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Example Consider the function f defined by

$$f(x) = \begin{cases} x^2 & \text{if } -2 \leq x < 0 \\ 2x & \text{if } 0 \leq x < 2 \\ 4 - x & \text{if } 2 \leq x \leq 6 \end{cases}$$

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(3) $f(2.5)$

and sketch the graph of f.

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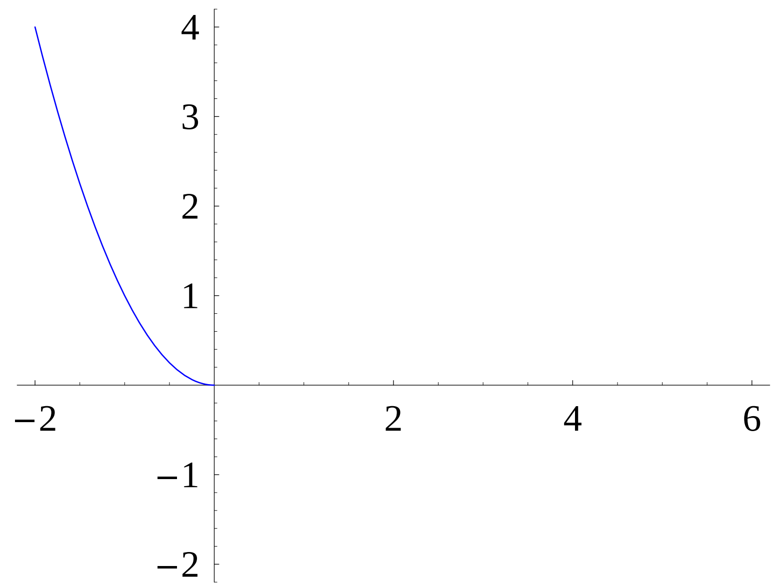

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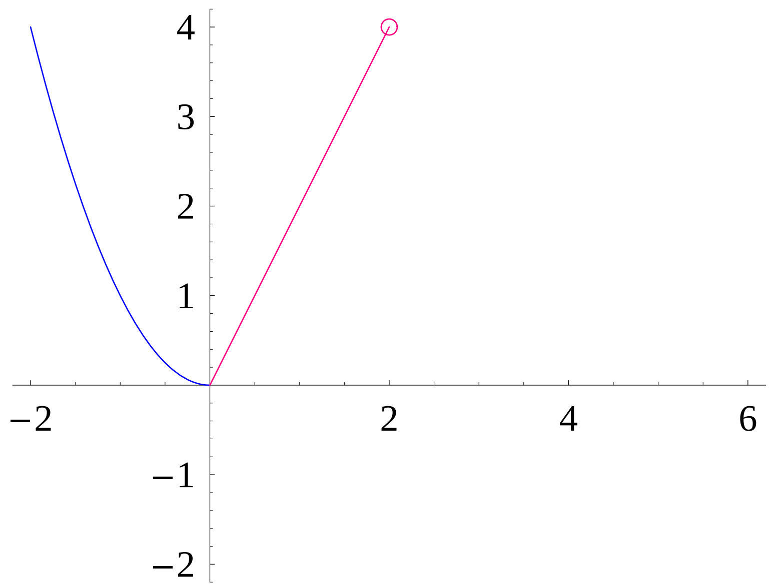

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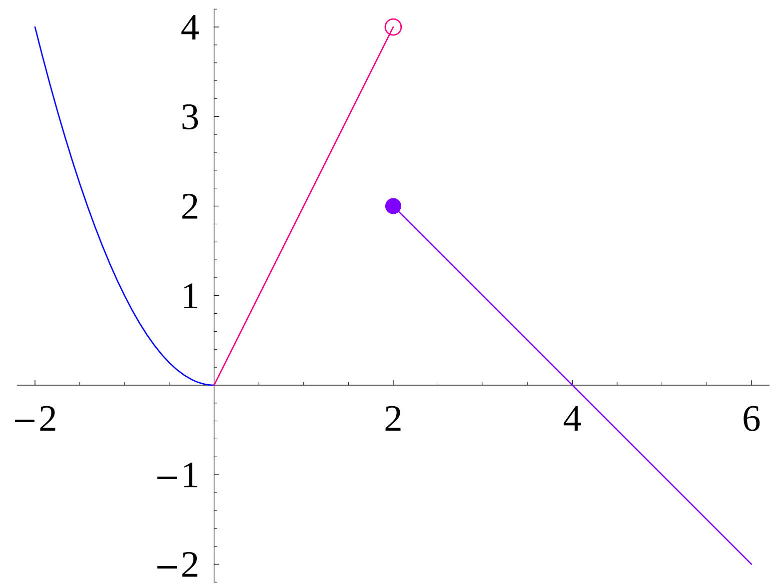

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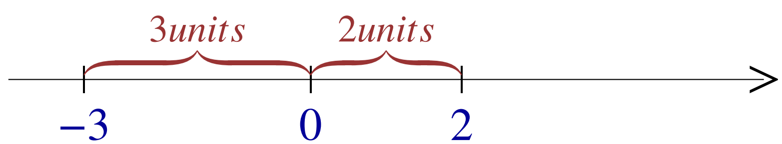

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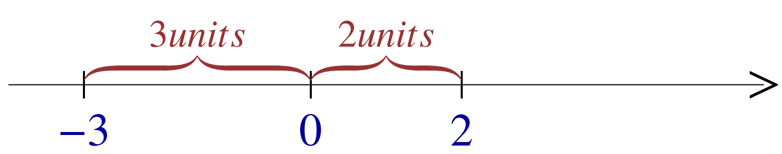

- $|a|$ is always *nonnegative*

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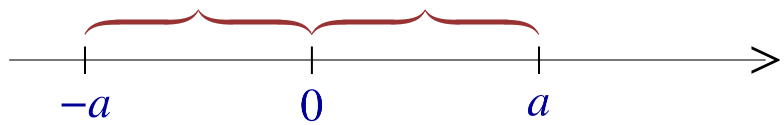

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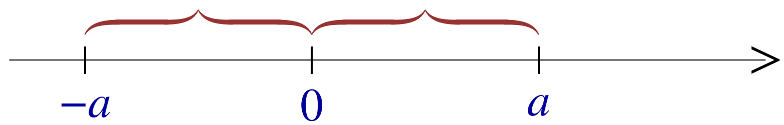

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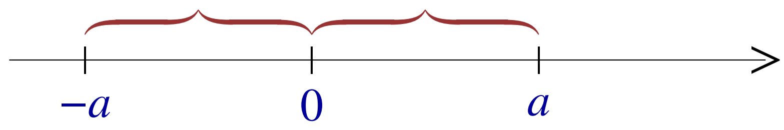

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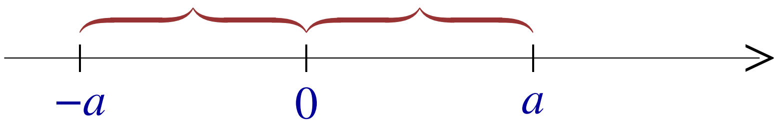

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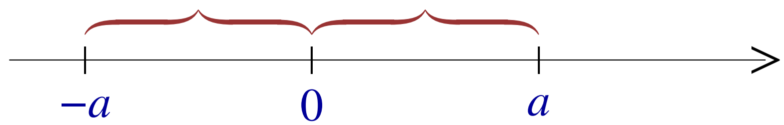

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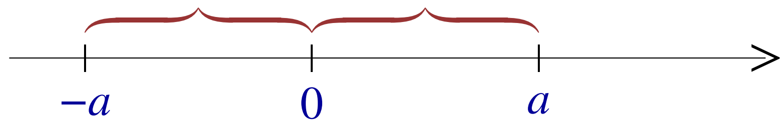

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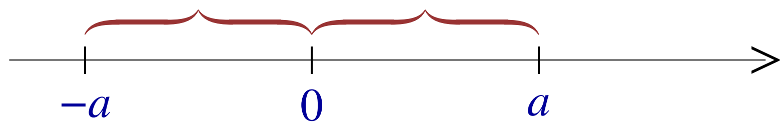

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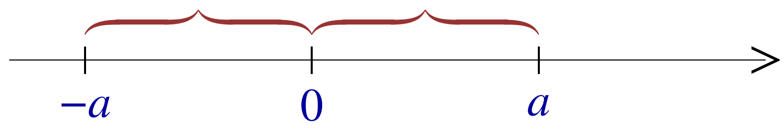

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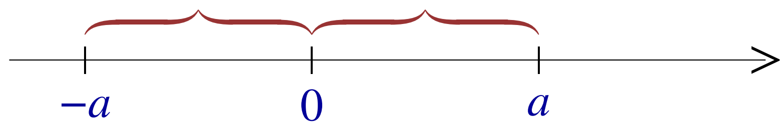

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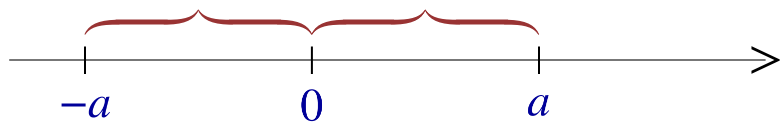

