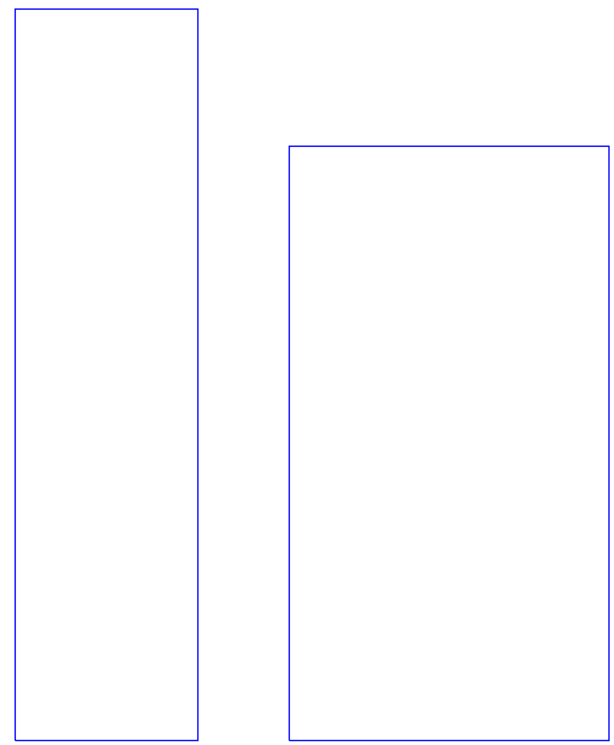

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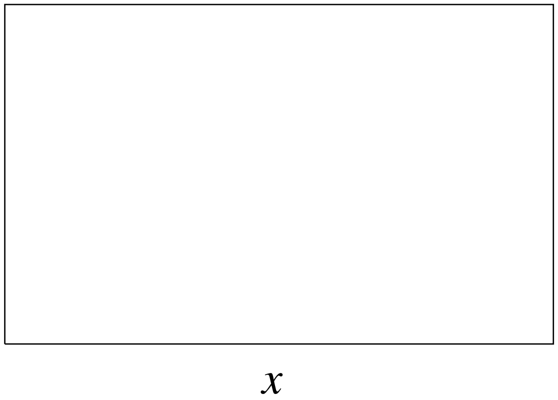

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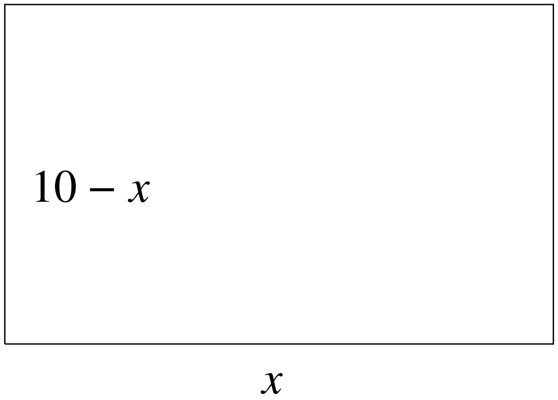

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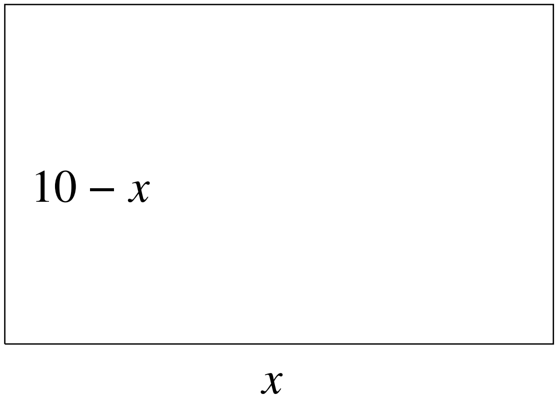

