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- Can also be applied to similar functions: powers of x need not be positive integers.
- Can be applied to *limits at infinity only* *but not* to limits at a point a (where $a \in \mathbb{R}$).

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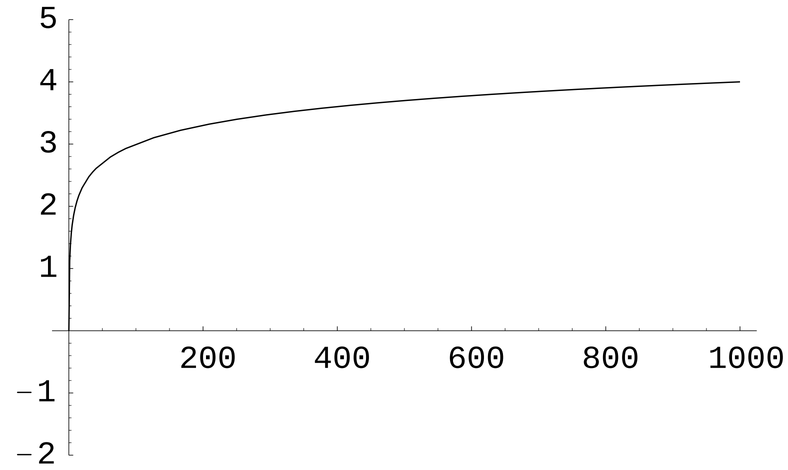

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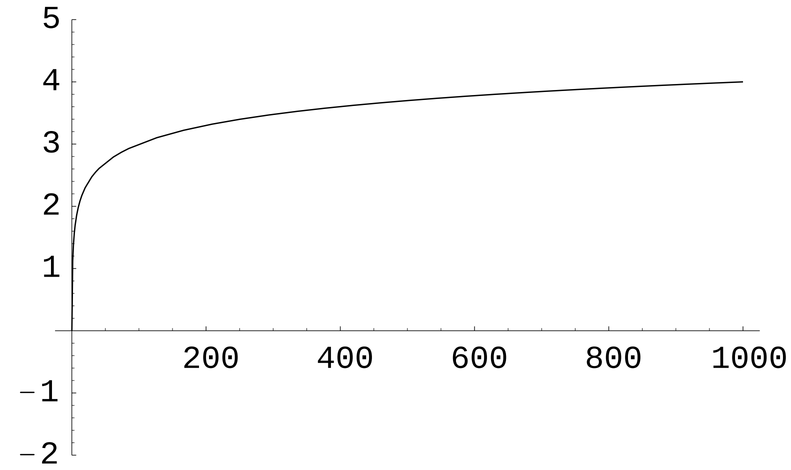

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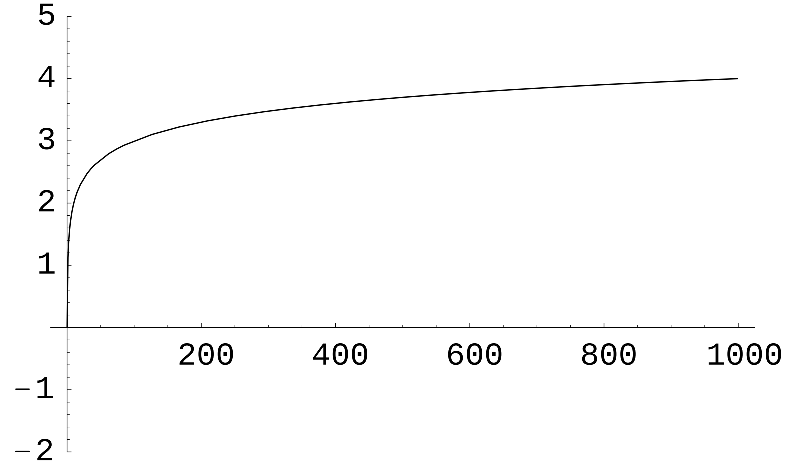

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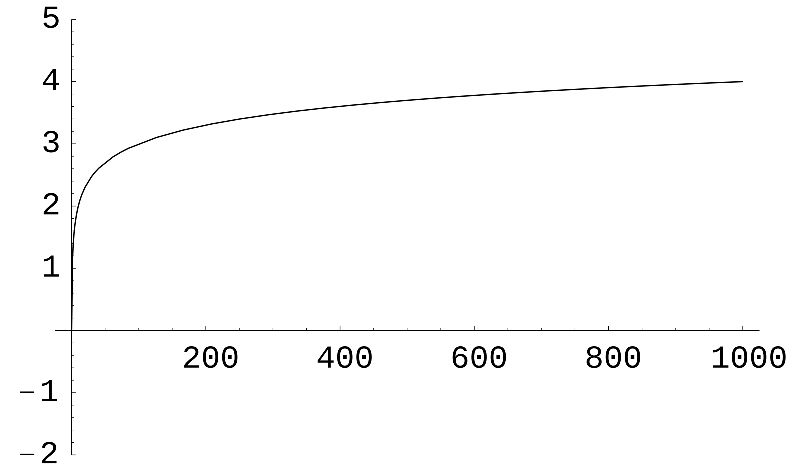

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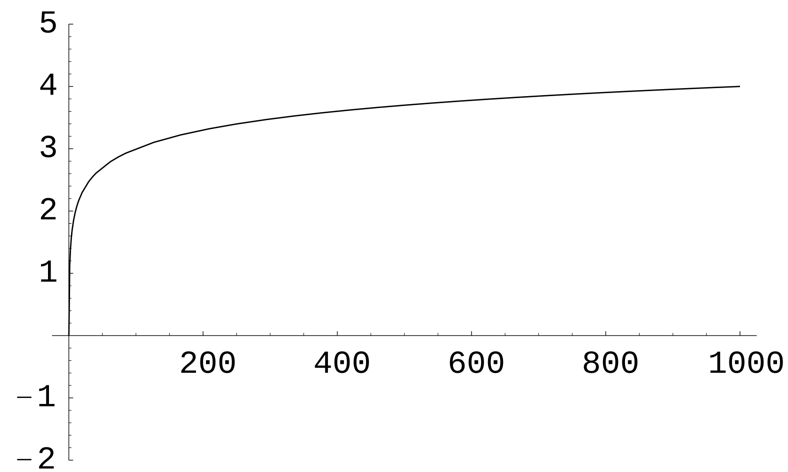

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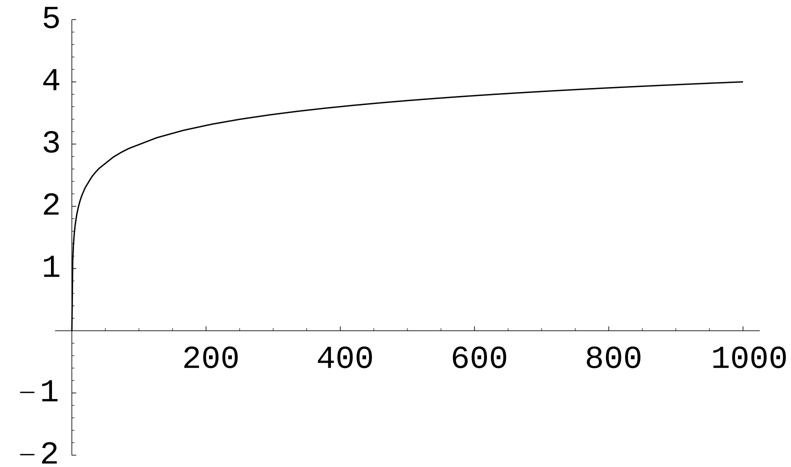

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However, *just a notation*; doesn't mean that limit exists.