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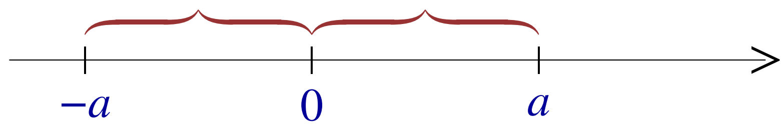

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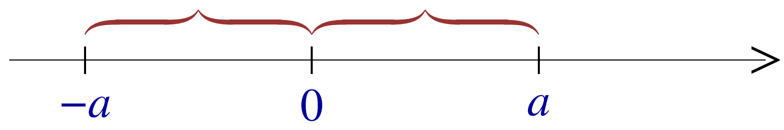

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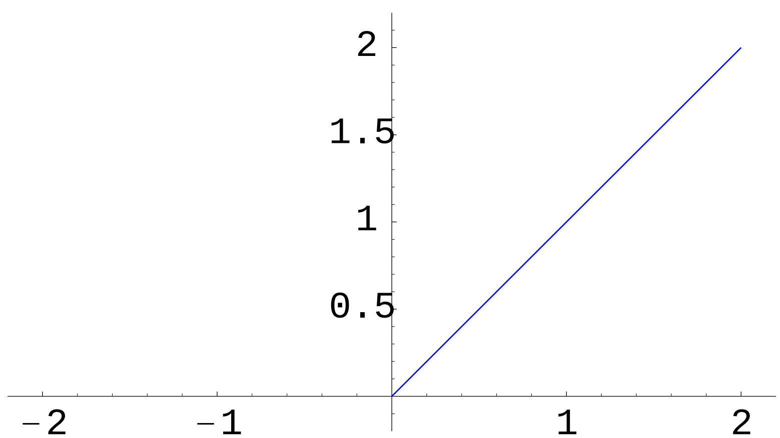

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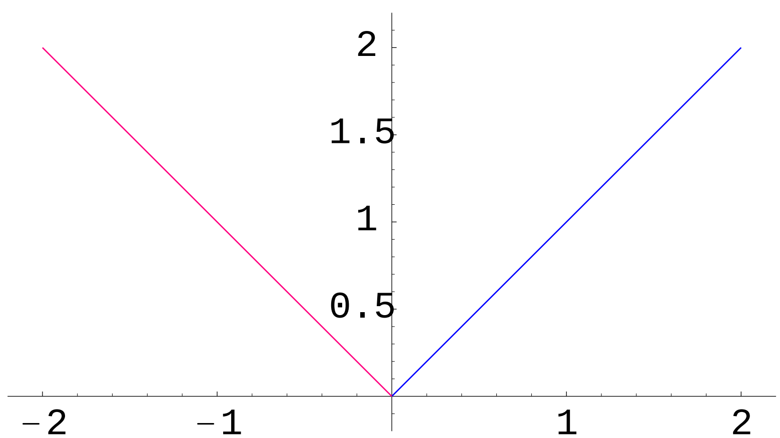

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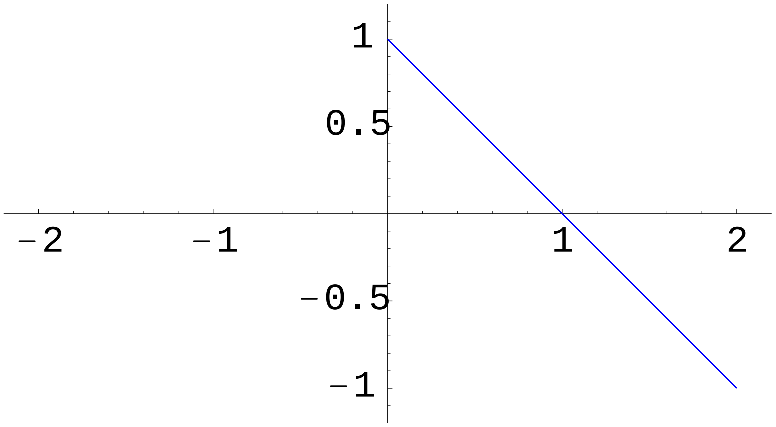

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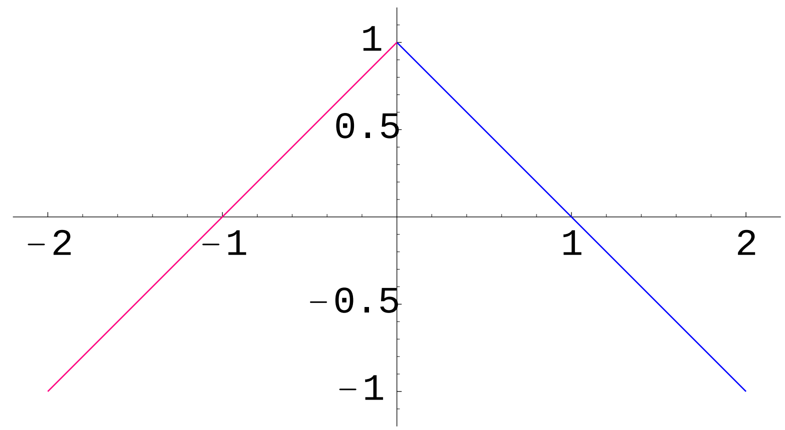

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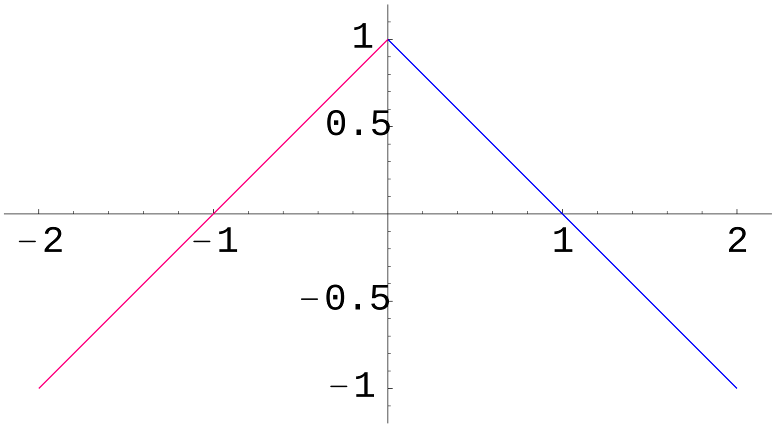

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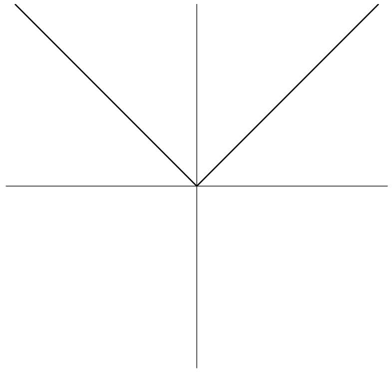

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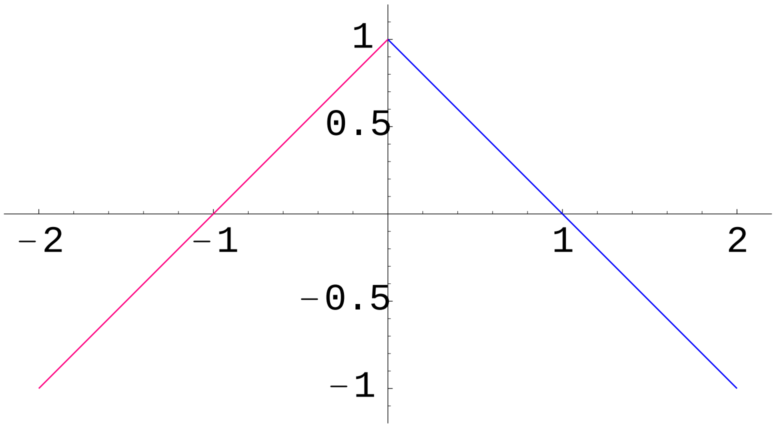

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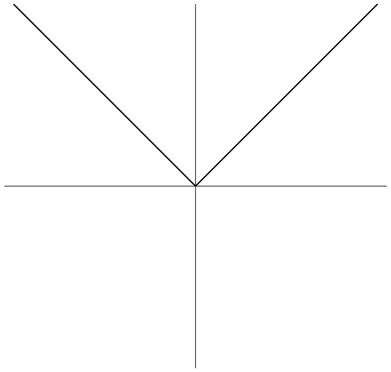

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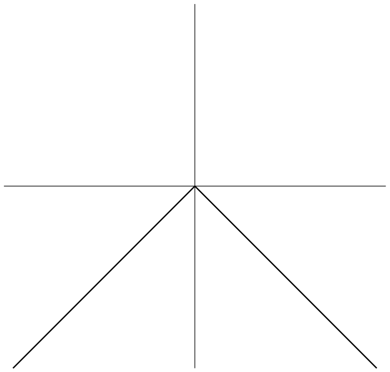