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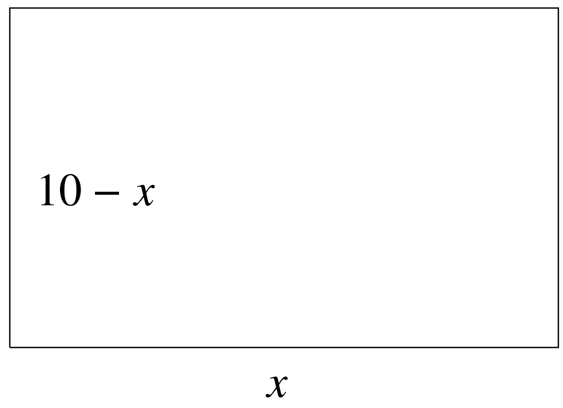

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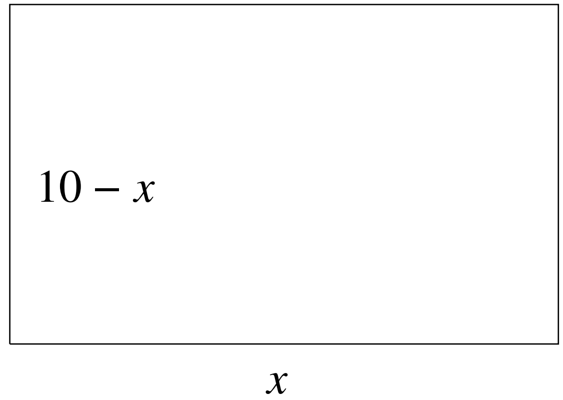

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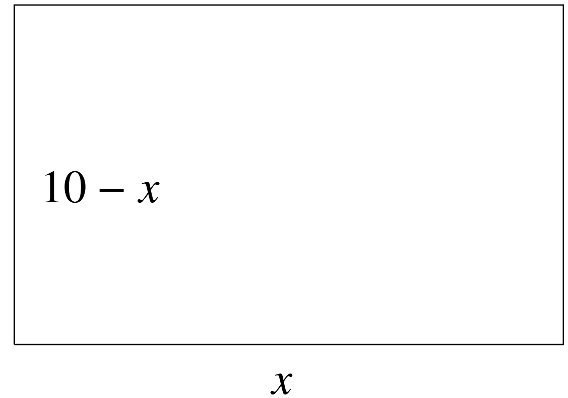

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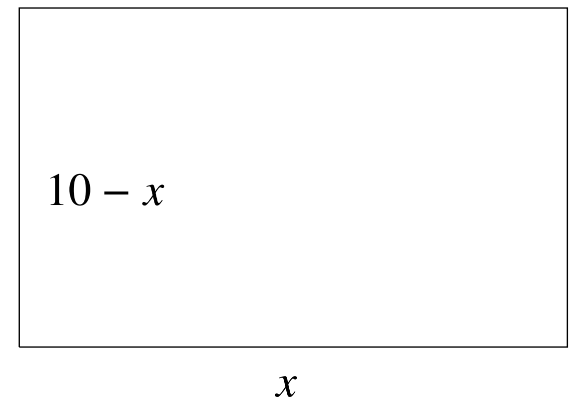

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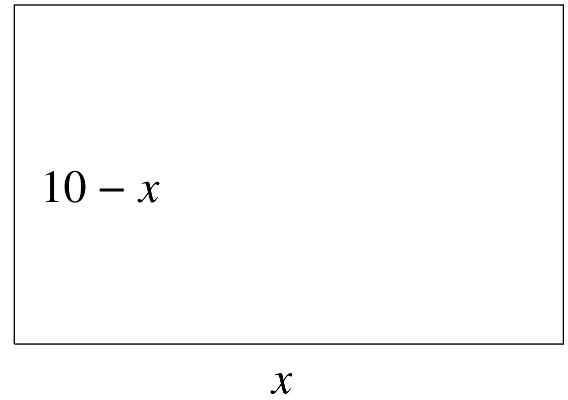