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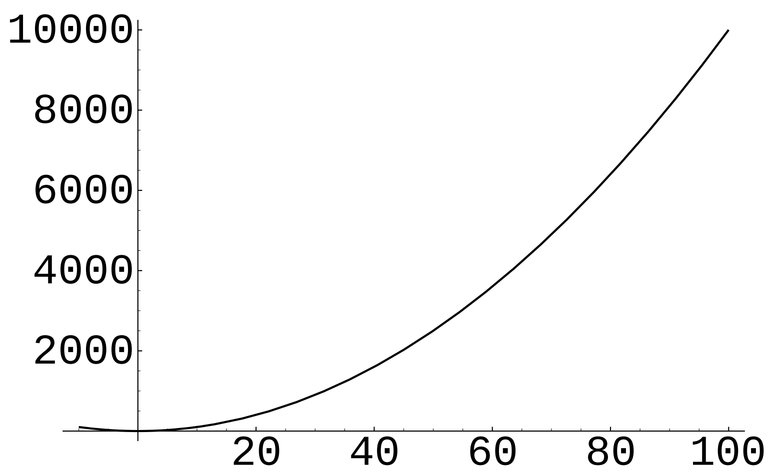

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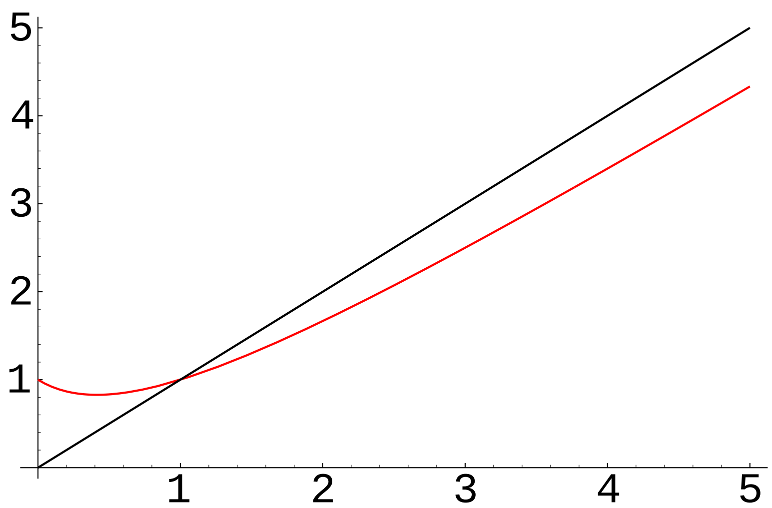

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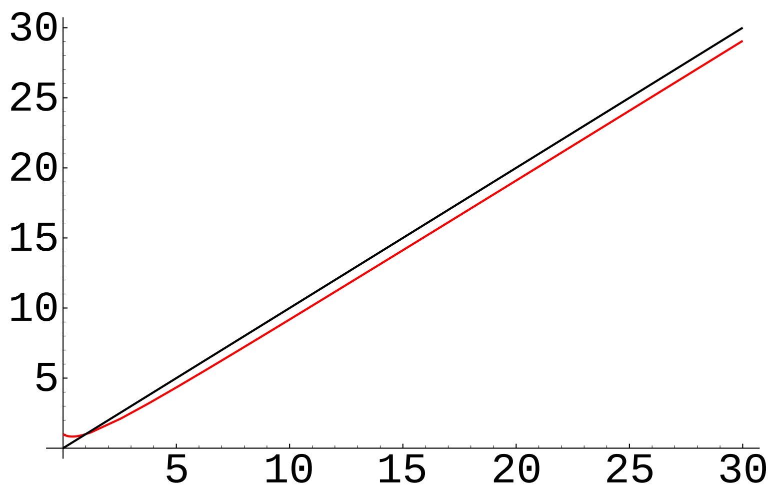

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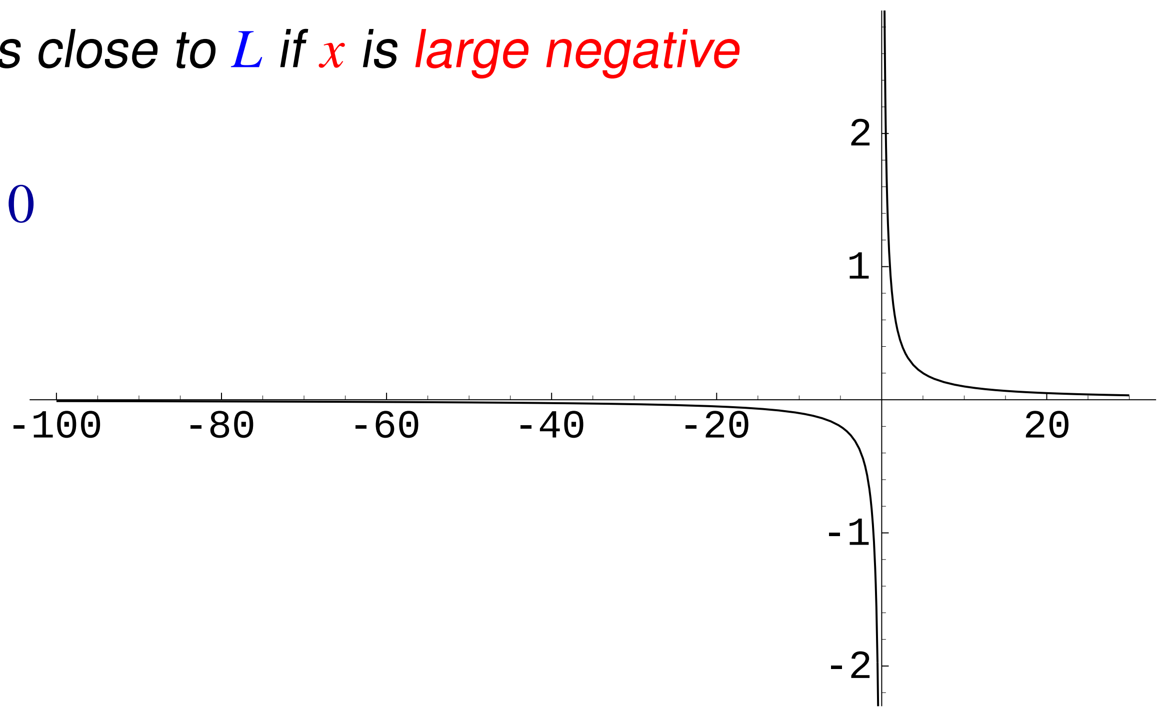

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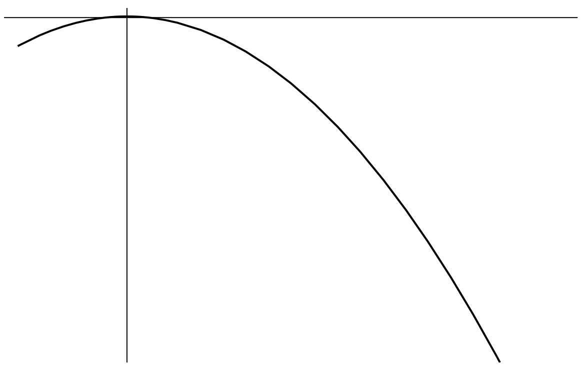

