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Proof (for the case where $r = n$ is a positive integer)

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Formula for expanding $(x+h)^n$

$$(x+h)^2 = x^2 + 2xh + h^2$$

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Show Pascal triangle

$$\vdots$$

$$(x+h)^n = x^n + nx^{n-1}h + (\ )x^{n-2}h^2 + \cdots + (\ )xh^{n-1} + h^n$$

where missing numbers are constants *depending on n and the position*

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\
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&= \lim_{h\to 0} \left(nx^{n-1} + (\ )x^{n-2}h + \cdots + h^{n-1}\right)
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General case, use logarithmic differentiation (Chapter 8).

Rule 6: (Product Rule) If f and g are differentiable (at x), then the product fg is also differentiable (at x) and

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Proof Put $F(x) = f(x)g(x)$.

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Trick – subtract and add $f(x+h)g(x)$ in the numerator:

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = \lim_{h\to 0}\frac{[f(x+h)g(x+h) - f(x+h)g(x)] + [f(x+h)g(x) - f(x)g(x)]}{h}$$

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$$= f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h\to 0}\frac{[f(x+h)g(x+h) - f(x+h)g(x)] + [f(x+h)g(x) - f(x)g(x)]}{h}$$

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$$= f(x)g'(x) + g(x)f'(x)$$

Fact If f is differentiable, then it is continuous.

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$$F'(x) = (3x^2 - 5)\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - 5x + 1) + (x^3 - 5x + 1)\frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 5)$$

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&= (3x^2 - 5)(3x^2 - 5) + (x^3 - 5x + 1)(6x) \\
&\vdots \\
&= 15x^4 - 60x^2 + 6x + 25
\end{aligned}
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Alternative Method Expand

$$\begin{aligned} F(x) &= \cdots\cdots \\ &= 3x^5 - 20x^3 + 3x^2 + 25x - 5 \end{aligned}$$

and then differentiate term by term.

Rule 7: (Quotient Rule) If f and g are differentiable (at x) and $g(x) \neq 0$, then the quotient $\dfrac{f}{g}$ is also differentiable (at x) and

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Proof Similar to the proof of the product rule.

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Proof Similar to the proof of the product rule.

Alternative notation $$D\left(\frac{f}{g}\right) = \frac{g\cdot Df - f\cdot Dg}{g^2}$$

Example Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{2x^2-3}{\sqrt{x}}\right)$.

Method 1

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Method 1 (Using quotient rule)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x^2-3}{\sqrt{x}}\right) = \frac{\sqrt{x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(2x^2-3)-(2x^2-3)\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{2}}}{(\sqrt{x})^2}$$

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$$= \sqrt{x}\cdot 4x$$

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$$= 3x^{\frac{1}{2}}+\frac{3}{2}x^{-\frac{3}{2}} = 3\sqrt{x}+\frac{3}{2x\sqrt{x}}$$

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$$= \frac{(2x+1)(2x+3-0) - (x^2+3x-4)(2+0)}{(2x+1)^2}$$

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$$\vdots$$

$$= \frac{2x^2+2x+11}{(2x+1)^2}$$

Remark

- In power rule:
  - put $r = 1$,

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- In Chapter 9, will discuss the *chain rule*.

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These are called *higher-order derivatives*.

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&= 30 \\
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f^{(n)}(x) &= 0 \qquad \text{for } n \geq 4
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Exercise Guess a formula for $f^{(n)}(x)$. *Involves $n!$, see supplementary notes*

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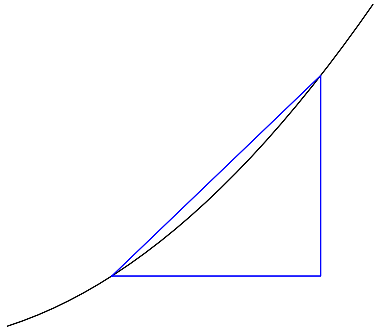
Geometric meaning

Physical meaning

Rate of Change $\frac{\mathrm{d}y}{\mathrm{d}x}$ is the rate of change of y with respect to x.