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A	↗		↘

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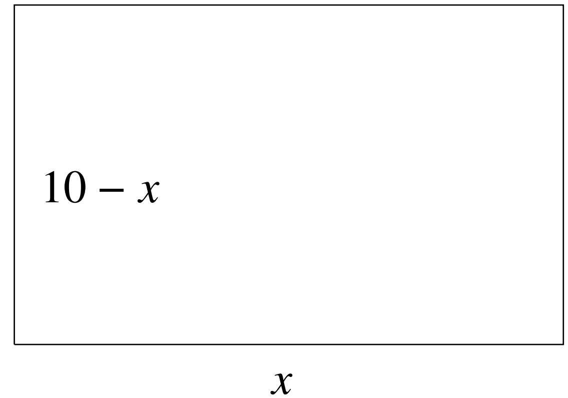

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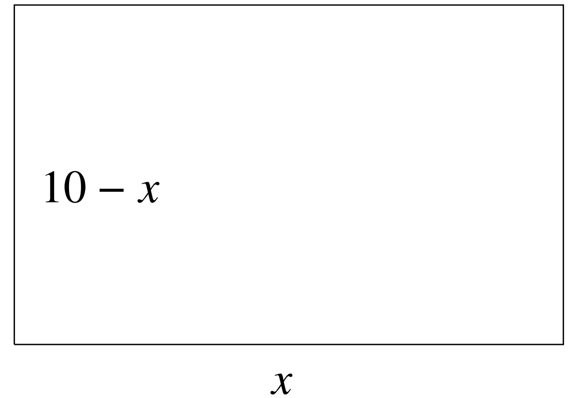

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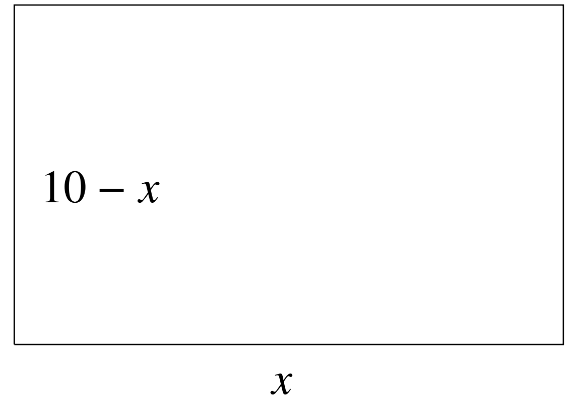

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- *Answer* It is a $5\text{cm} \times 5\text{cm}$ rectangle (in fact, a square).

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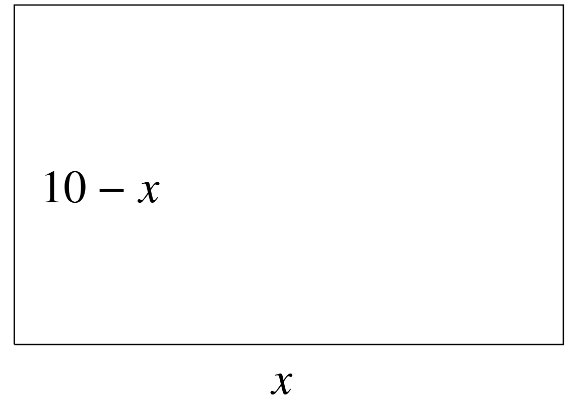

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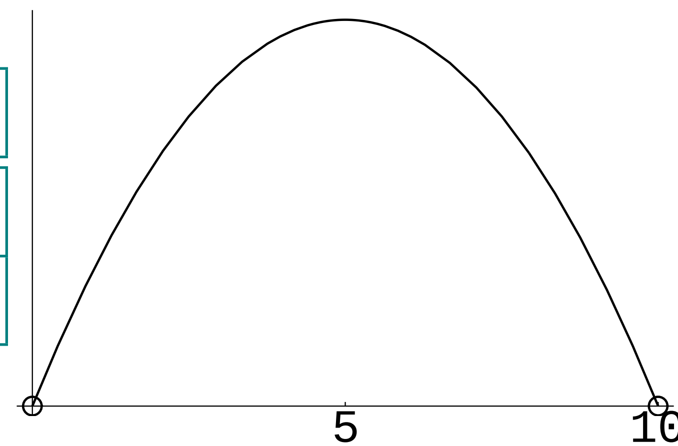

- *Answer* It is a $5\text{cm} \times 5\text{cm}$ rectangle (in fact, a square).