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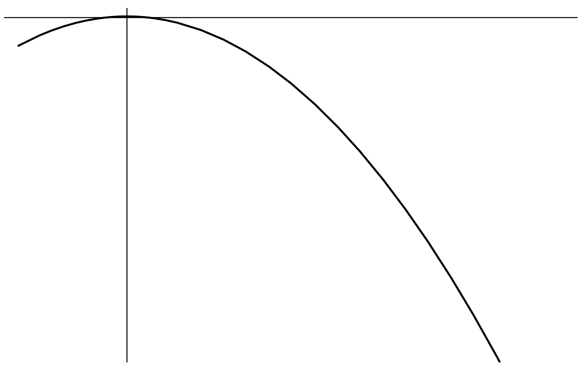

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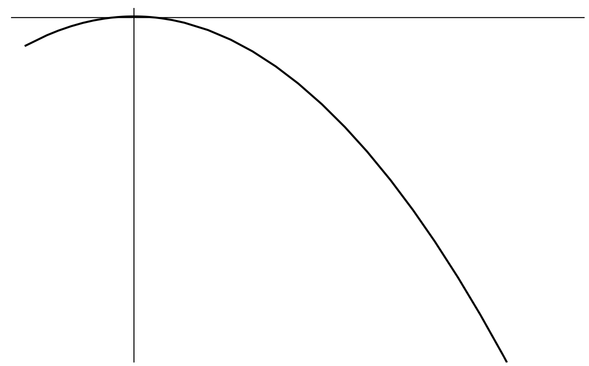

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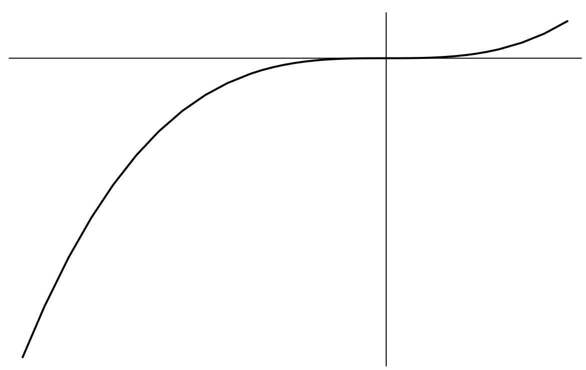

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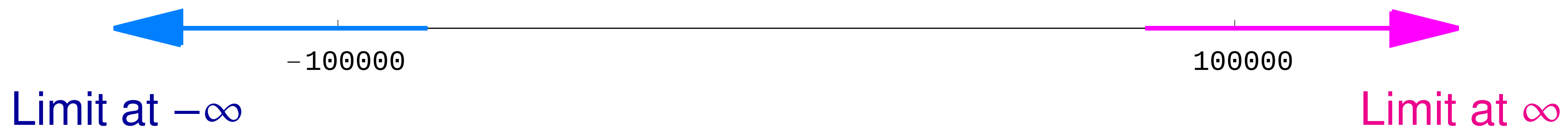

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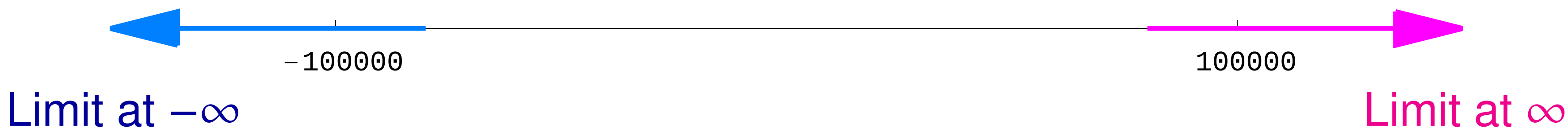

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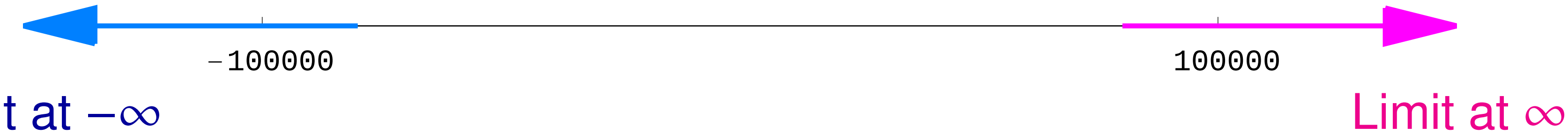

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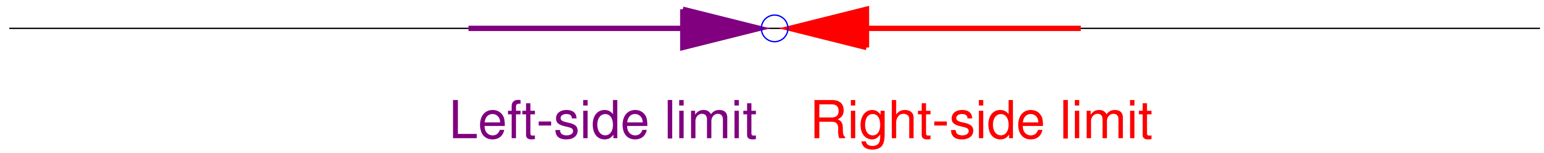

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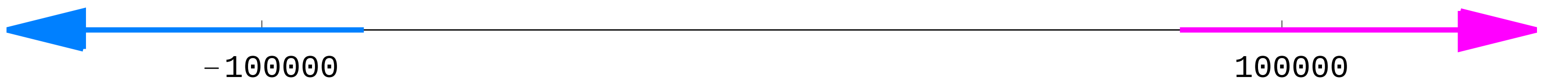

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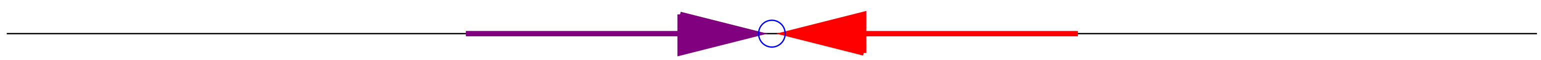

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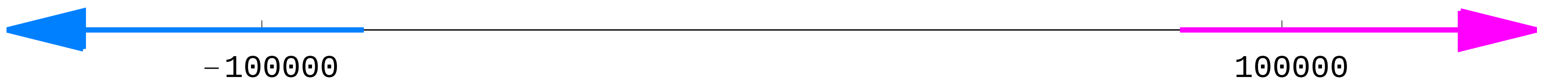

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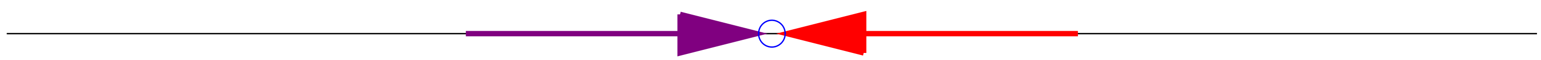

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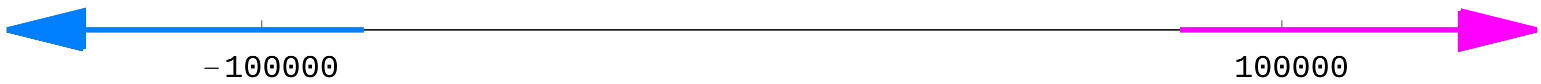