Geometric meaning $y = f(x)$ represents a curve in the xy-plane:



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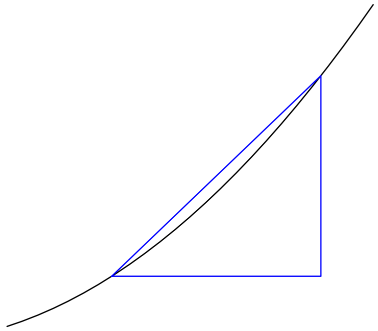

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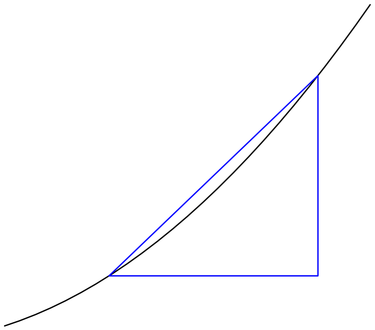

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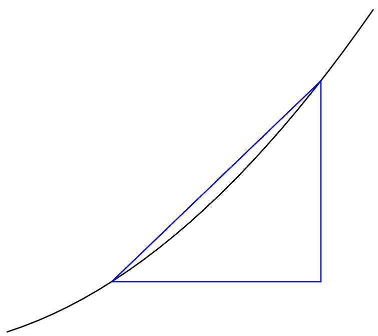

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Example Problem 1 in Chapter 3, $s = t^2$, velocity at $t = 2$?

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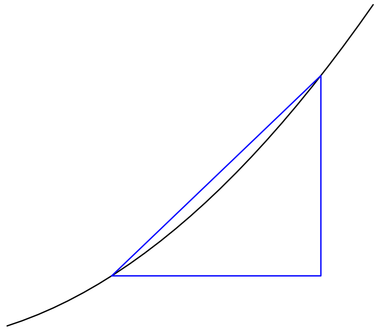

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Physical meaning When x is time and y is displacement

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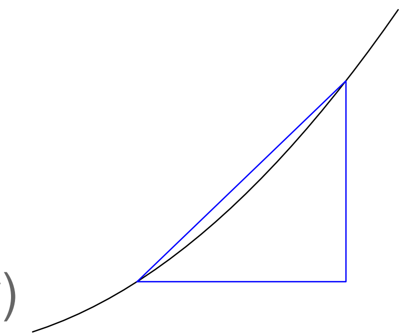

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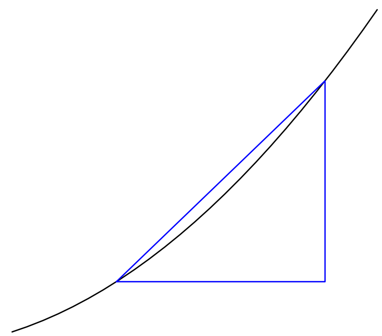

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$$y = x^3 - 2x^2 + 3x - 7$$

Find an equation for the line tangent to the curve at the point $A = (2, -1)$.

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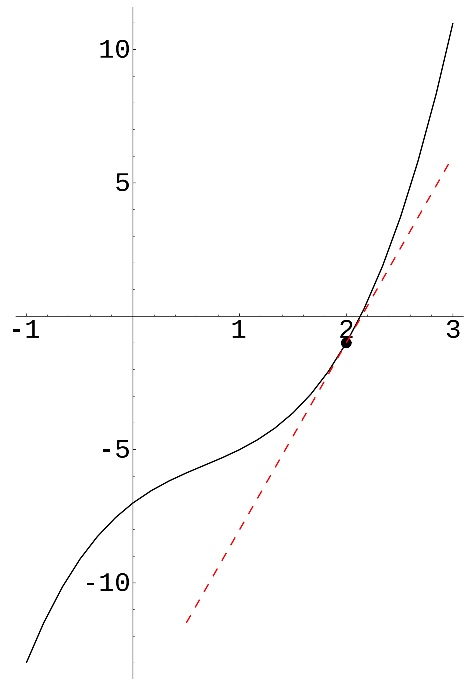

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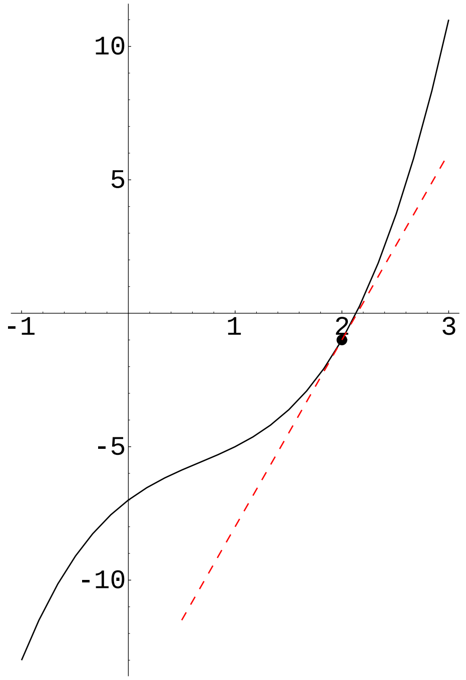