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x	0	5	10
A	0	25	0

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- Find critical point(s): get $x = 5$
- Compare values $A(5)$, $A(0)$, $A(10)$
- Maximum area is 25

  (attained when $x = 5$).

x	0	5	10
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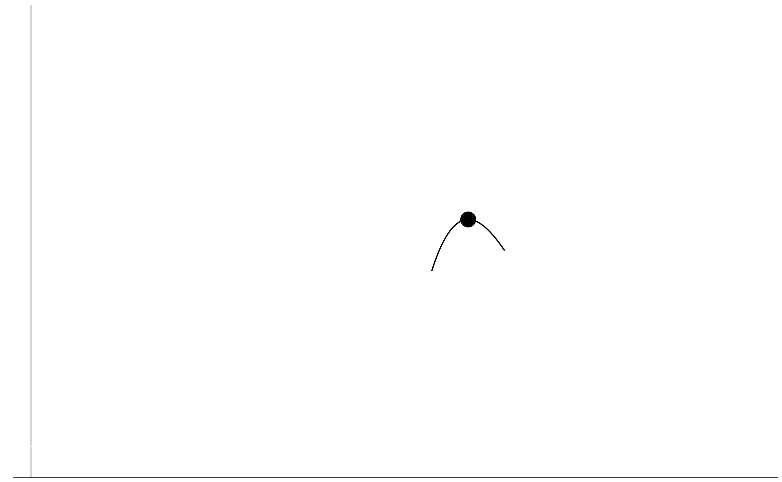

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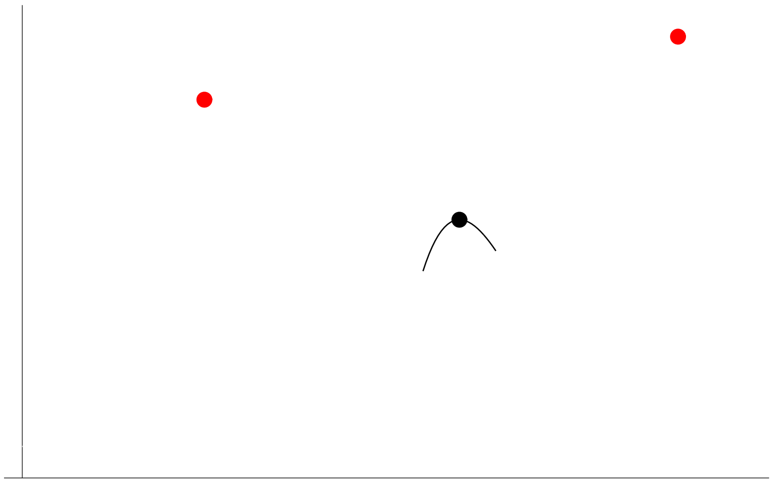

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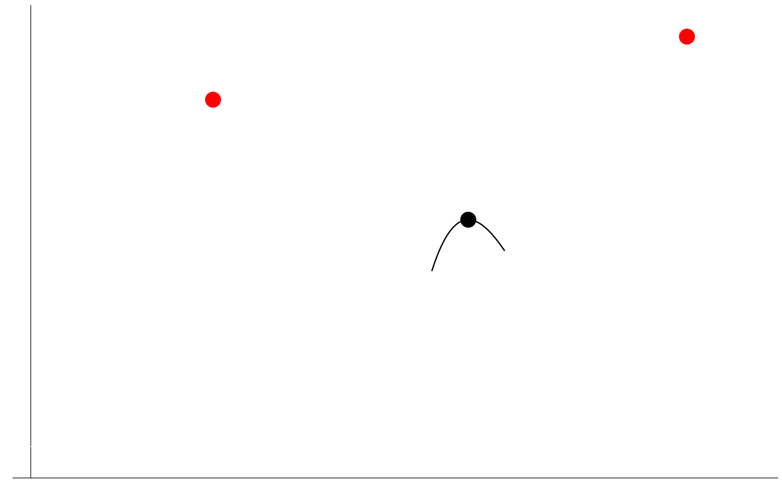