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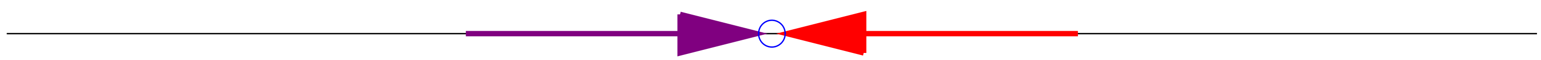

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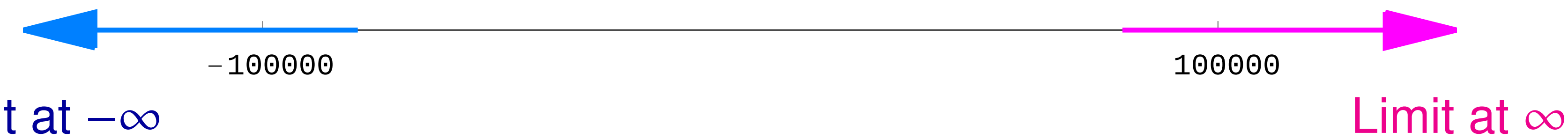

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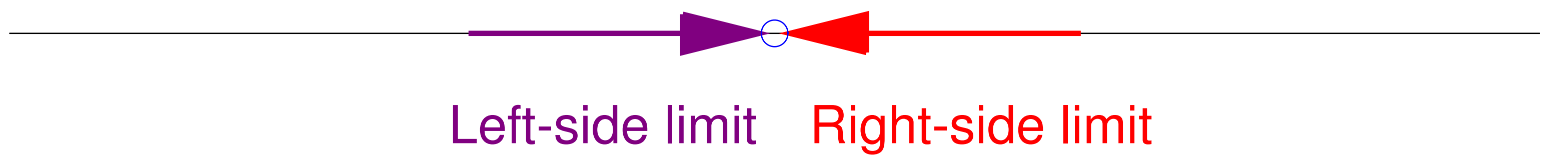

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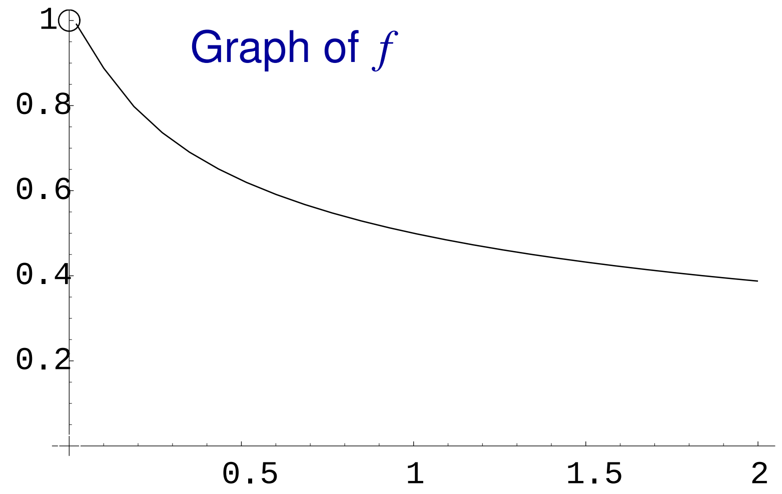

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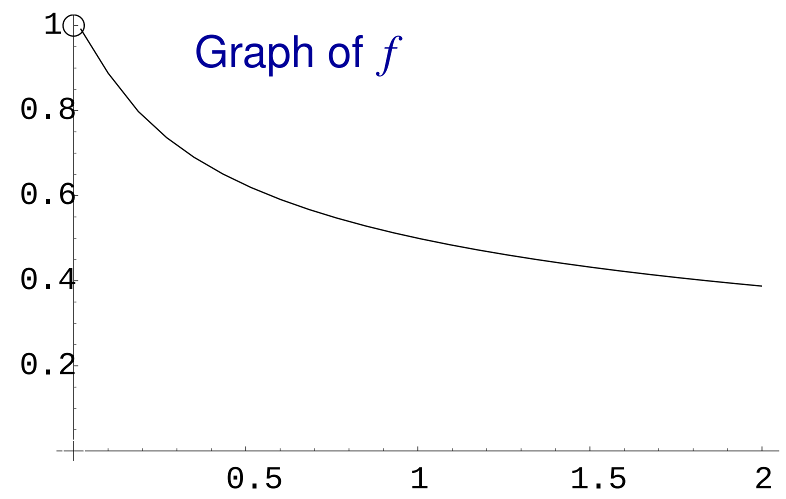

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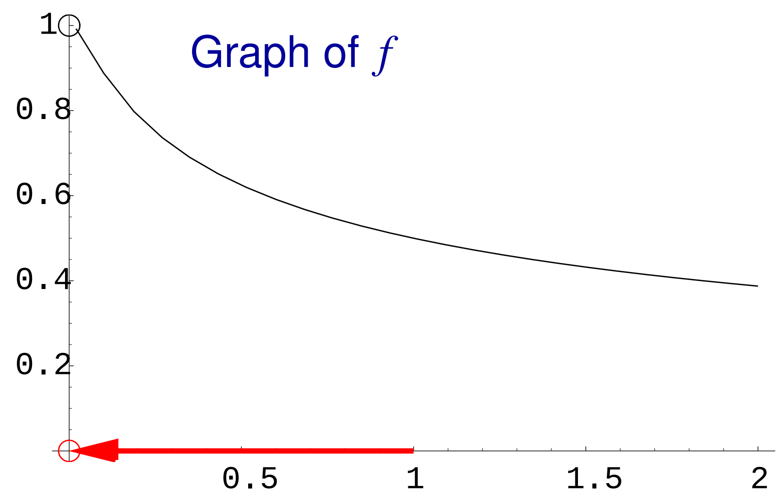

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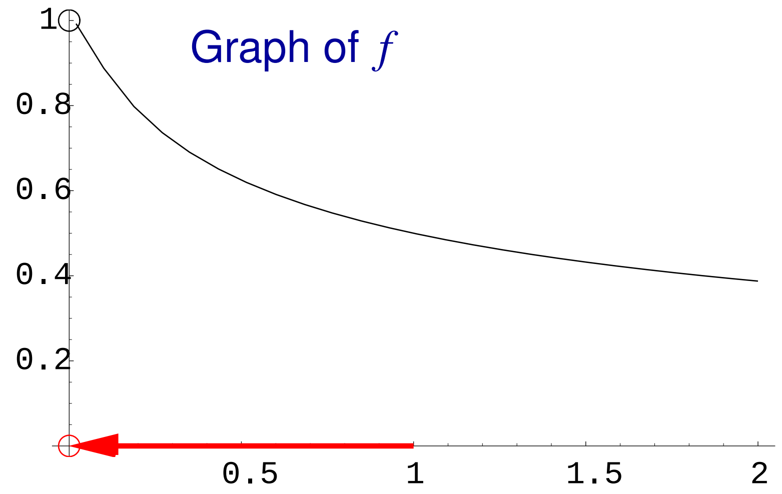

