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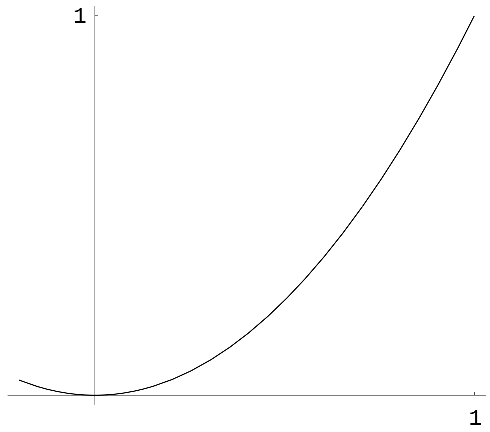

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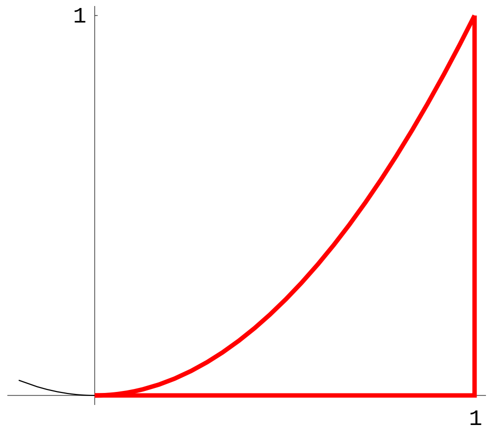

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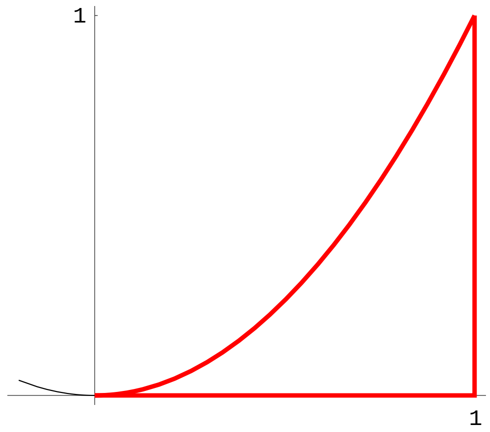


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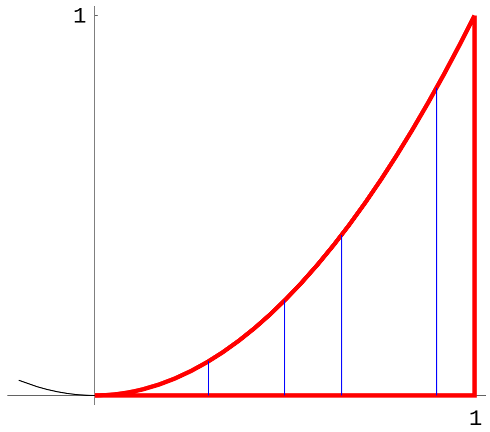


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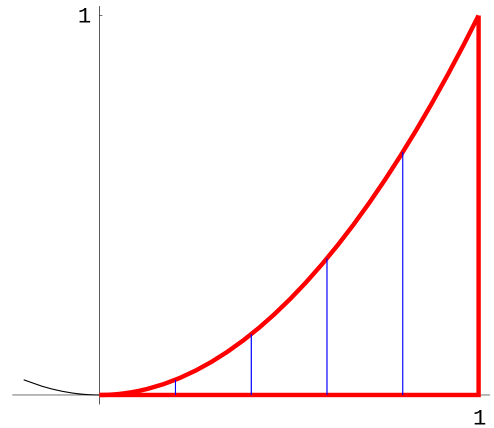


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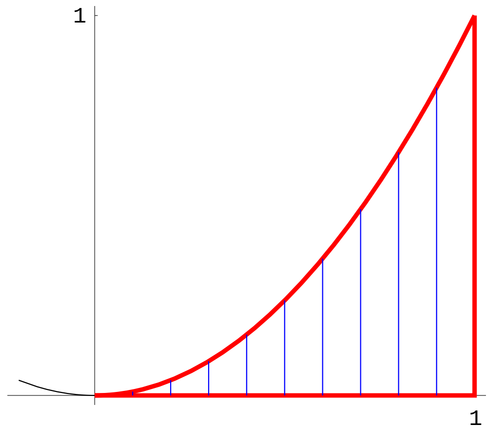


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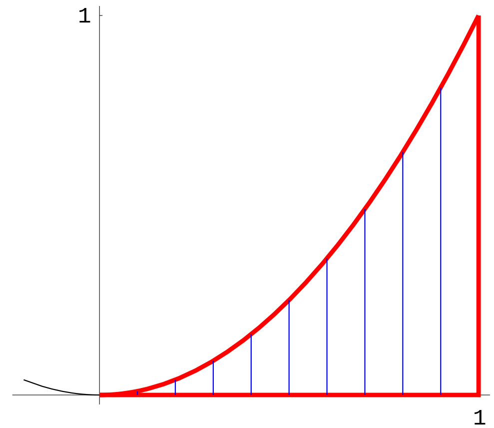


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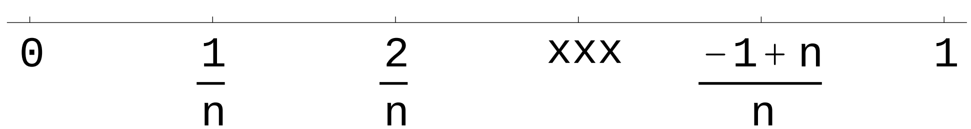
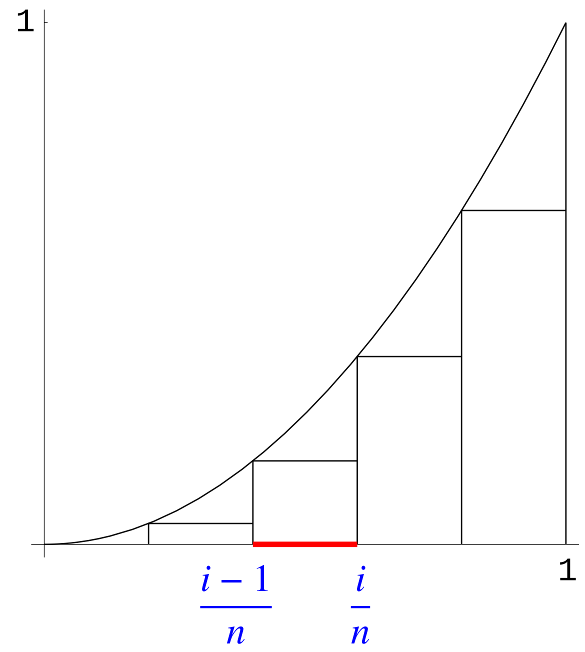
Similar to Problem 1, use approximation to find/define the area.