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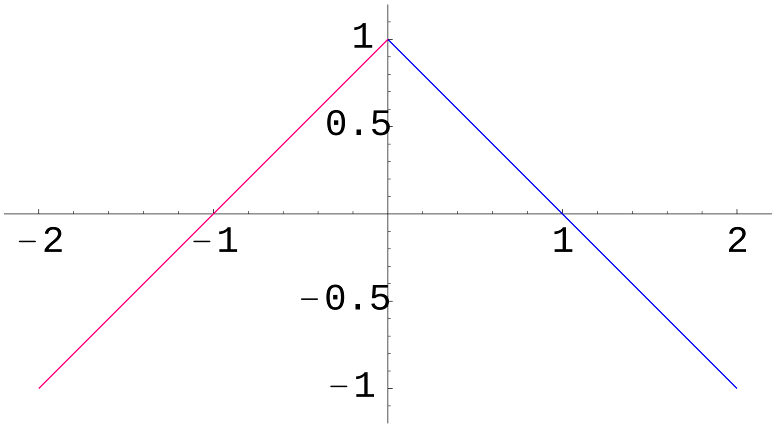

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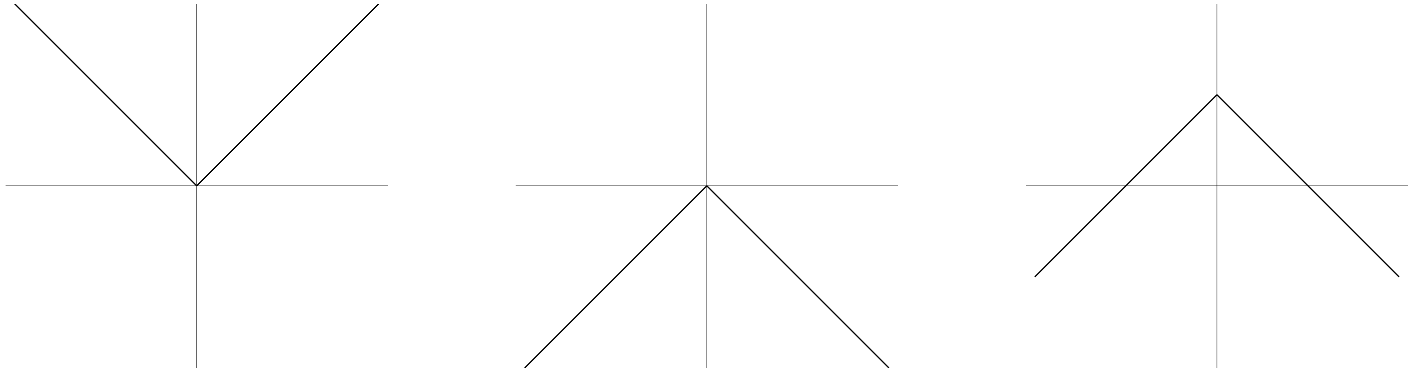

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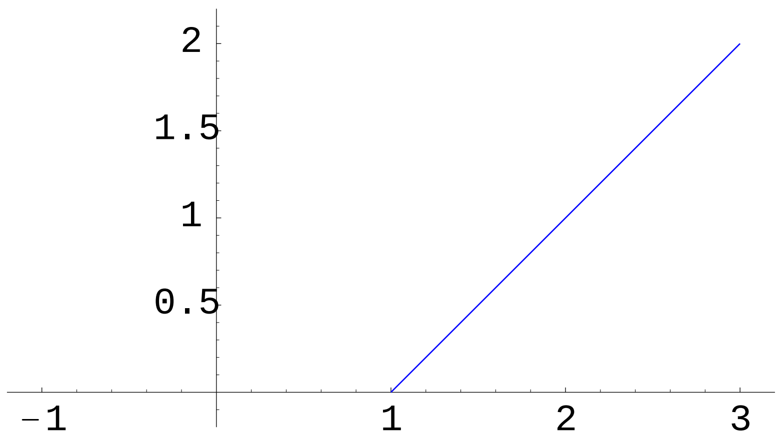

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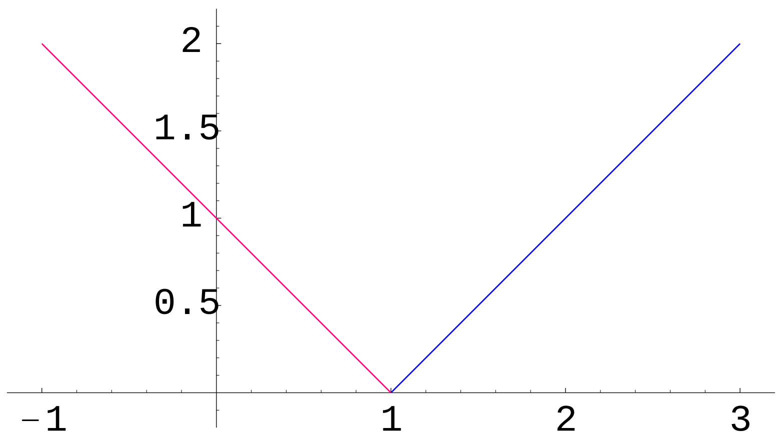

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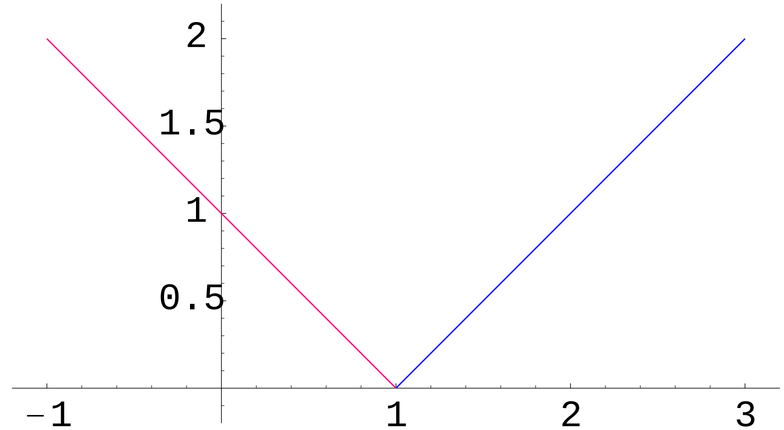

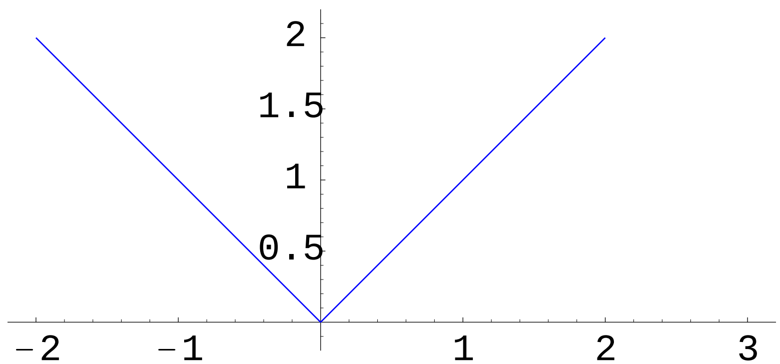

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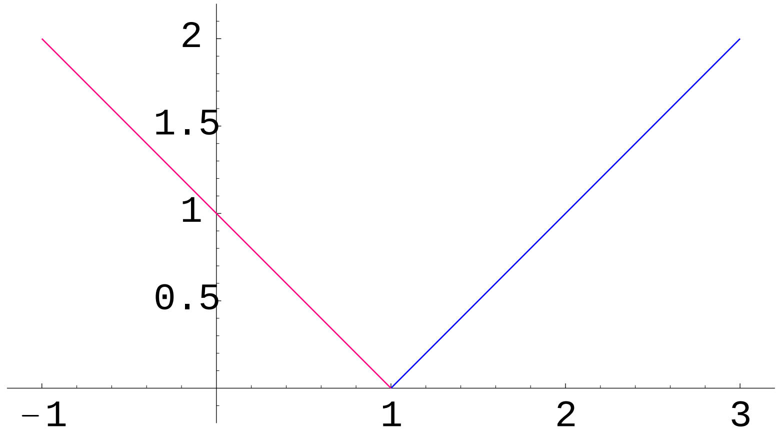

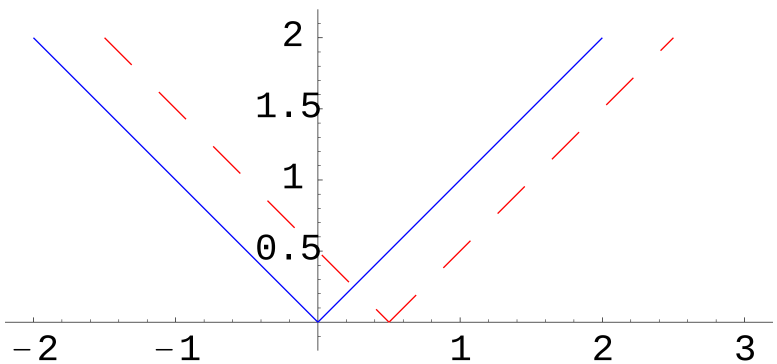

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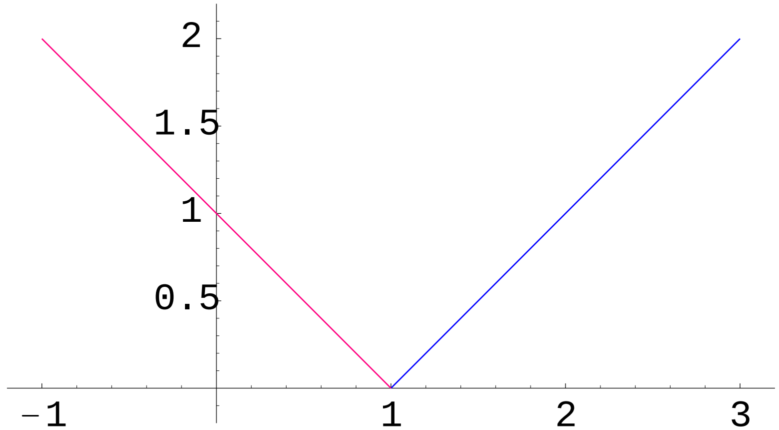