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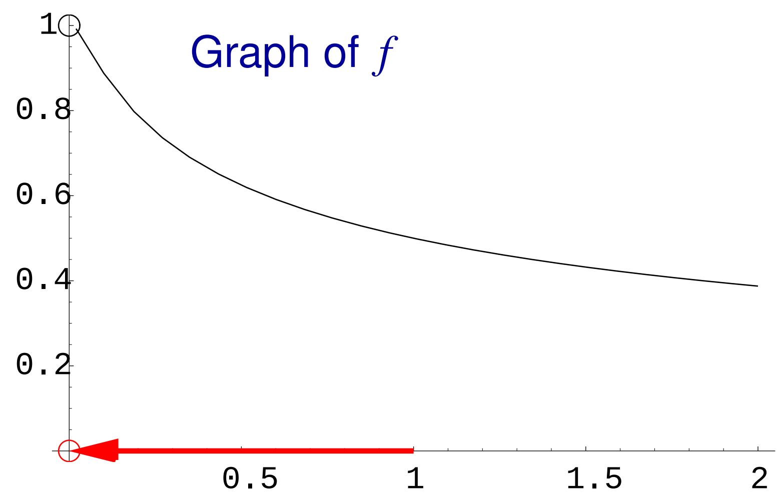

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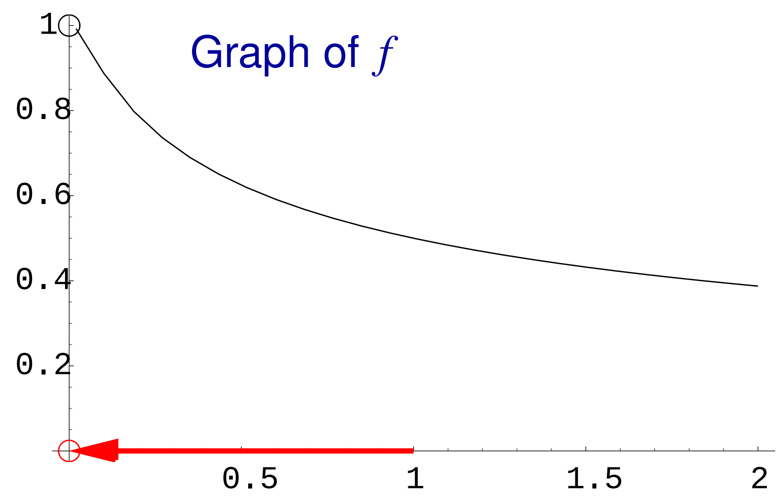

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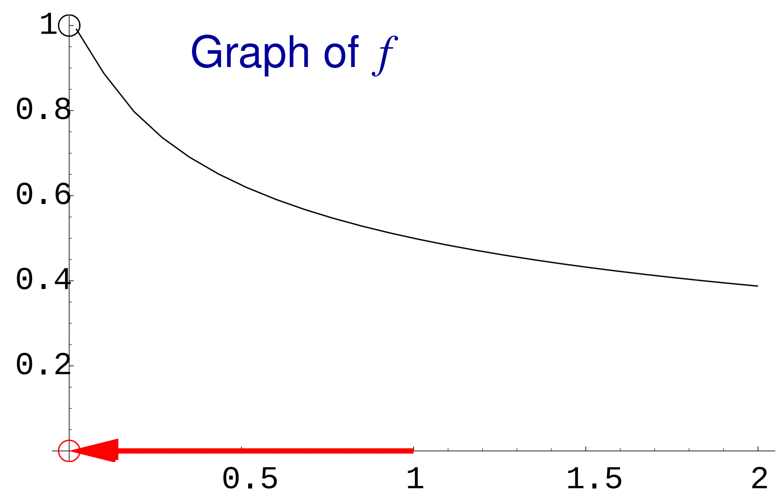

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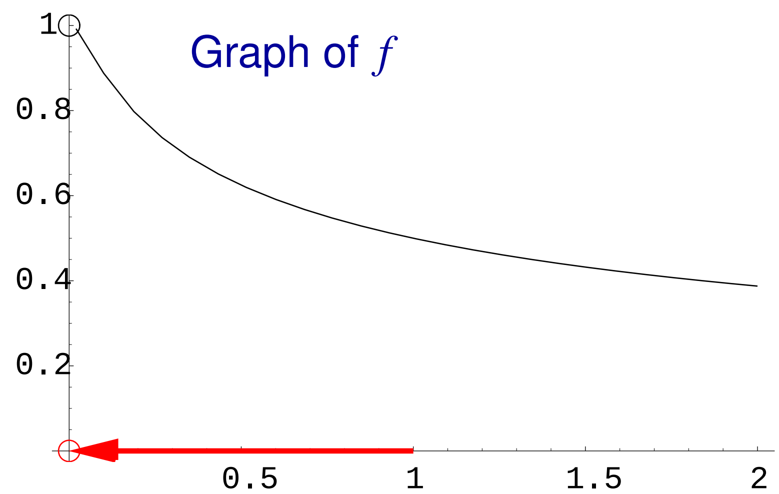

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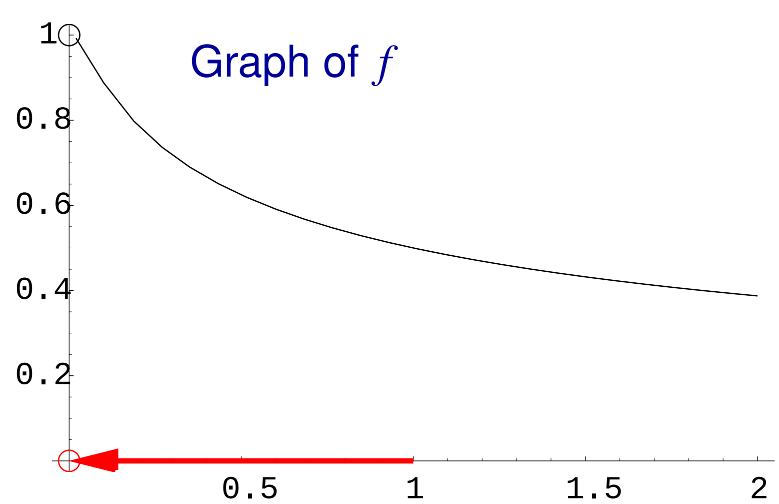

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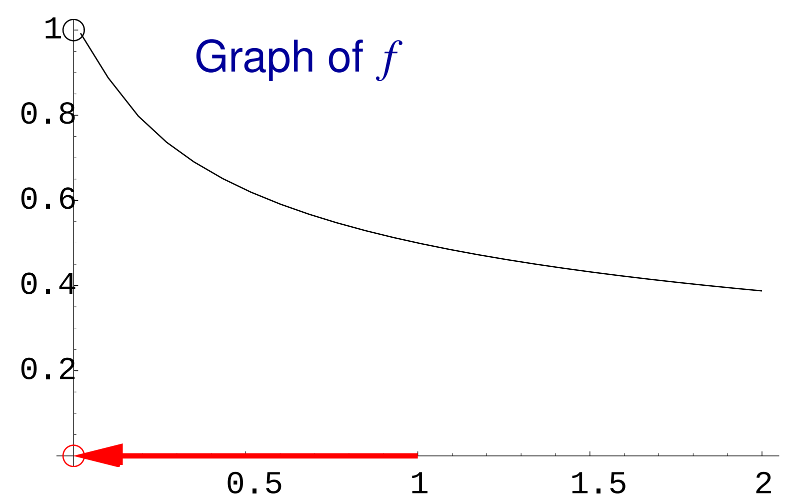