- Divide $[0, 1]$ into finitely many subintervals of equal lengths

$$\left[0, \tfrac{1}{n}\right], \quad \left[\tfrac{1}{n}, \tfrac{2}{n}\right], \quad \left[\tfrac{2}{n}, \tfrac{3}{n}\right], \quad \ldots, \quad \left[\tfrac{n-1}{n}, 1\right]$$



- For each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$, consider the rectangular region with base on the subinterval and height $\left(\frac{i-1}{n}\right)^2$ *(the largest region that lies under the curve)*.



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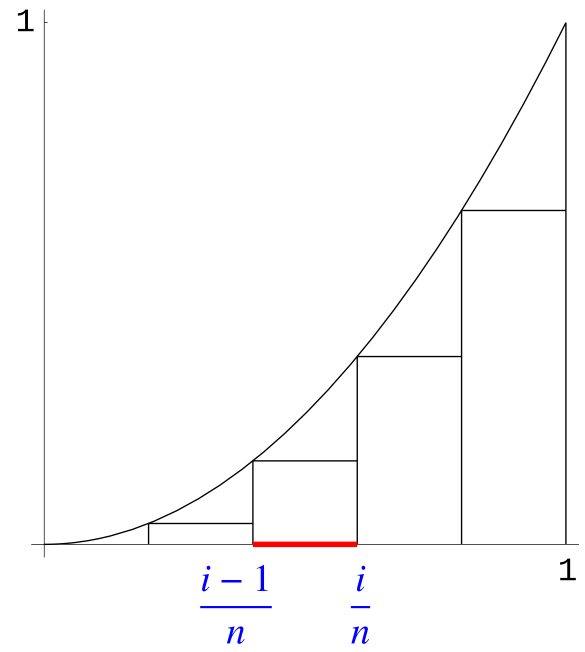

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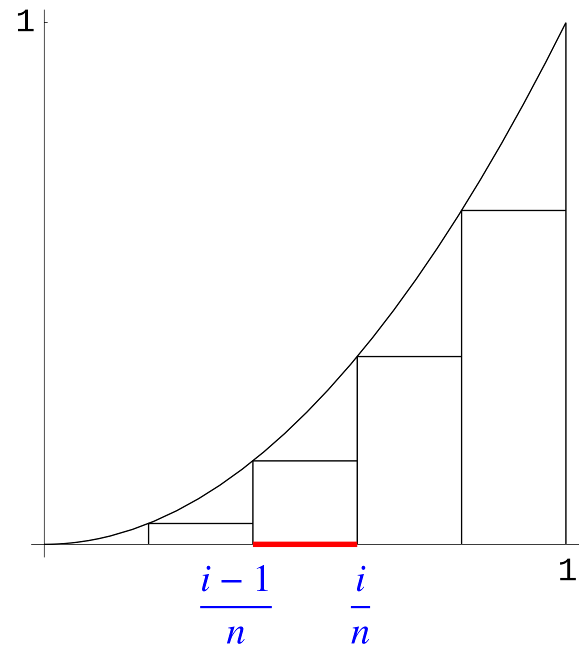

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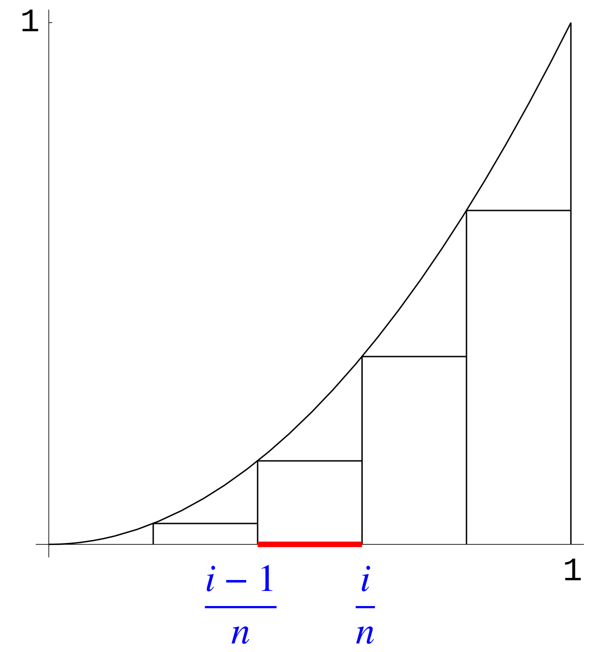
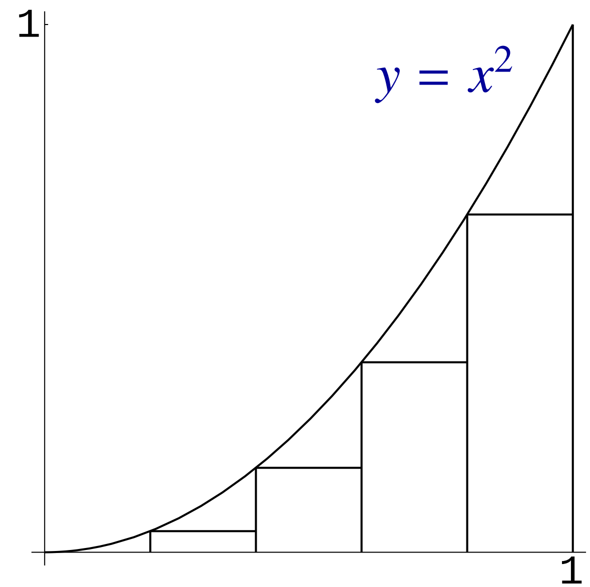
- Show area.html