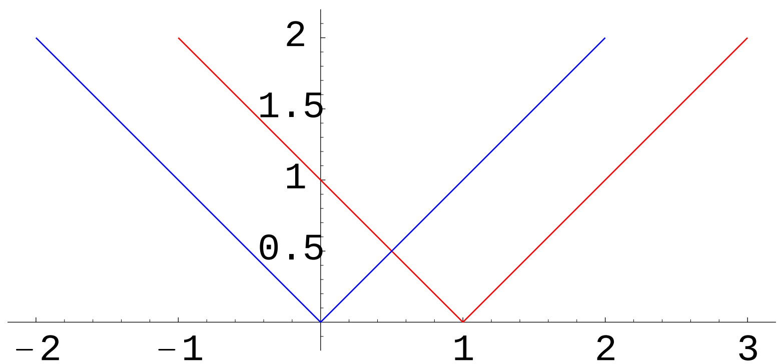

Square-root Function $f(x) = \sqrt{x}$

Graph of $y = \sqrt{x}$?

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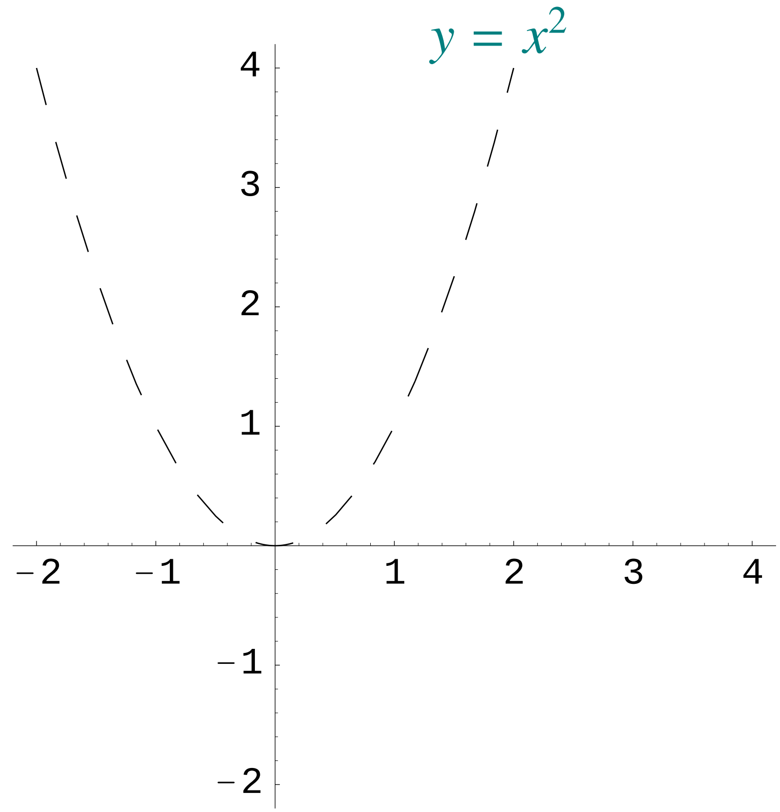

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- First, square both sides to get $y^2 = x$.

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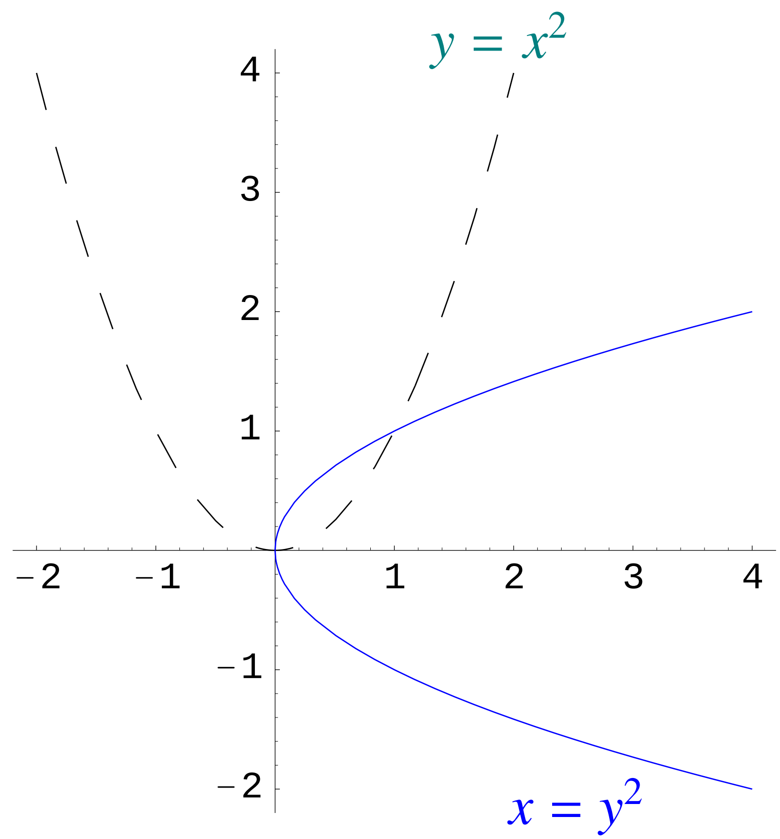

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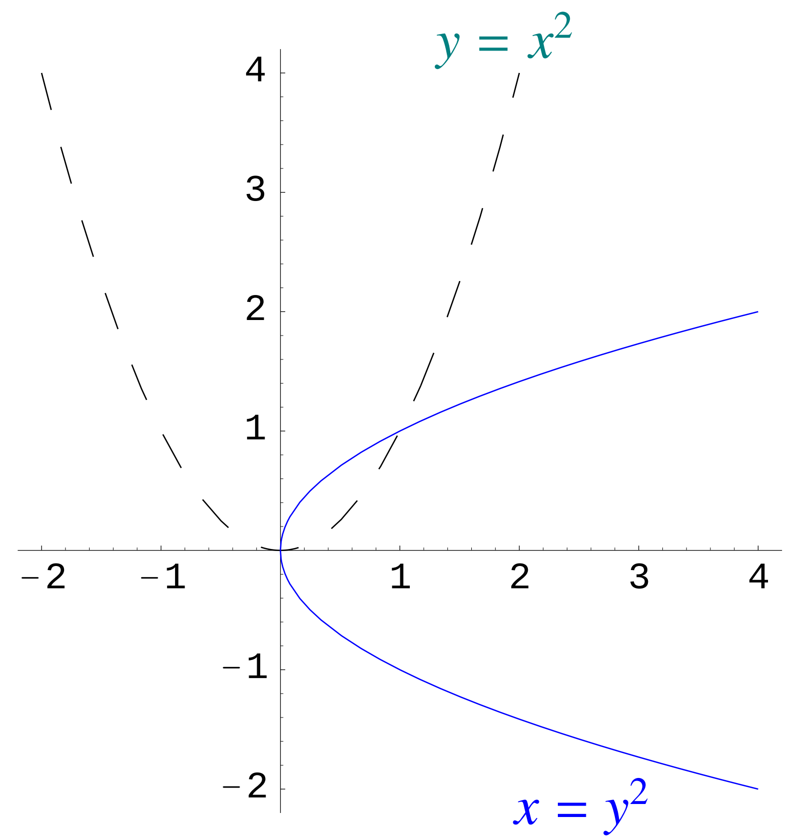

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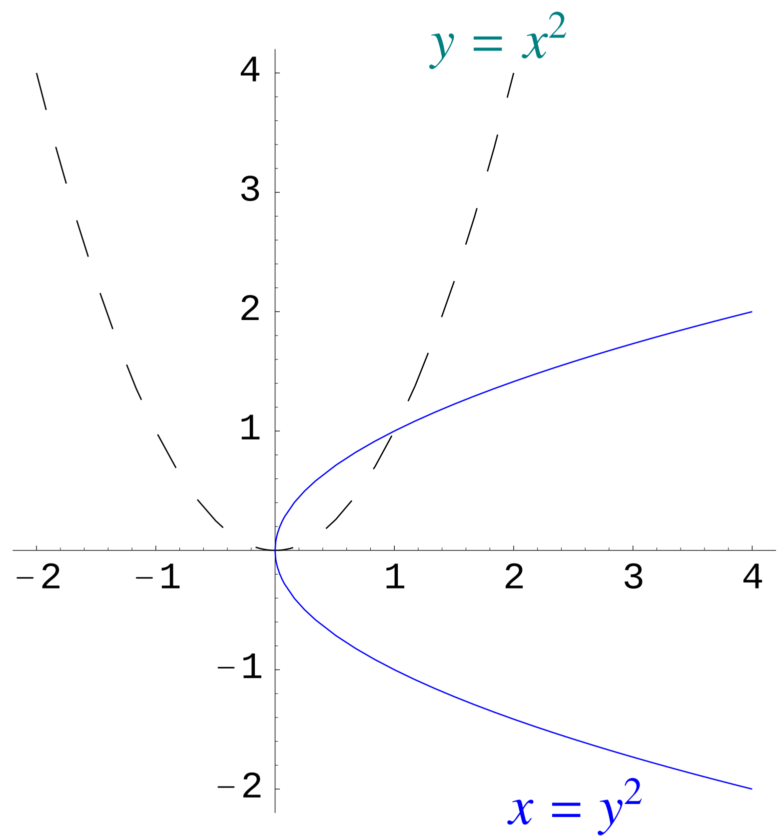