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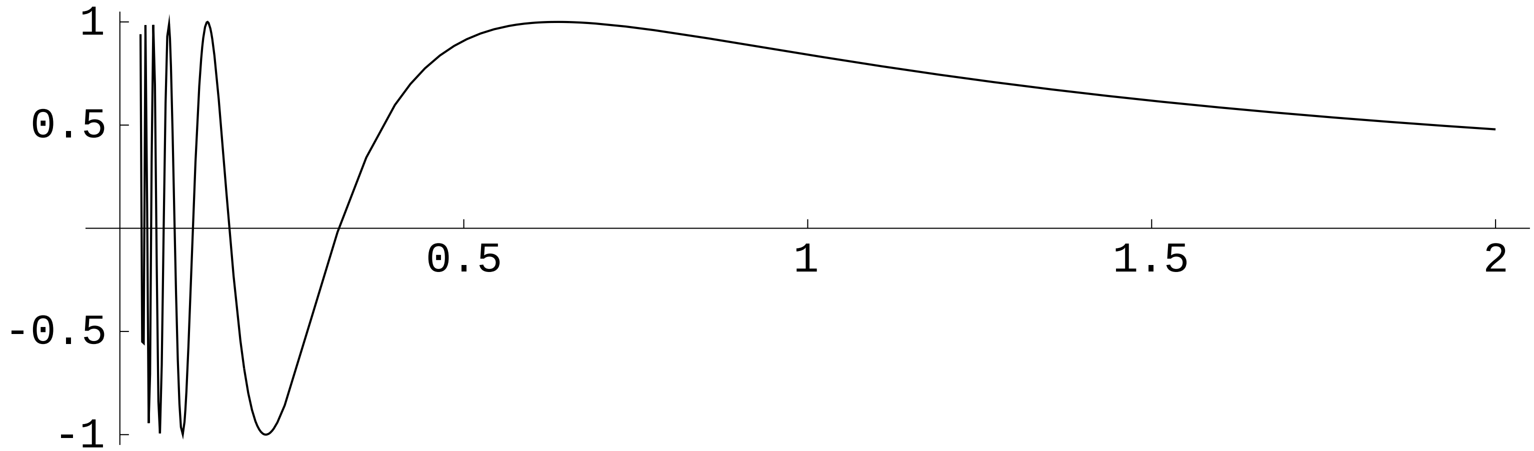

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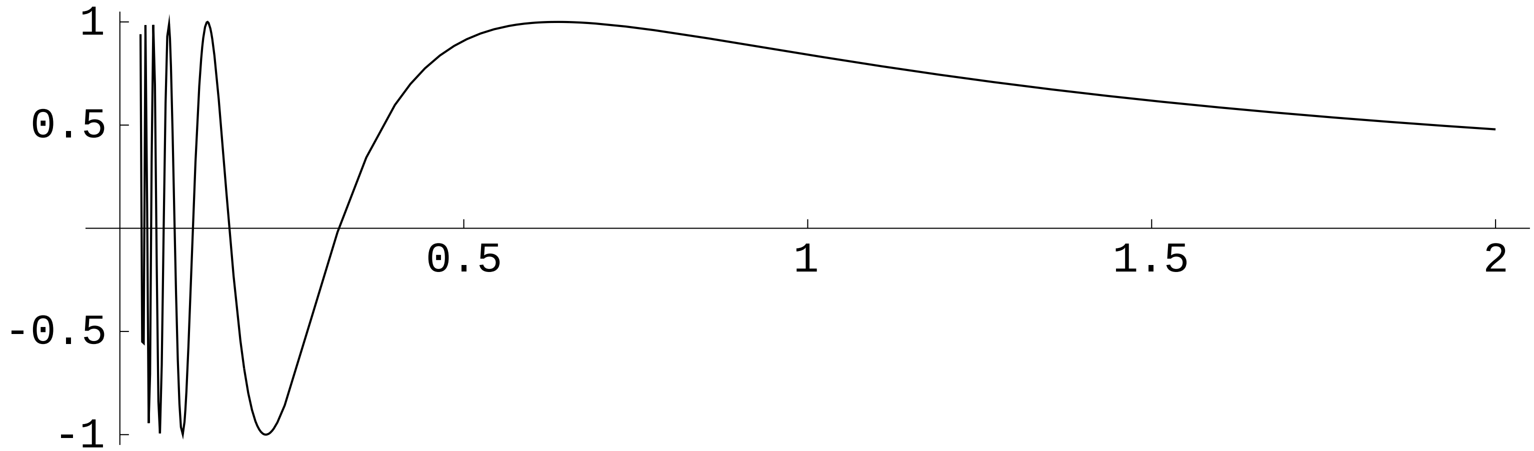

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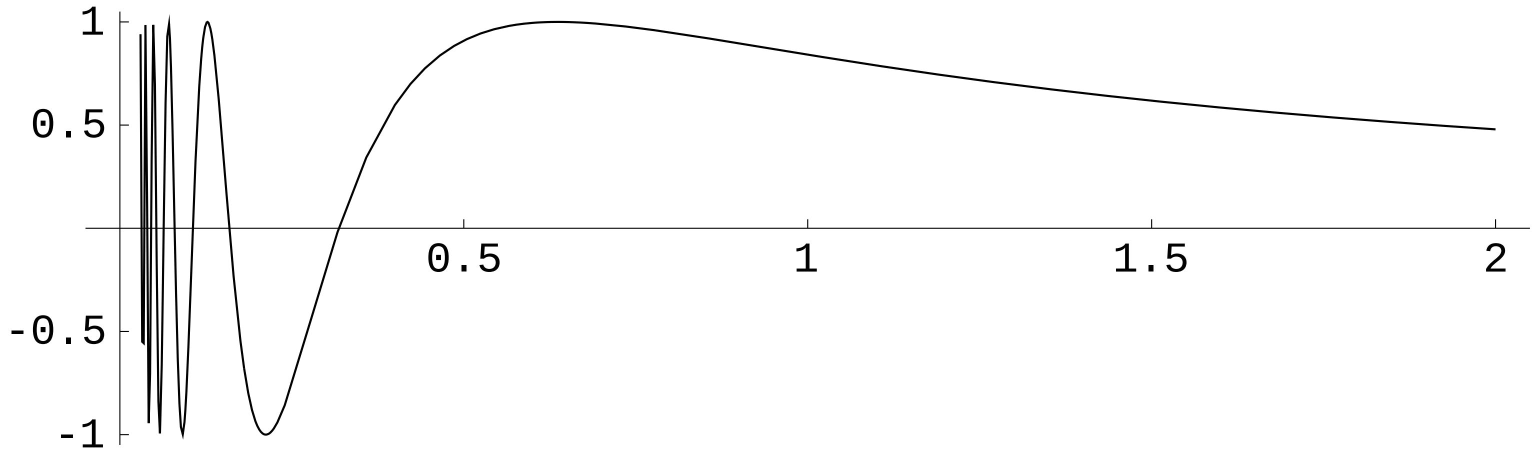

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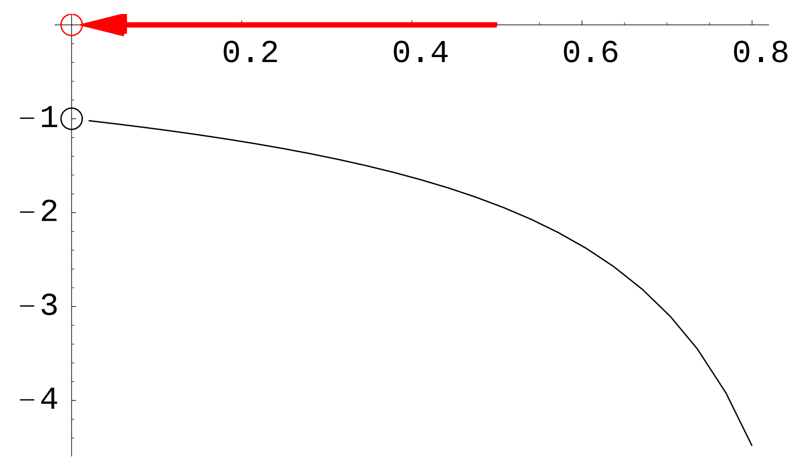