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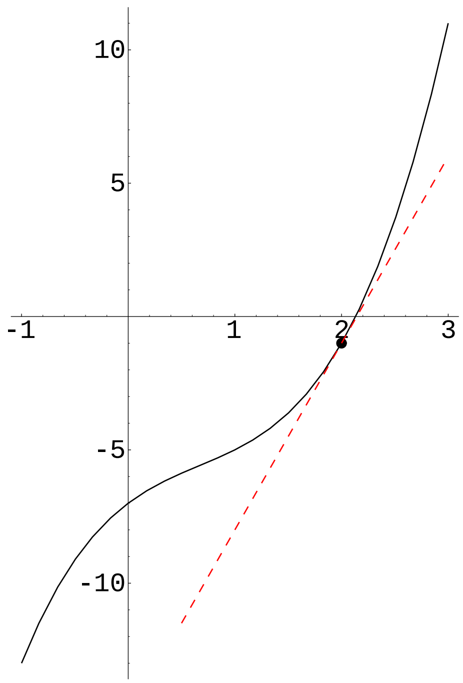

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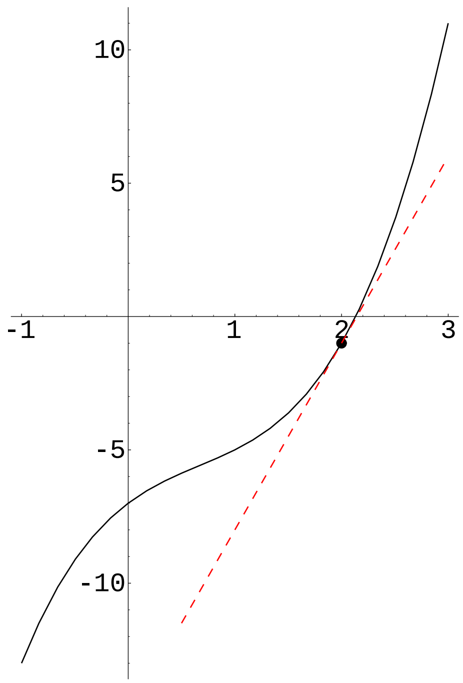

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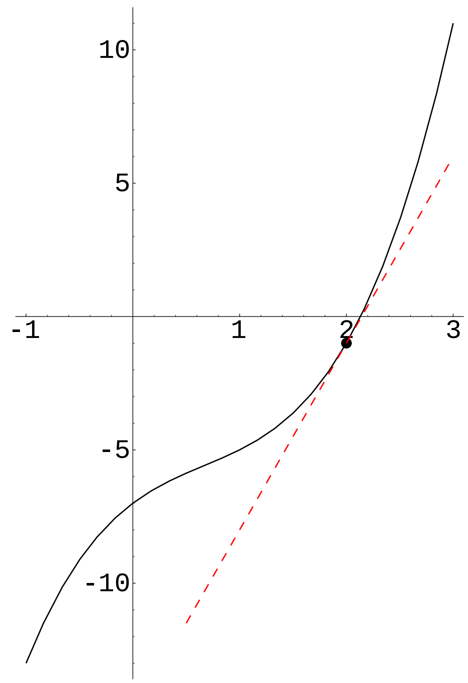

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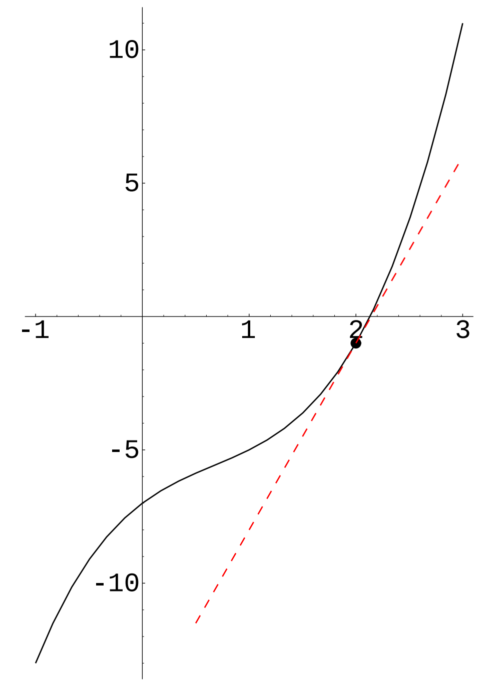