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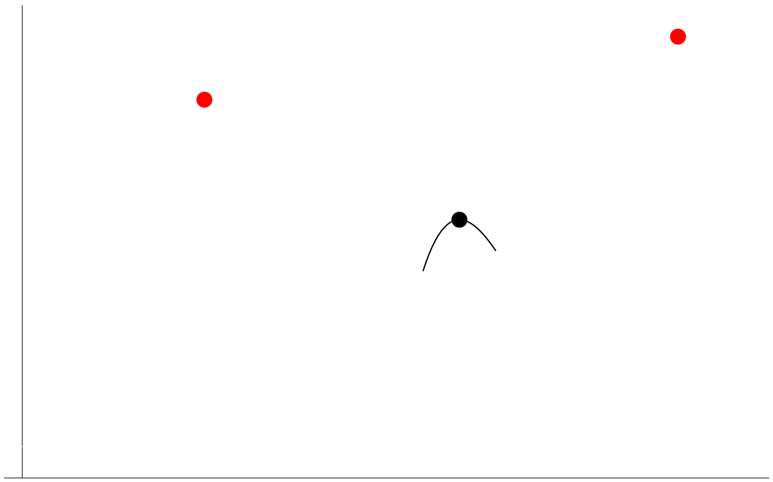

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- $A'(x) = 10 - 2x$
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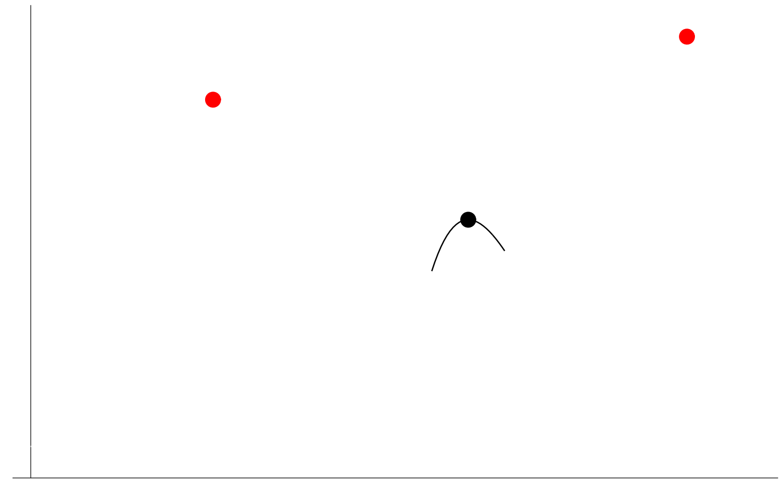

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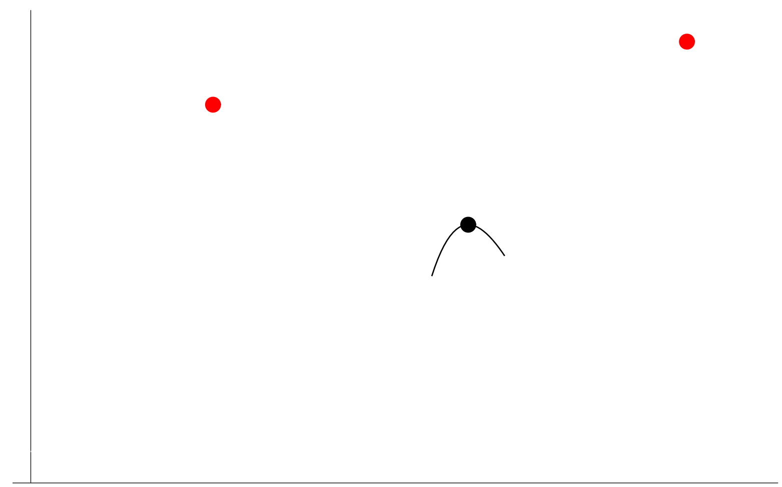

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- $A'(x) = 10 - 2x$
- Critical pt: 5
- $A''(x) = -2$
- Because $A''(5) = -2 < 0$ and 5 is the only critical point in $(0, 10)$, A has max when $x = 5$, *continue ...*