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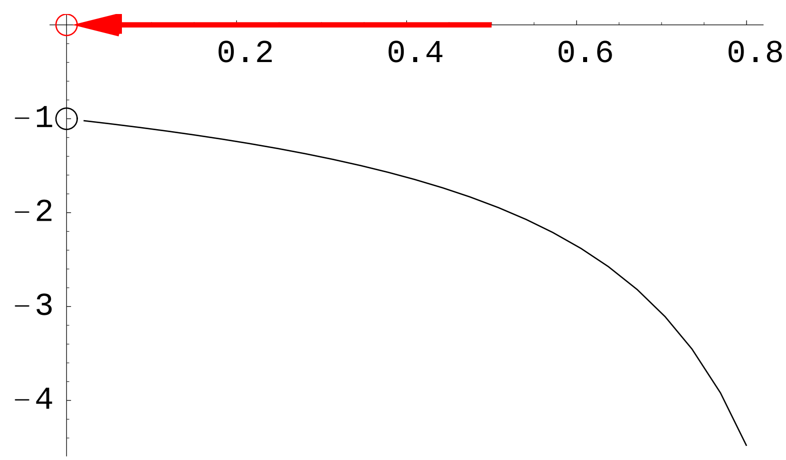

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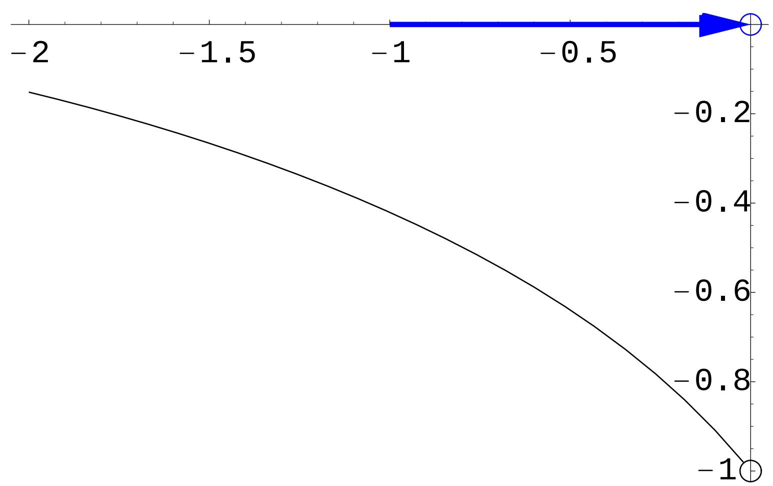

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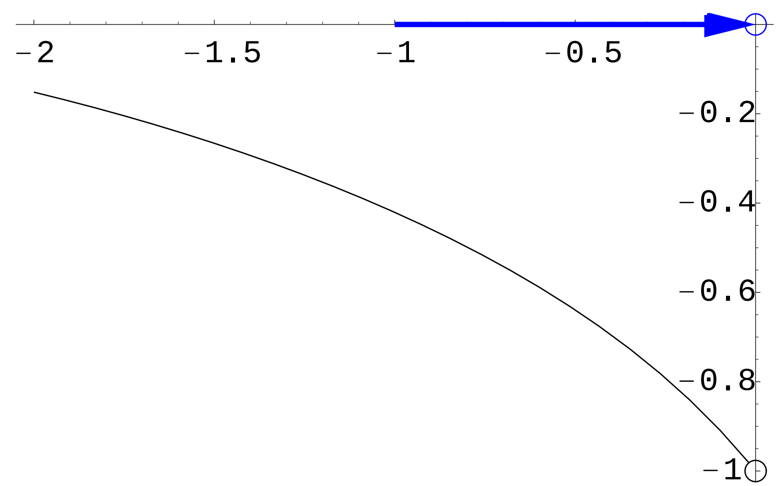

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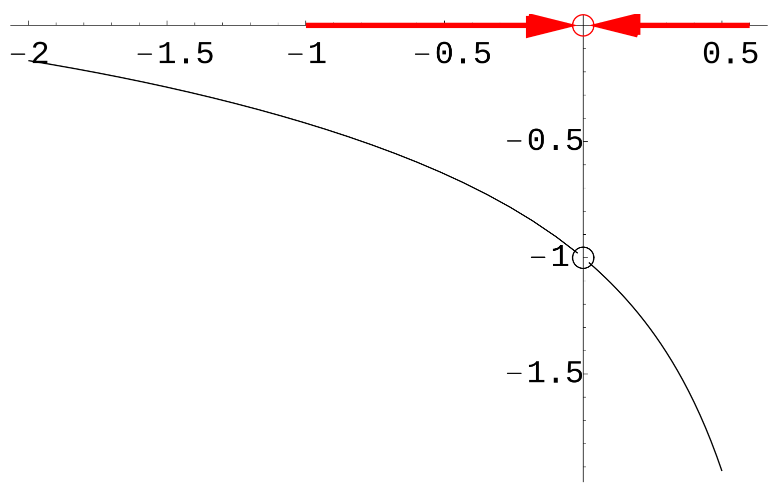
Reason Both the left-side limit and the right-side limit are -1 *(see examples in Section 4)*.



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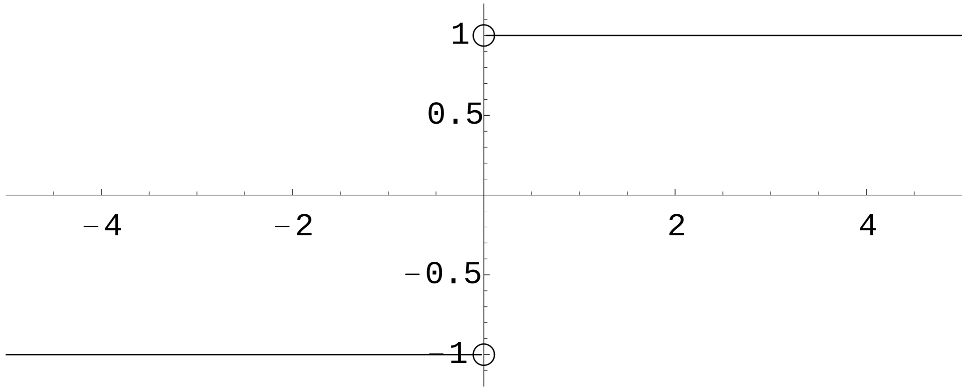