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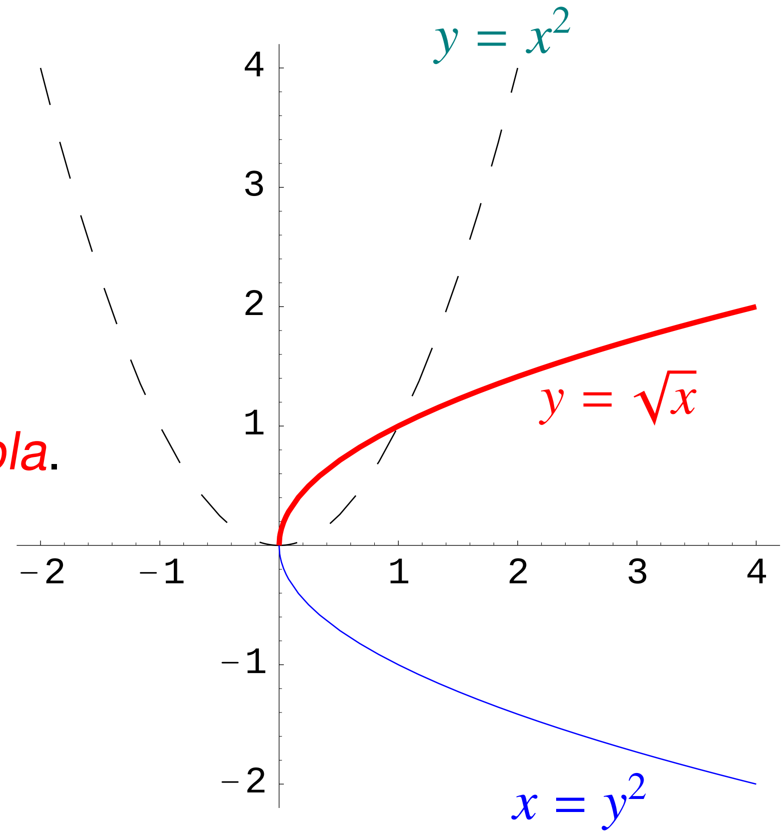

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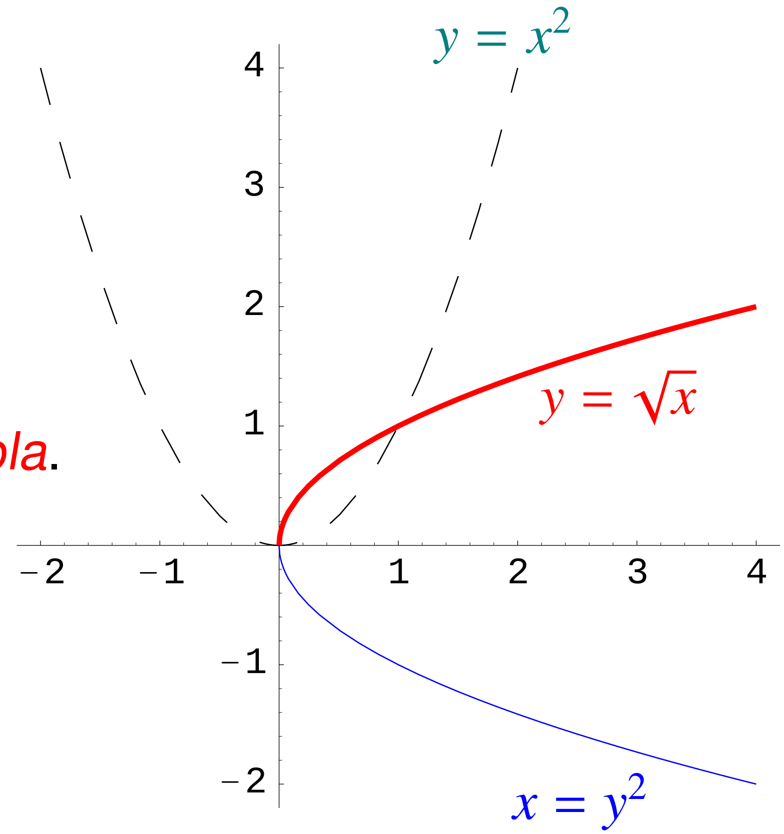

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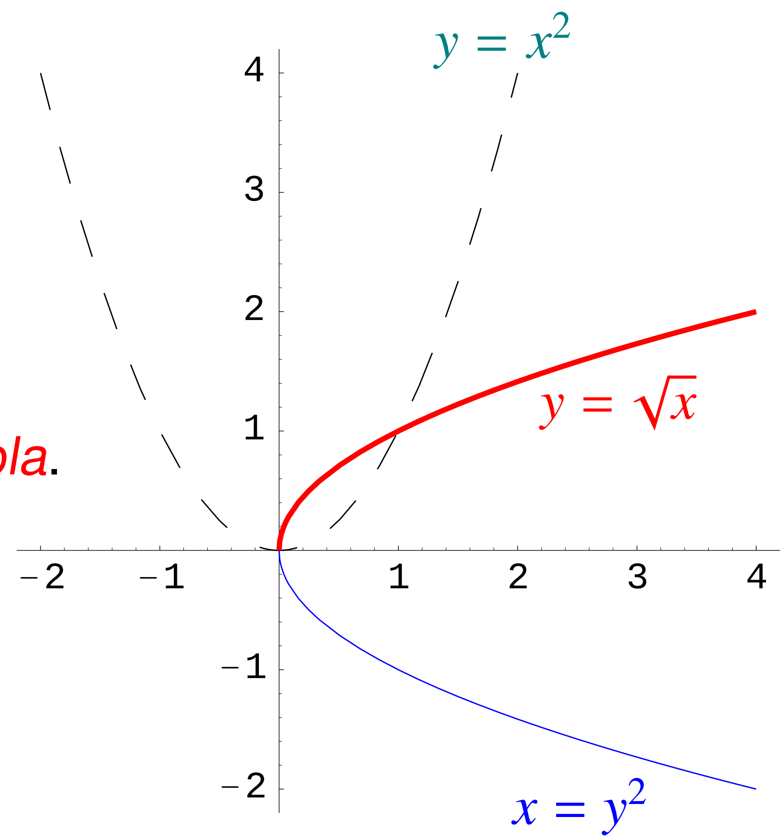

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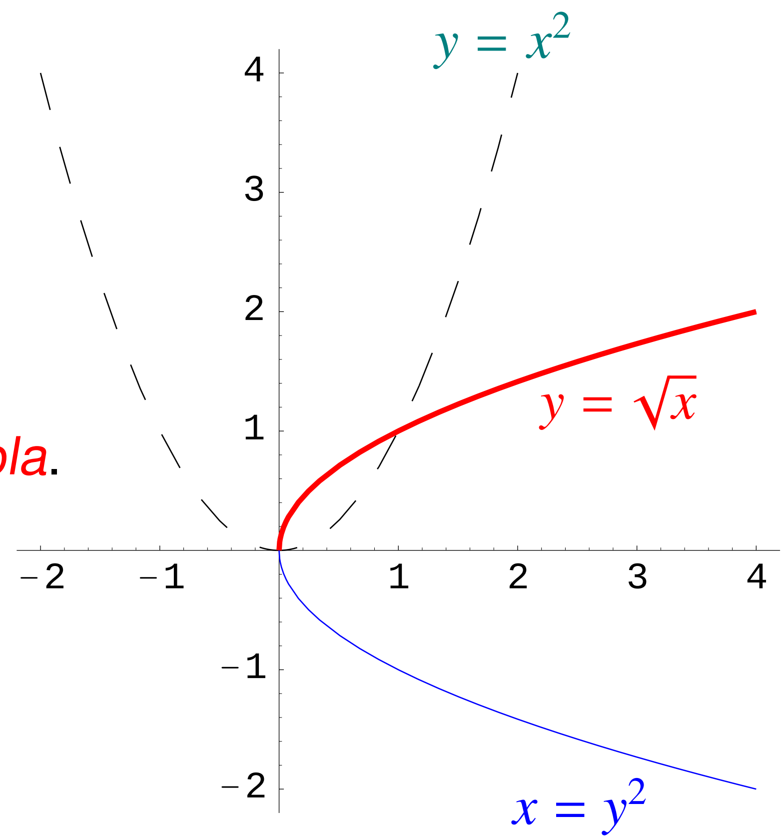