Sum of the areas of the rectangular regions (correct to 3 decimal places)

| $n$ | Sum of areas |
|---|---|
| 2 | 0.125 |
| 3 | 0.185 |
| 4 | 0.219 |
| 5 | 0.240 |
| ⋮ | |
| 10 | 0.285 |
| ⋮ | |
| 100 | 0.328 |
| ⋮ | |
| 500 | 0.332 |
| ⋮ | show area.xls |



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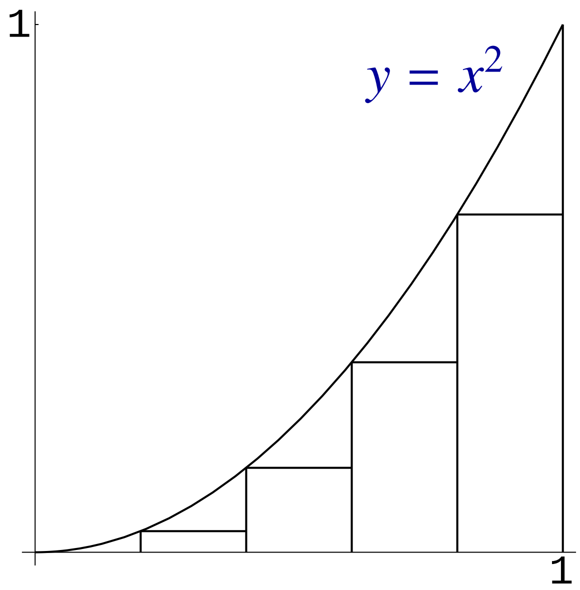


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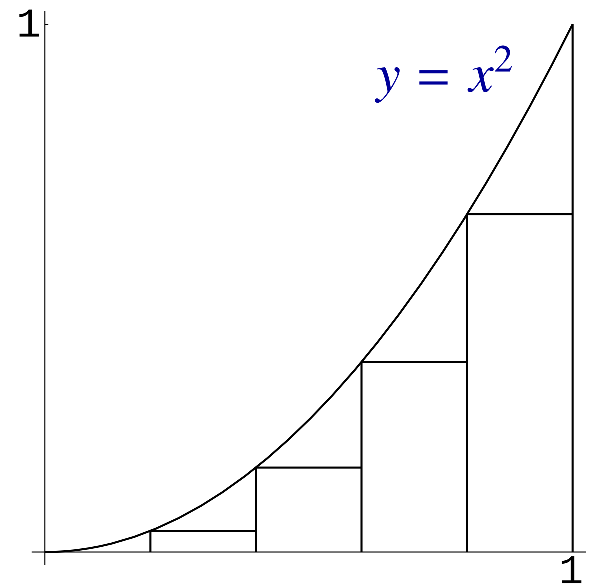


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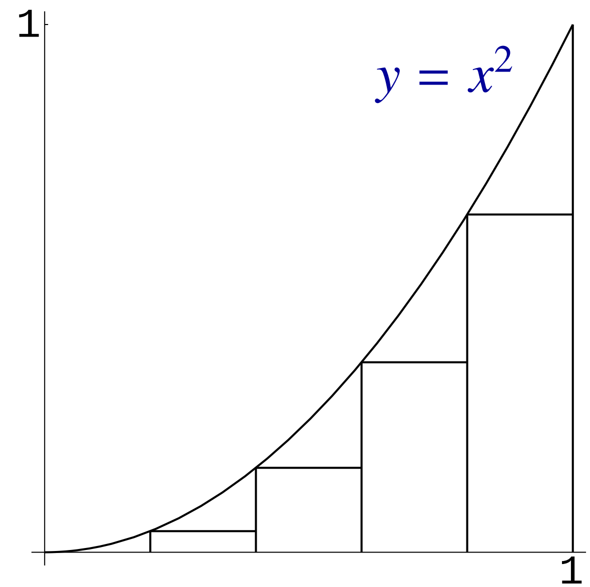


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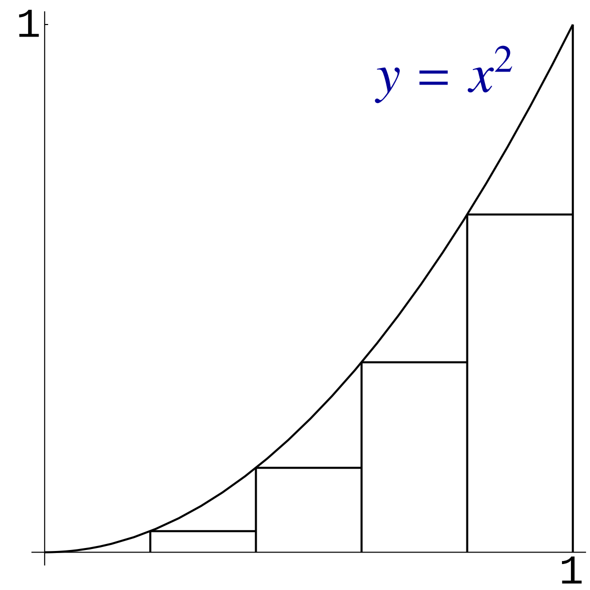


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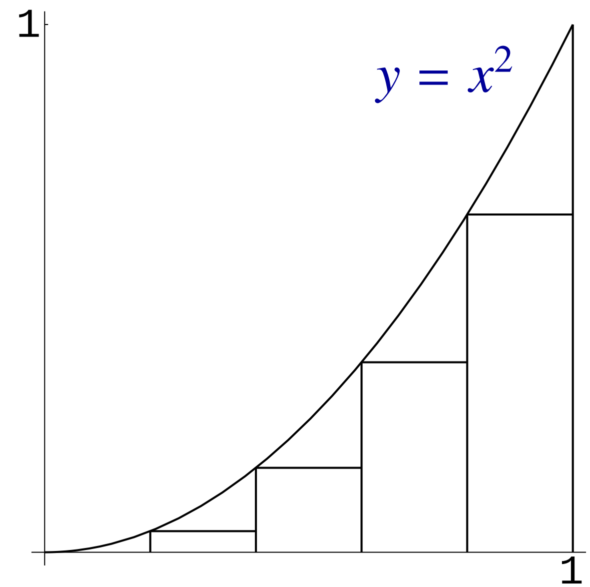