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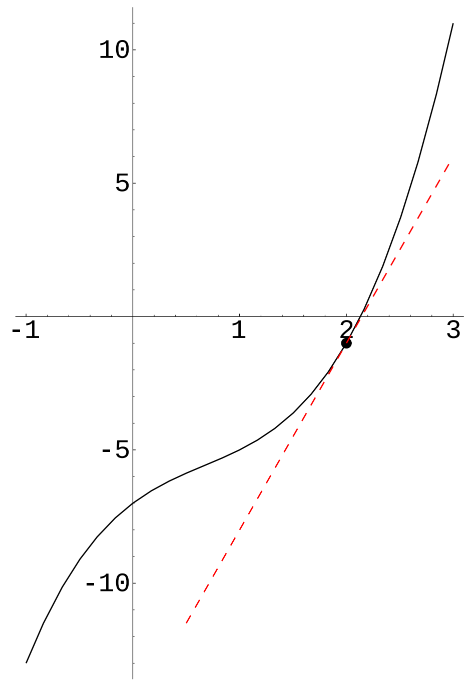

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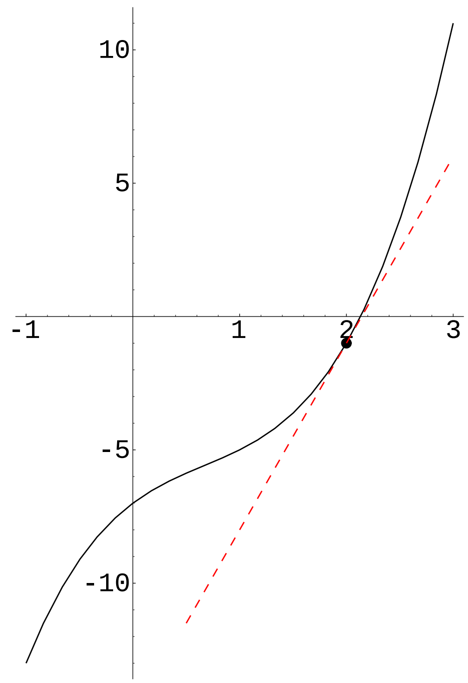

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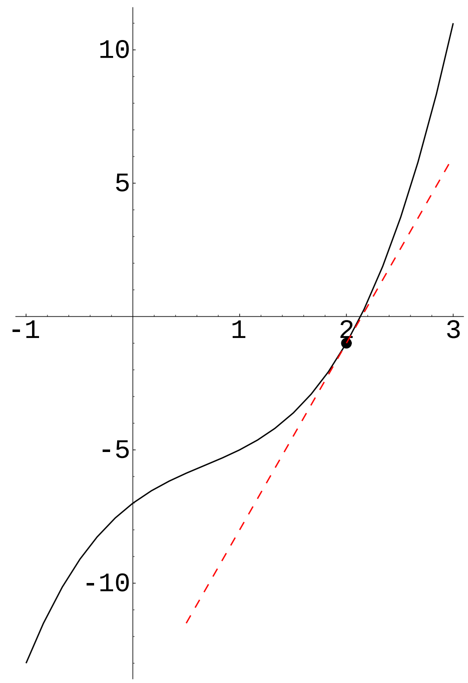

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$7x - y - 15 = 0$ general linear form

Example Find the vertex of the parabola $y = -x^2 - 2x + 3$.