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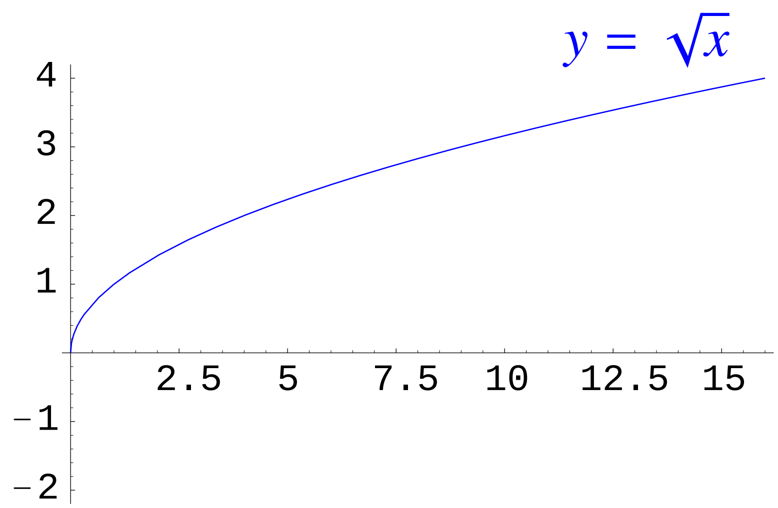

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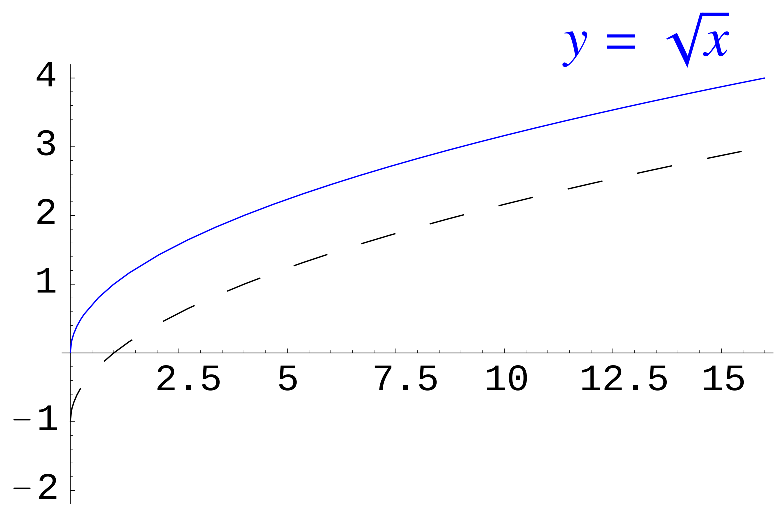

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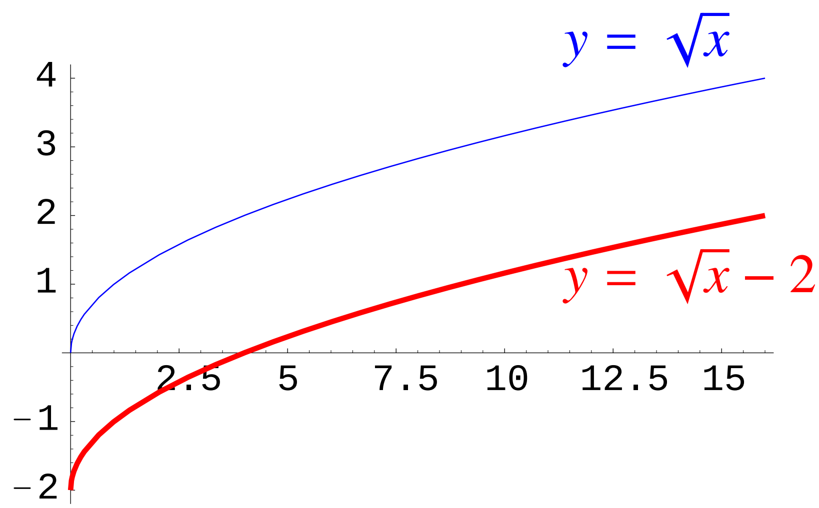

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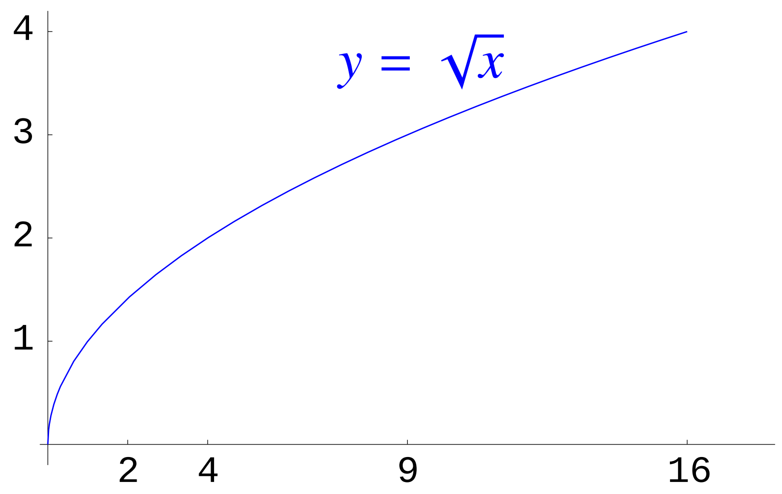

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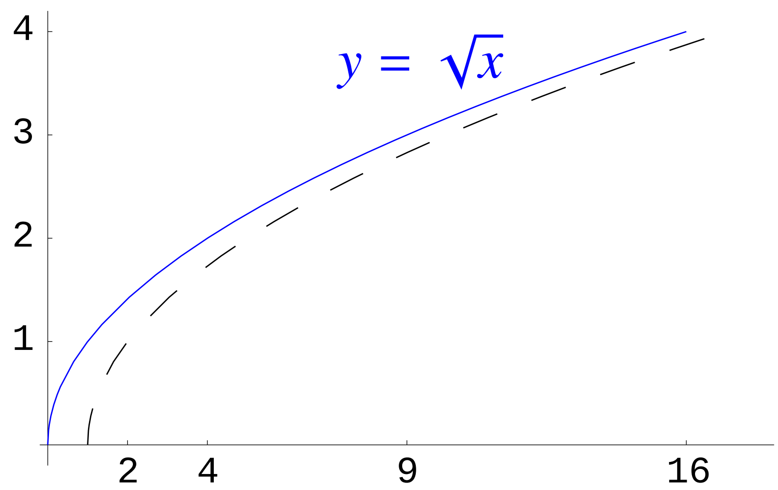

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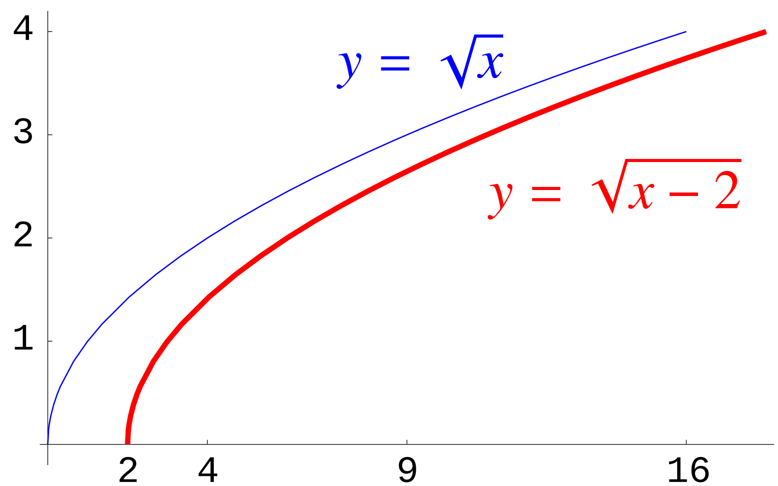

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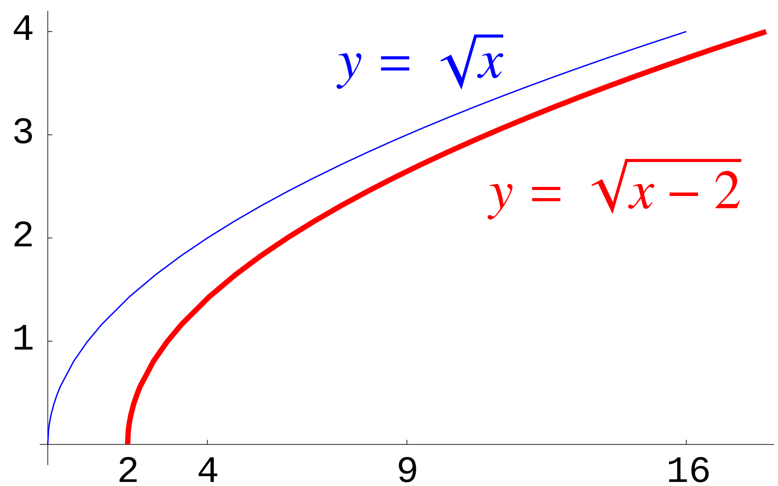

$\sqrt{x-2}$ is defined for $x \geq 2$ only.

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Example Let $f(x) = \sqrt{x}$ and $g(x) = x - 2$. Find the following (if defined)

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Remark In order that $(f \circ g)(x)$ be defined, need $g(x) \in \text{dom}\,(f)$.