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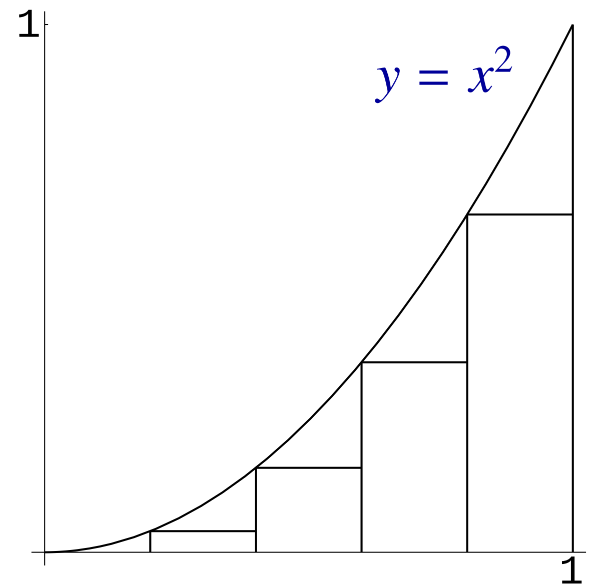


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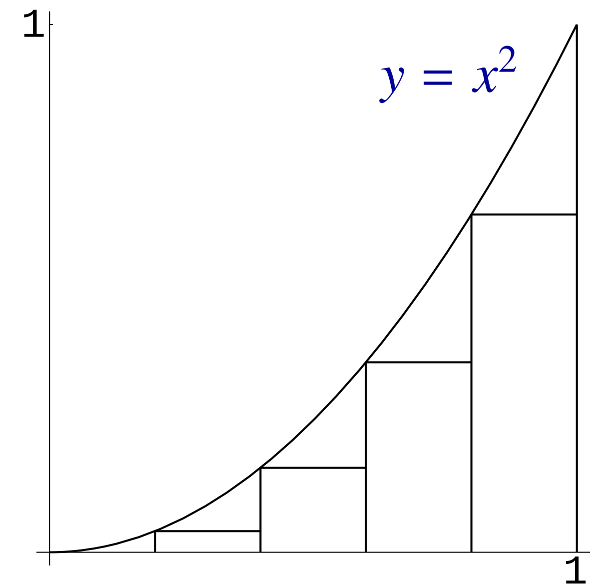


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$$= \frac{6}{25}$$

In general, if there are $n$ subintervals, the sum $S_n$ is

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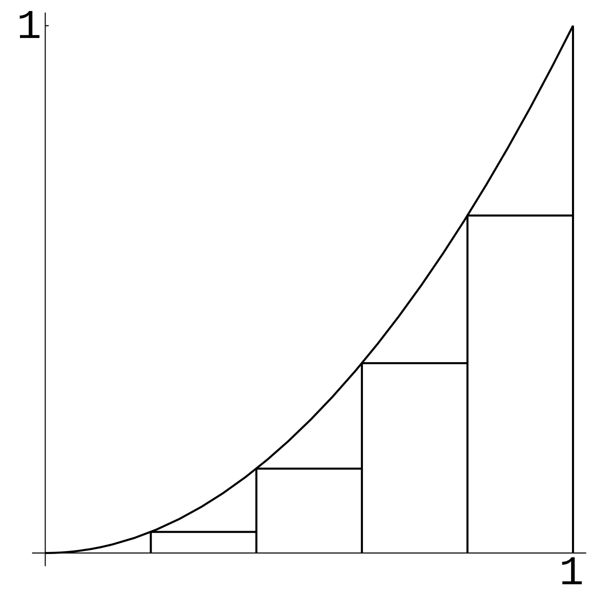

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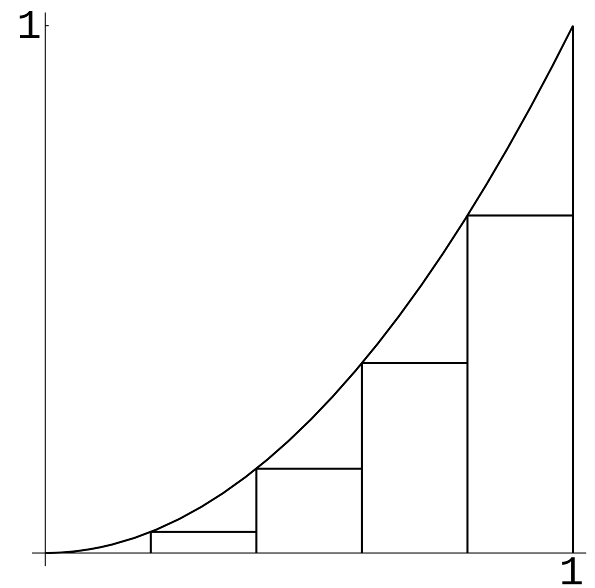

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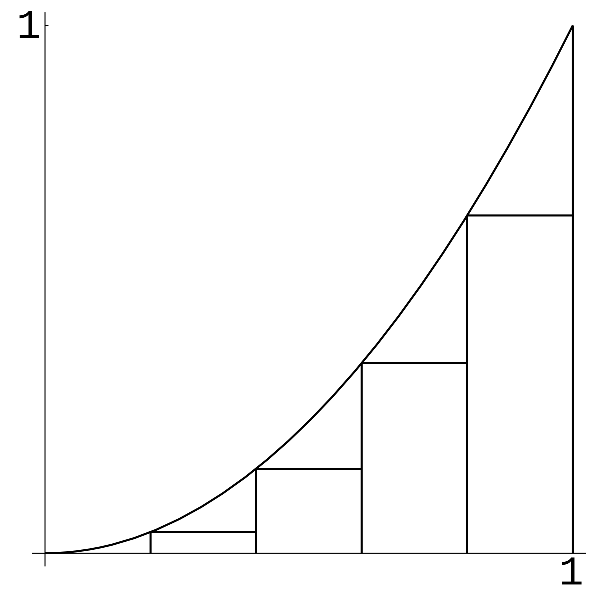