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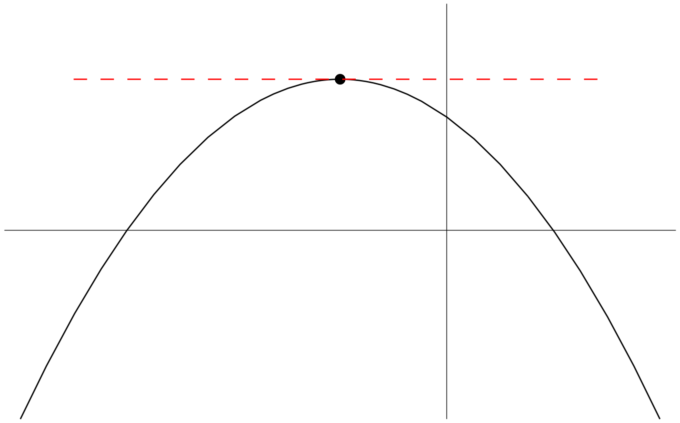

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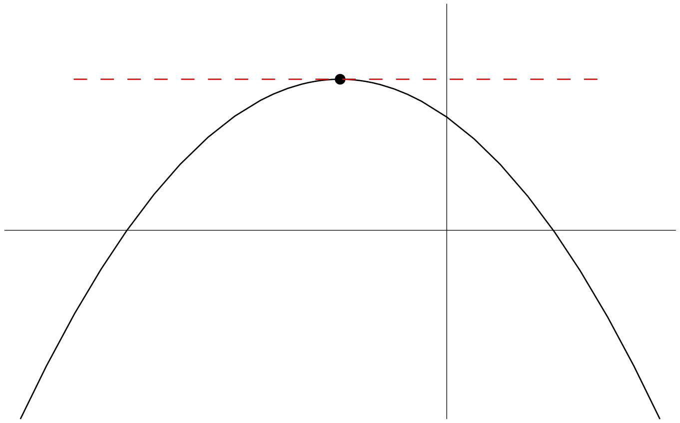

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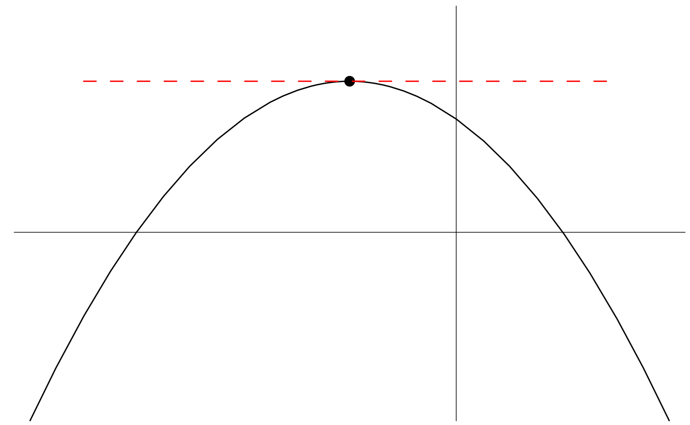

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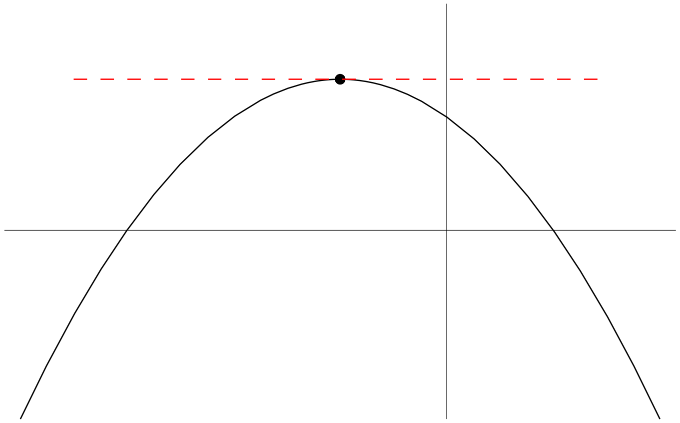

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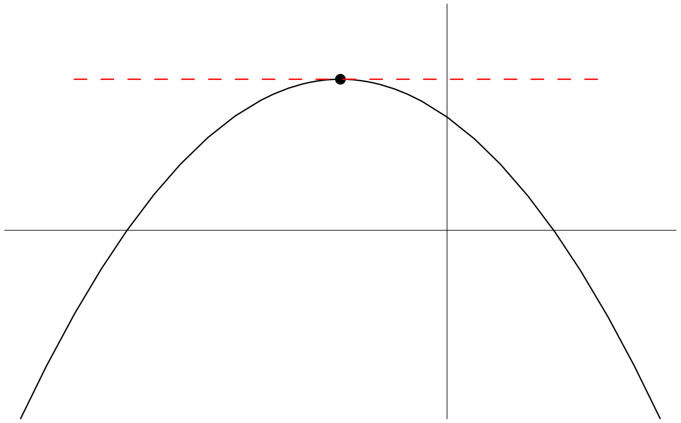