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(1) $$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+1) \\ &= (2x+1)^2\end{aligned}$$

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\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f(2x + 1) \\
&= (2x + 1)^2 \\
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Note $f \circ g \neq g \circ f$ in general.

Exercise Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions given by

$$f(x) = 2x + 1, \qquad g(x) = \frac{x-1}{2}.$$

Find

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Definition A function φ from a set X into itself satisfying $\varphi(x) = x$ *for all* $x \in X$ is called the *identity function* on X.

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Remark The above example means that $(f \circ g)$ and $(g \circ f)$ are the identity functions on $\mathbb{R}$.

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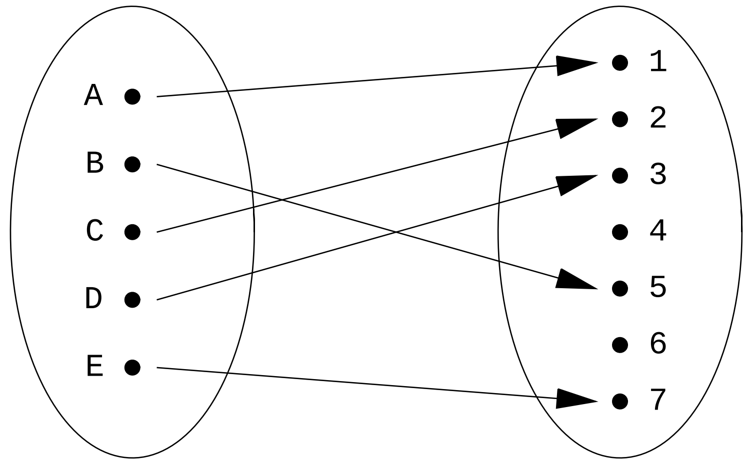
Question Given a function $f : X \longrightarrow Y$, *when can we find* a function $g : Y \longrightarrow X$ such that $(g \circ f)(x) = x$ *for all* $x \in X$?

Example Let $f : X \longrightarrow Y$ be represented by



Is there any $g : Y \longrightarrow X$ satisfying $(g \circ f)(x) = x$ for all $x \in X$?

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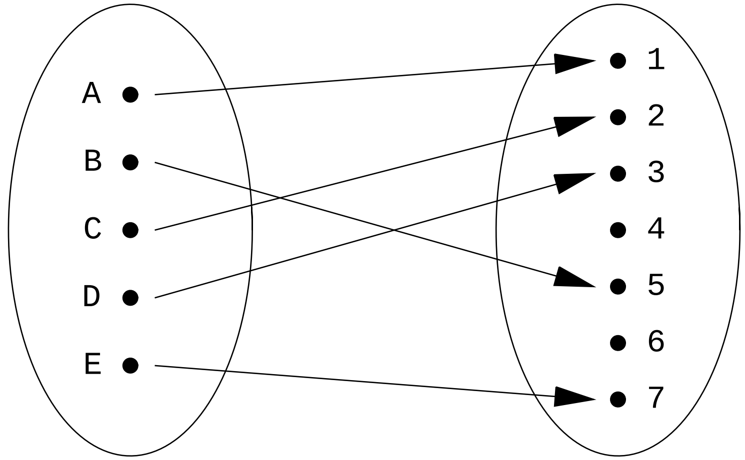

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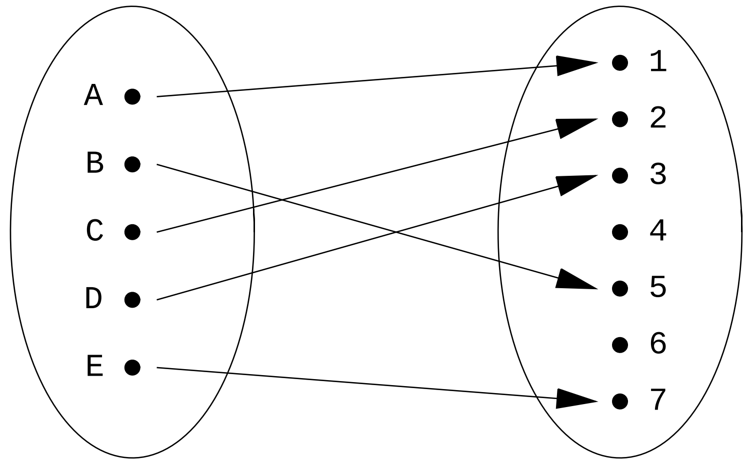

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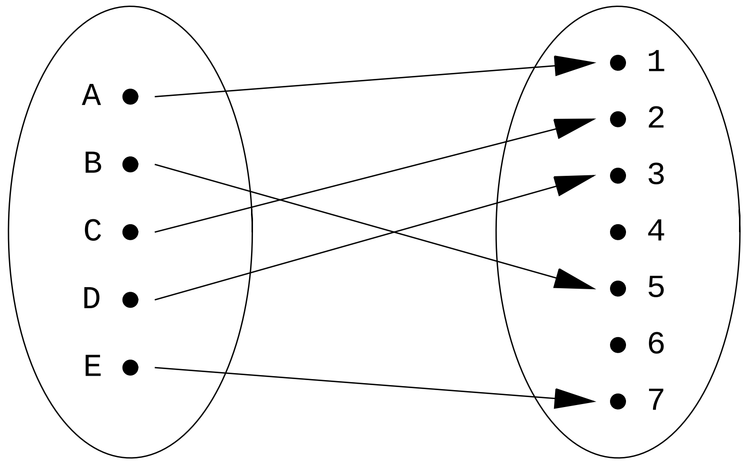

Is there any $g : Y \longrightarrow X$ satisfying $(g \circ f)(x) = x$ for all $x \in X$?

Solution Yes. Define

$$g(1) = A \qquad g(2) = C$$

Then

$$(g \circ f)(A) = A$$

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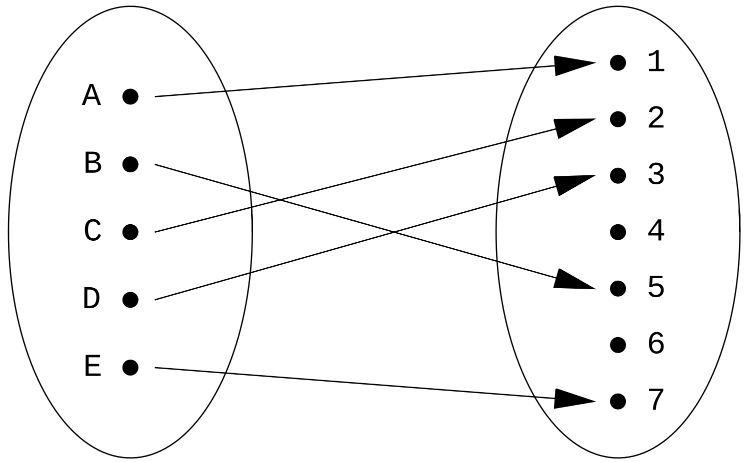

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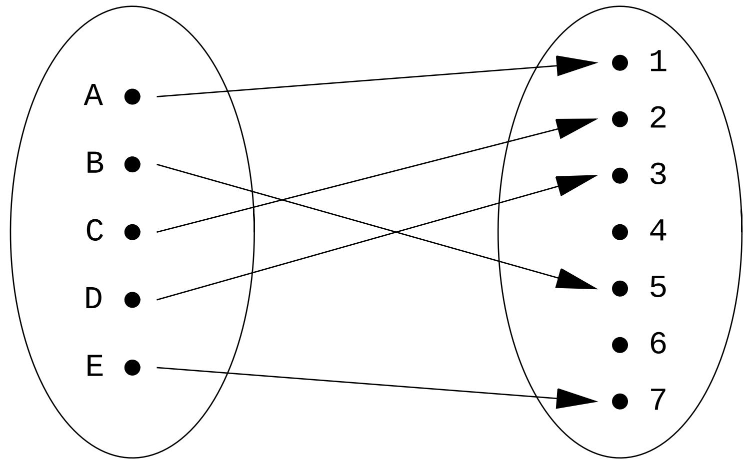
$$(g \circ f)(C) = C$$

Example Let $f : X \longrightarrow Y$ be represented by



Is there any $g : Y \longrightarrow X$ satisfying $(g \circ f)(x) = x$ for all $x \in X$?