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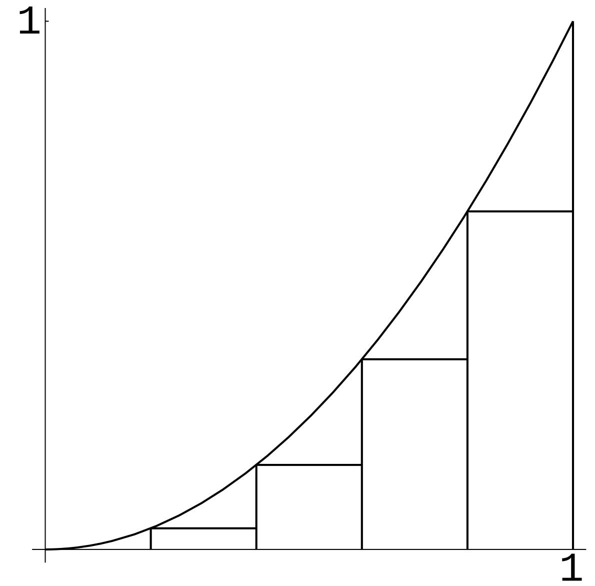

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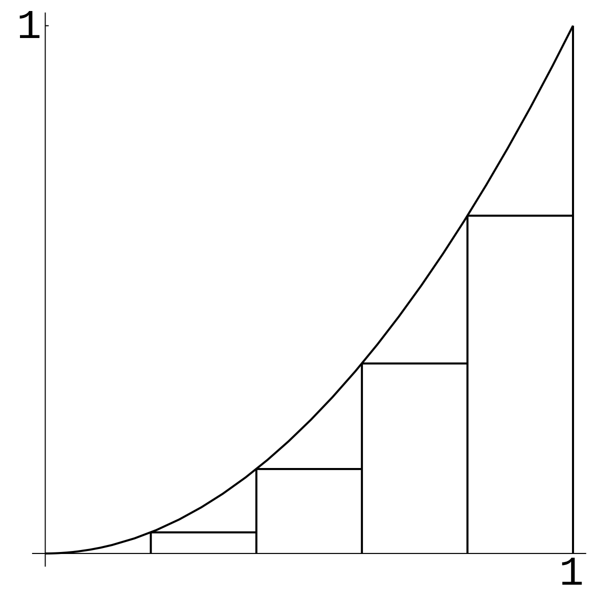

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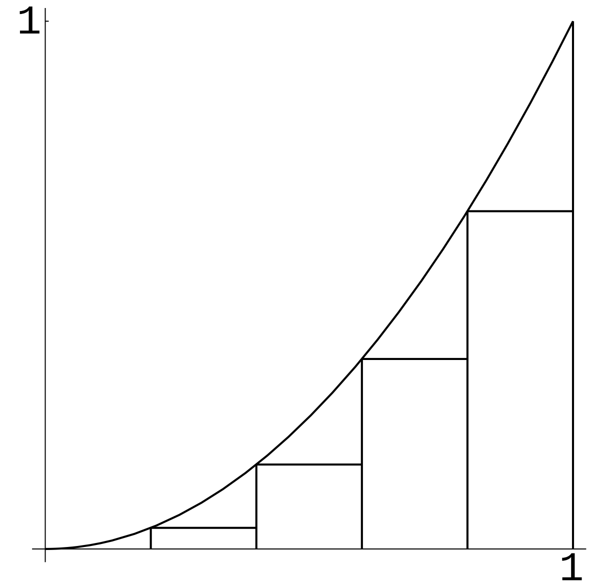


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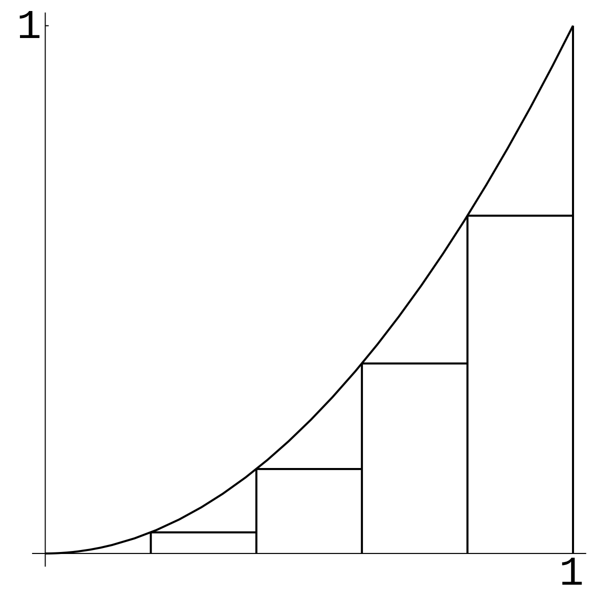


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**Conclusion** The required area is $\frac{1}{3}$.

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or $(a_n)_{n=0}^{\infty}$ or simply $(a_n)$.

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Show sequence.html

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- The points $(n, a_n)$ approach the horizontal line $y = L$.

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**Remark** The precise meaning of the "+" part of (1) is

*if both $(a_n)$ and $(b_n)$ are convergent and their limits are $L$ and $M$ respectively, then $(a_n + b_n)$ is also convergent and its limit is $L + M$.*

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- Can't use the property of limit for quotients because the *limit of the numerator* *(and the denominator)* *does not exist*.