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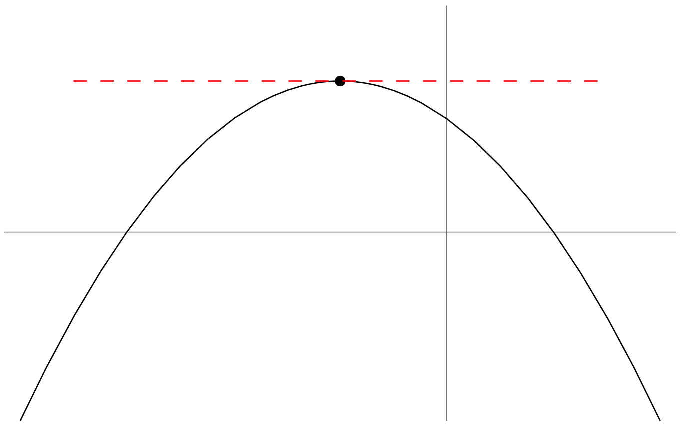

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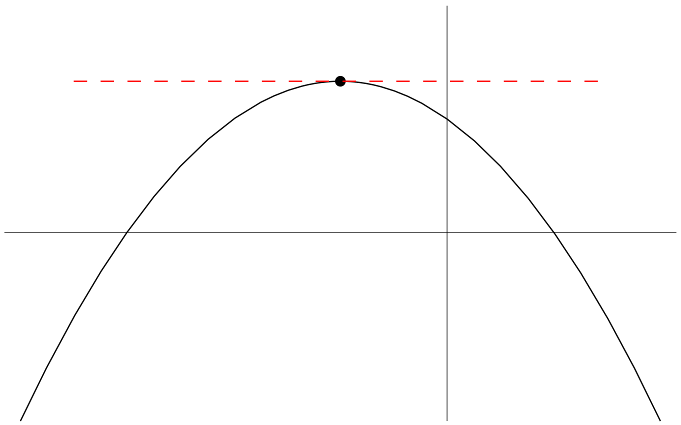

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y-coordinate of vertex $= -x^2 - 2x + 3\Big|_{x=-1}$

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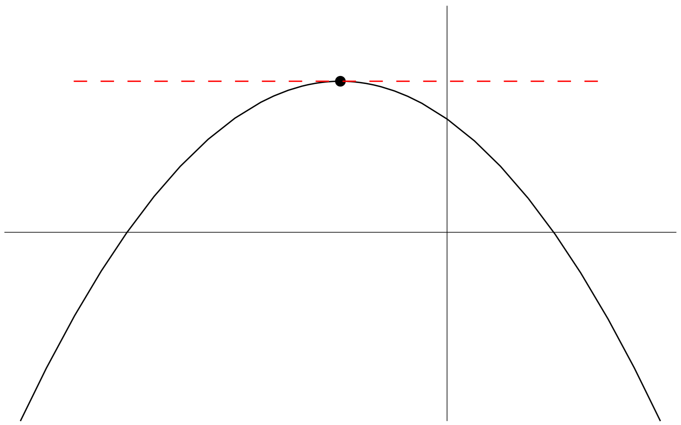

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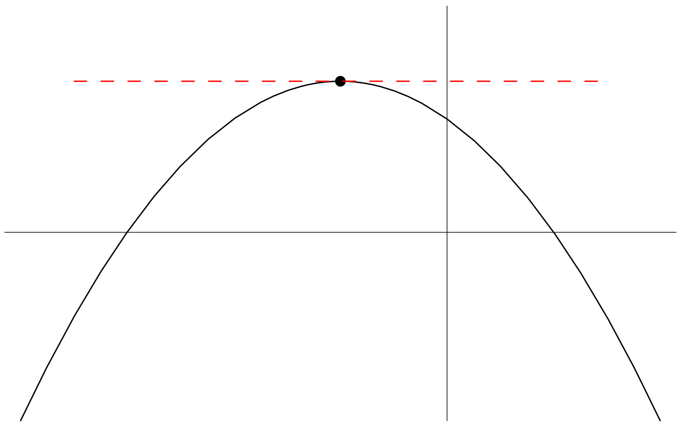

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The vertex is $(-1, 4)$