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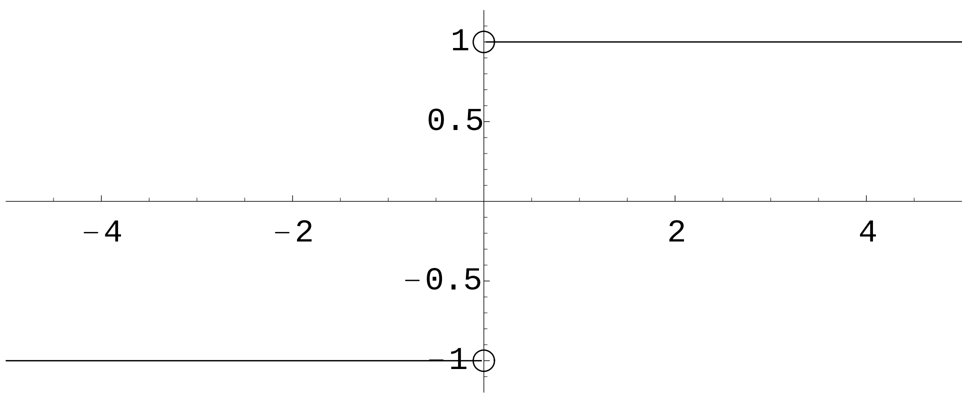

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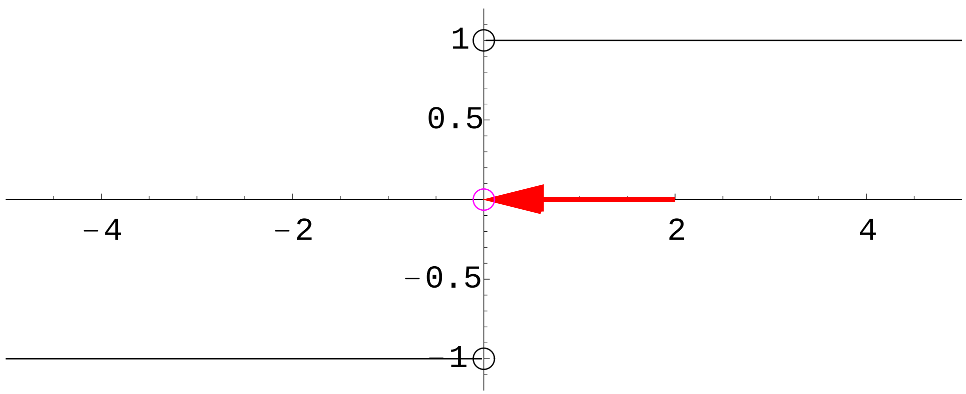

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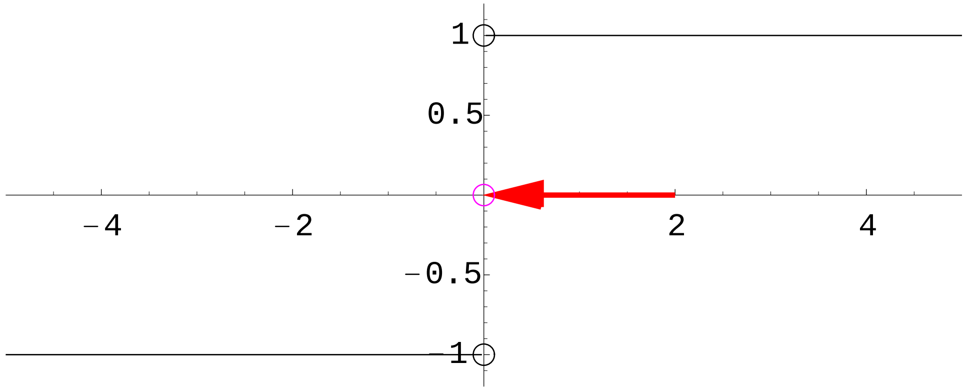

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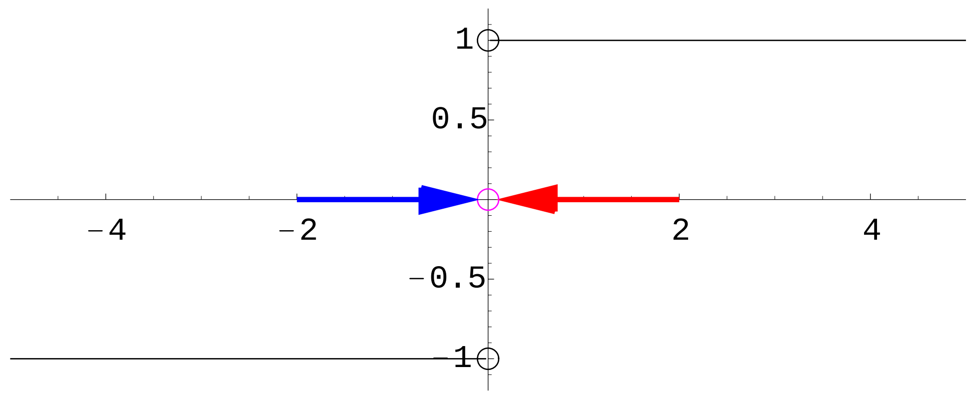

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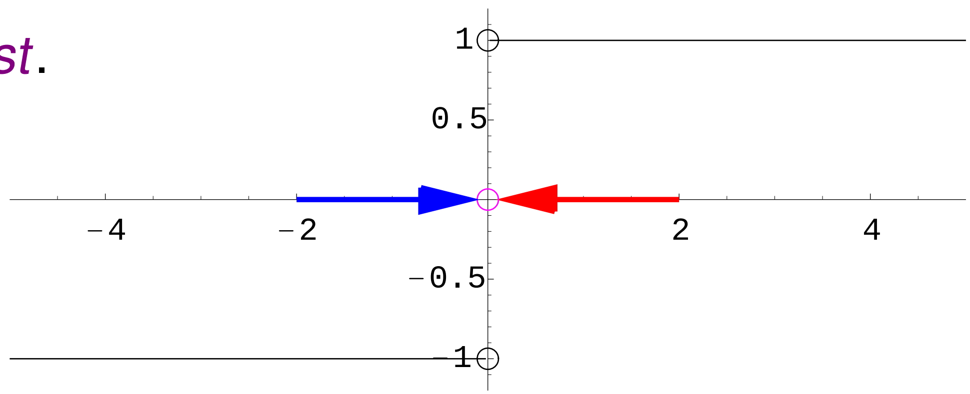

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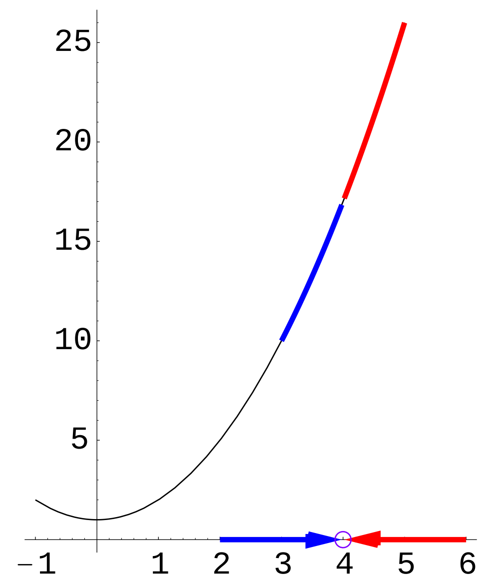

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$$= \frac{2-1}{2^2+2-2}$$

$$= \frac{1}{4}$$

Example Find $\lim_{x\to 2} \dfrac{x-1}{x^2+x-2}$, if exist.

Solution

$$\lim_{x\to 2} \frac{x-1}{x^2+x-2} = \frac{\lim_{x\to 2}(x-1)}{\lim_{x\to 2}(x^2+x-2)} = \frac{2-1}{2^2+2-2} = \frac{1}{4}$$

Remark Can use substitution ∵ *rational functions are* continuous *in their domain.*