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- Can't use the property of limit for quotients because the *limit of the numerator (and the denominator) does not exist*.
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- *Trick – divide the numerator and the denominator by* $n^2$

*Solution* $$\lim_{n\to\infty} \frac{n^2+1}{2n^2-3} = \lim_{n\to\infty} \frac{(n^2+1)/n^2}{(2n^2-3)/n^2}$$

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**Shortcut** Throw away the numbers $1$ and $-3$ in the numerator and denominator respectively.

$$\lim_{n\to\infty}\frac{n^2+1}{2n^2-3} = \lim_{n\to\infty}\frac{n^2}{2n^2} =$$

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**Shortcut** Throw away the numbers $1$ and $-3$ in the numerator and denominator respectively.

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*Solution*
$$\lim_{n\to\infty}\frac{n^2+1}{2n^2-3} = \lim_{n\to\infty}\frac{(n^2+1)/n^2}{(2n^2-3)/n^2}$$

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*Reason* If $n$ is very large, then compared with $n^2$, $1$ is very small.

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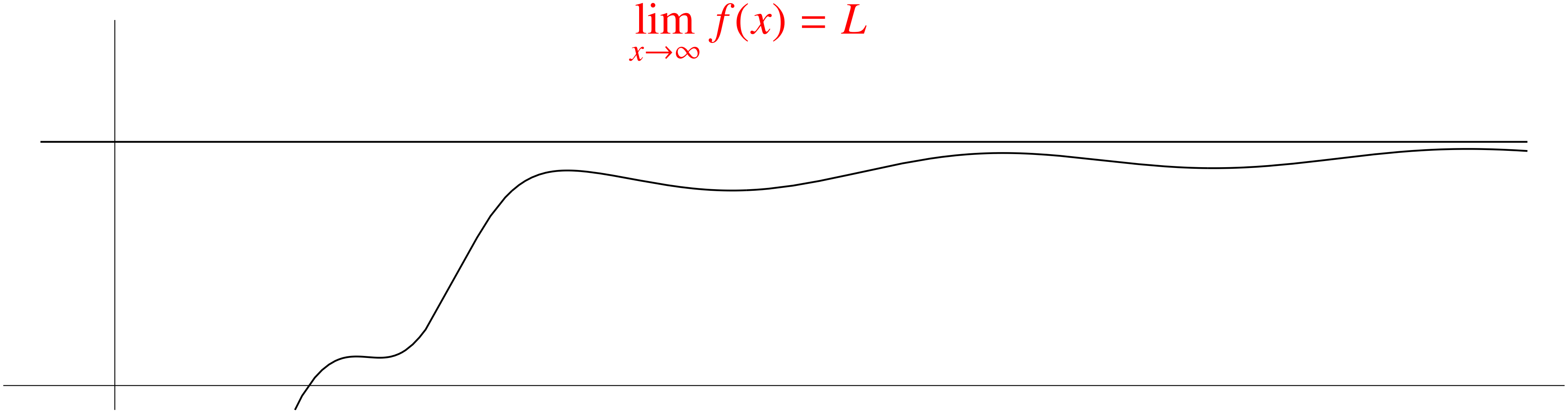

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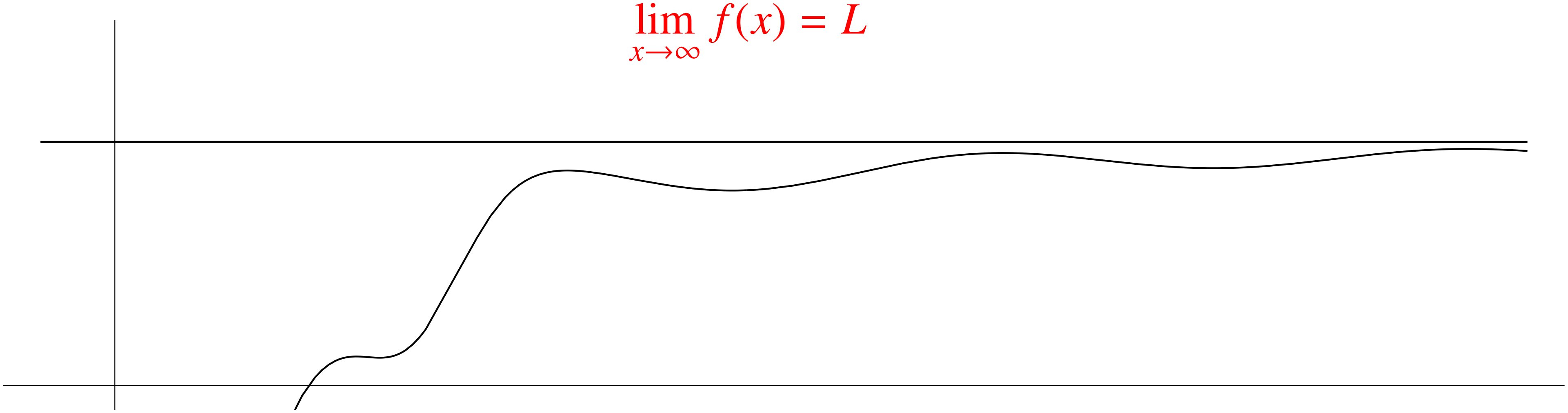

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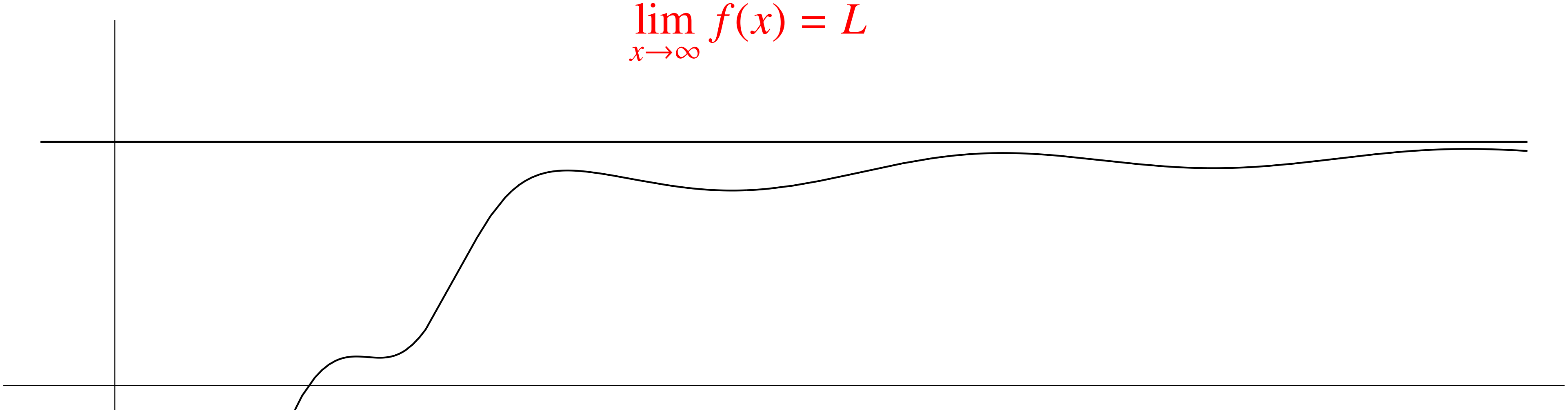

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**Remark** For sequences, just say limit.

For functions, say limit at infinity $\because$ *there are other types of limits*

**Example** Find the following limits, if exist:

(1) $$\lim_{x\to\infty}\left(1-\frac{2}{x^3}\right)$$

(2) $$\lim_{x\to\infty}(2^{-x}+3)$$

(3) $$\lim_{x\to\infty}\frac{\sin x}{x}$$

(4) $$\lim_{x\to\infty}\frac{x^2+1}{3x^3-4x+5}$$

(5) $$\lim_{x\to\infty}\frac{x^3+1}{3x^3-4x+5}$$

(6) $$\lim_{x\to\infty}(1+\log x)$$

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*Solution*

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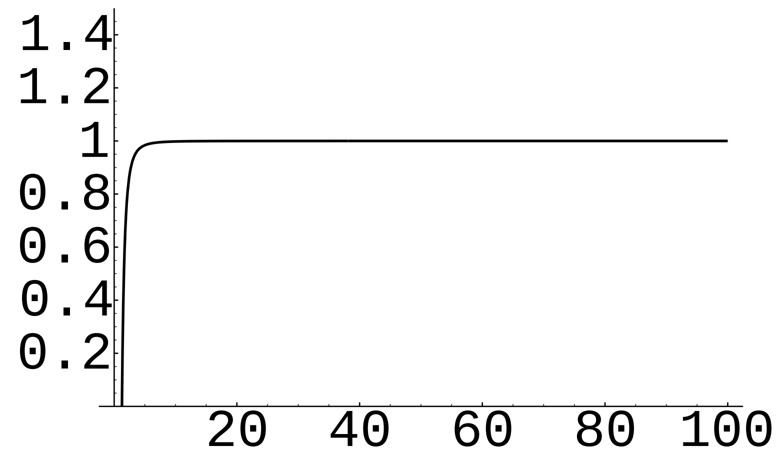


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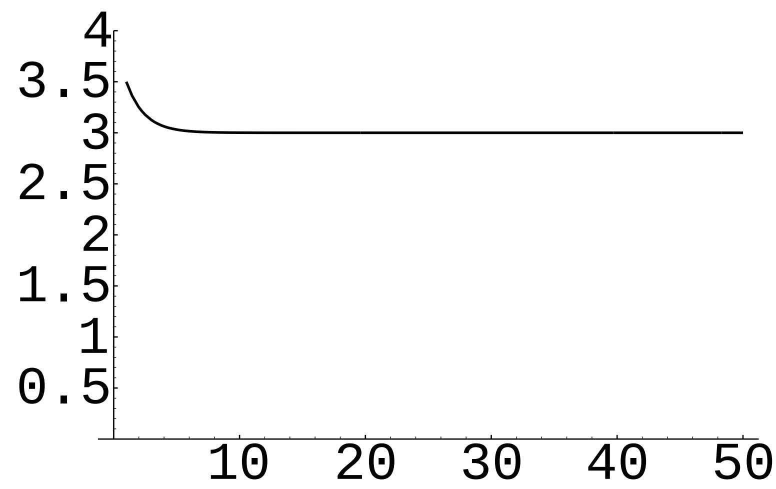

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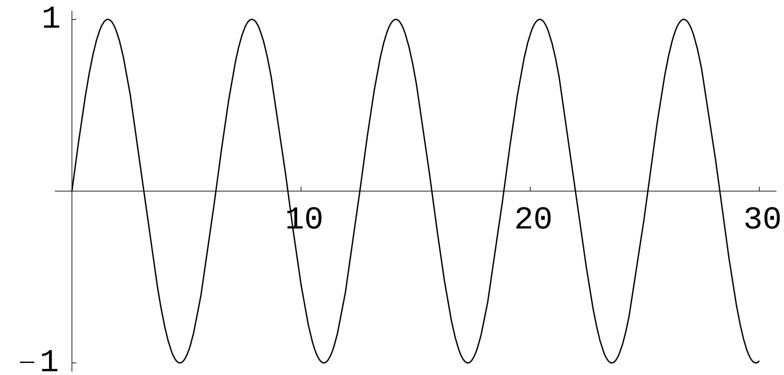

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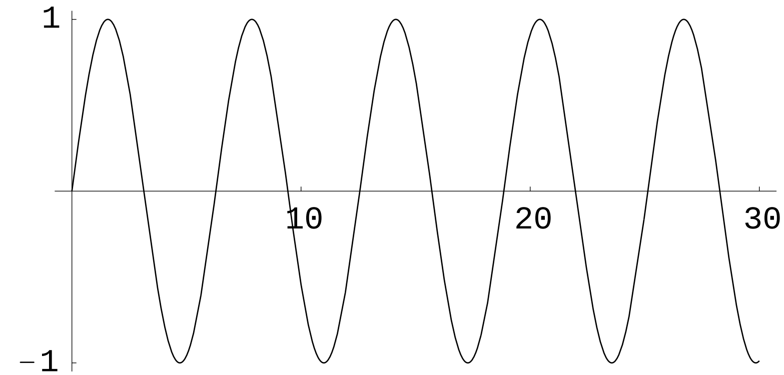


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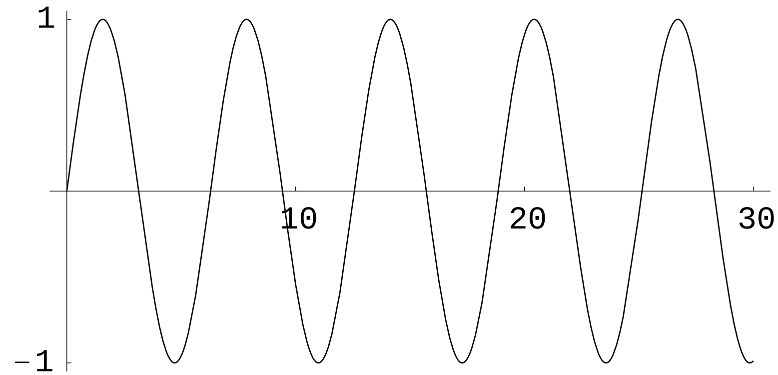


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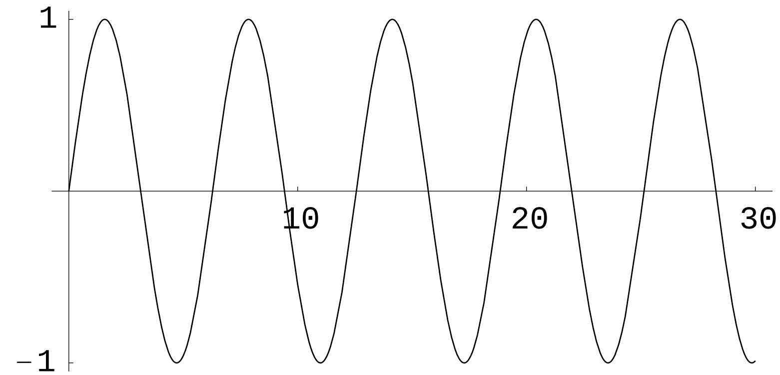


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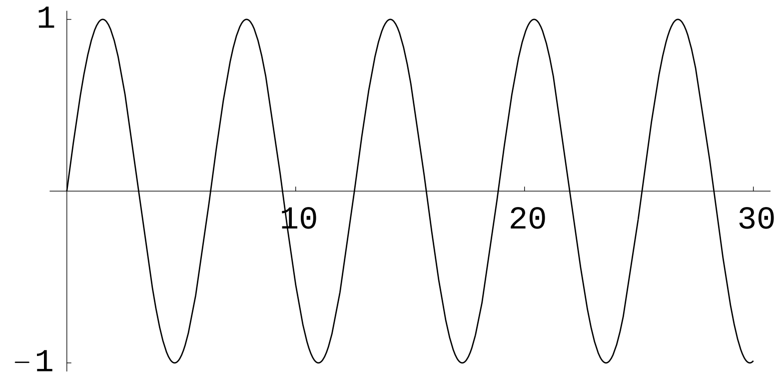


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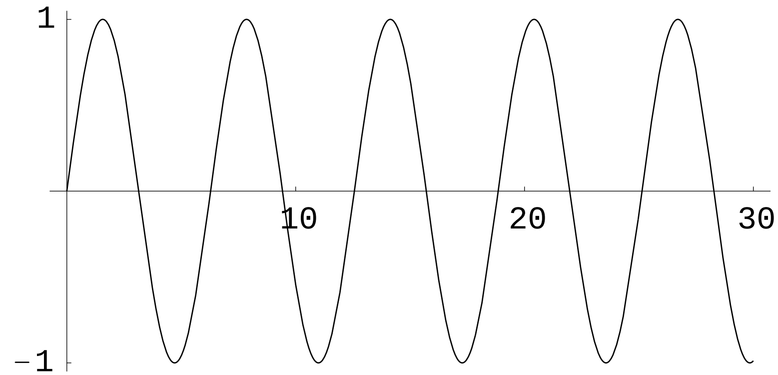


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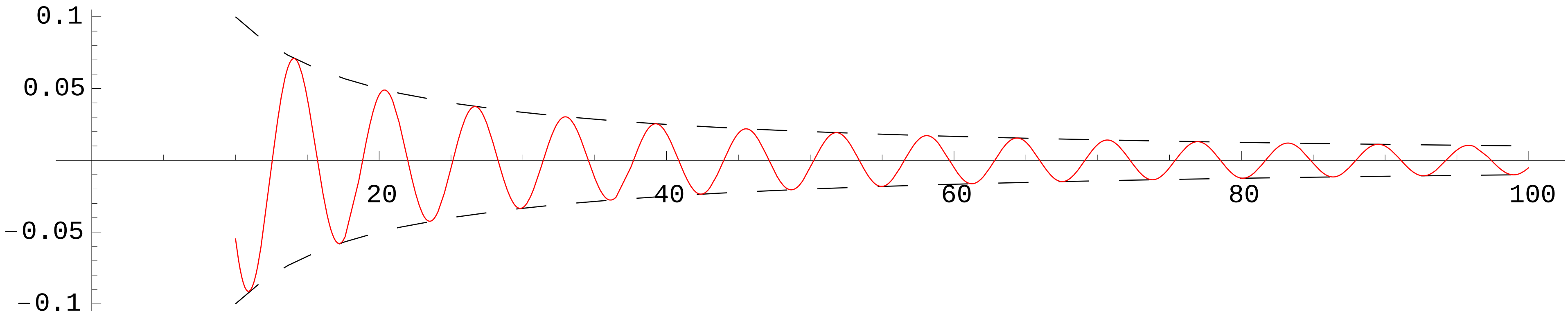

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**Remark** To find limits at infinity *for rational functions*

- Divide numerator and denominator by $x^p$ where $p =$ deg of denom.

**Leading Terms Rule** Let

$$R(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0}{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}$$

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