Solution Yes. Define

$$g(1) = A \qquad g(2) = C \qquad g(3) = D$$

Then

$$(g \circ f)(A) = A$$

$$(g \circ f)(C) = C$$

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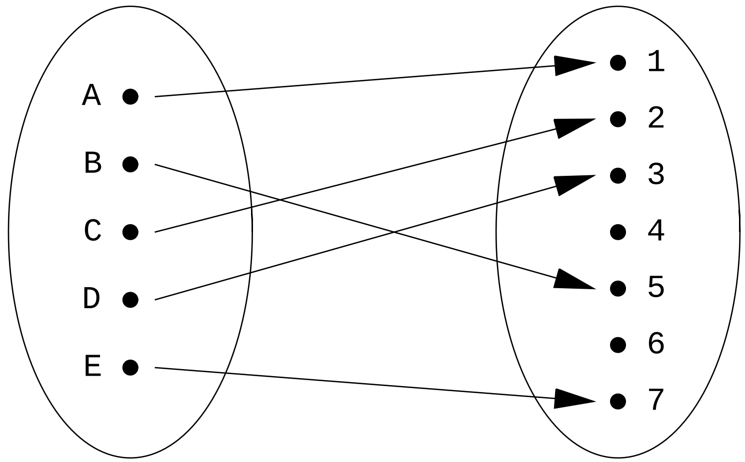

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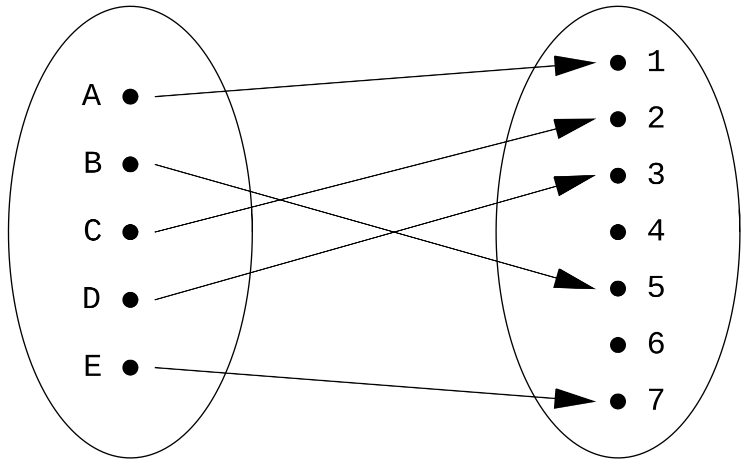

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Solution Yes. Define

$$\begin{aligned} g(1) &= A \\ g(2) &= C \\ g(3) &= D \\ g(4) &= A \end{aligned}$$

Then

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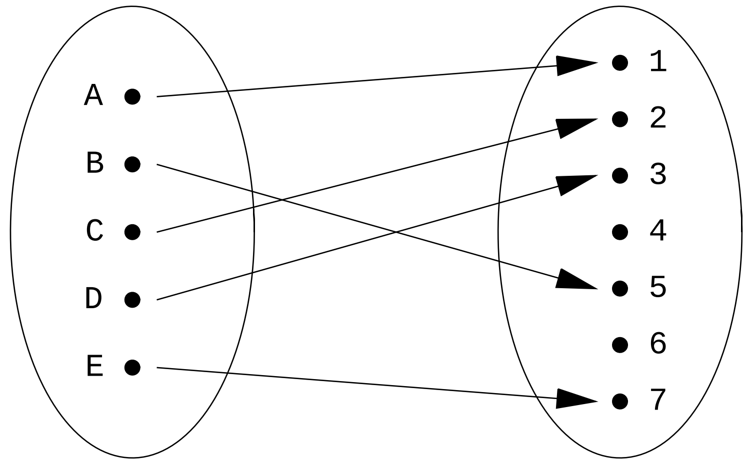

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Solution Yes. Define

$$
\begin{aligned}
g(1) &= A \\
g(2) &= C \\
g(3) &= D \\
g(4) &= A \\
g(5) &= B
\end{aligned}
$$

Then

$$
\begin{aligned}
(g \circ f)(A) &= A \\
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(g \circ f)(D) &= D
\end{aligned}
$$

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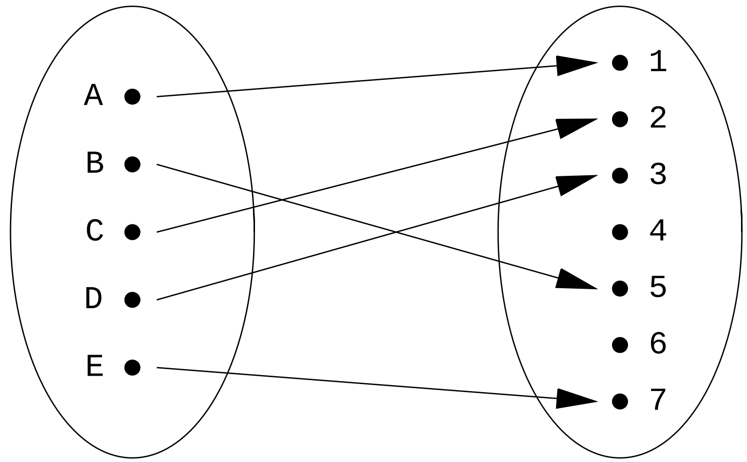

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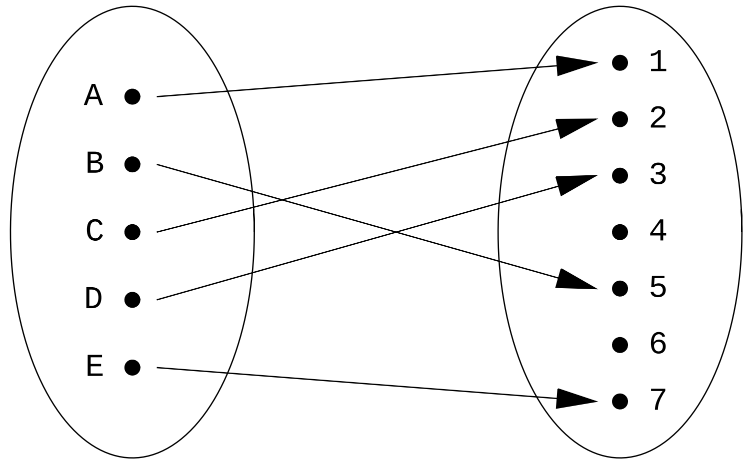

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$$\begin{aligned} g(1) &= A \\ g(2) &= C \\ g(3) &= D \\ g(4) &= A \\ g(5) &= B \\ g(6) &= A \end{aligned}$$

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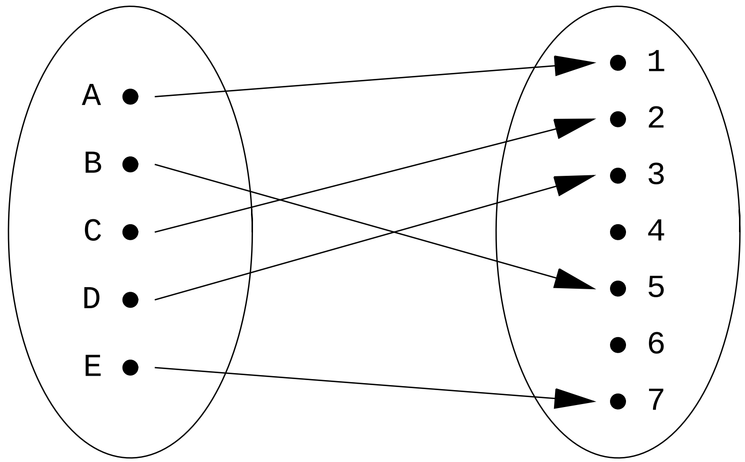

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Solution Yes. Define

$$\begin{aligned} g(1) &= A \\ g(2) &= C \\ g(3) &= D \\ g(4) &= A \\ g(5) &= B \\ g(6) &= A \\ g(7) &= E \end{aligned}$$

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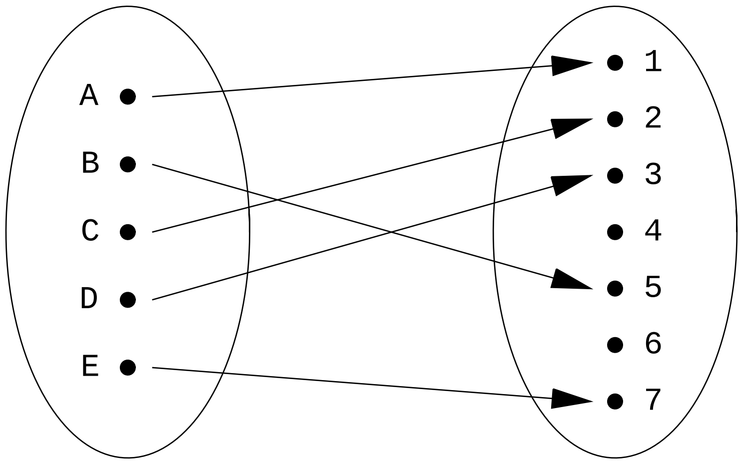

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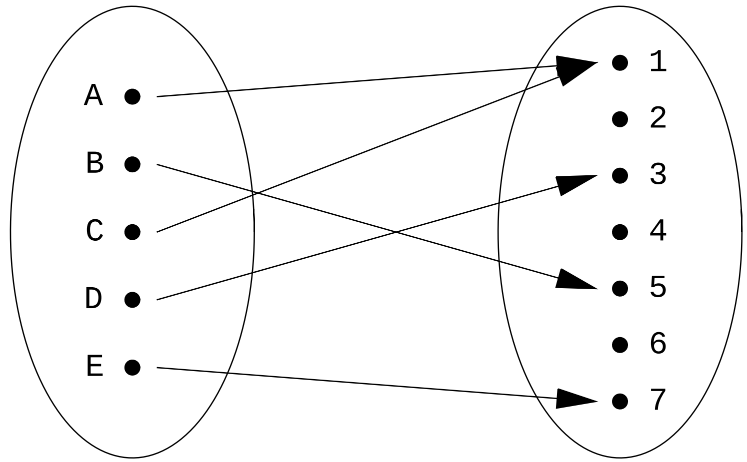

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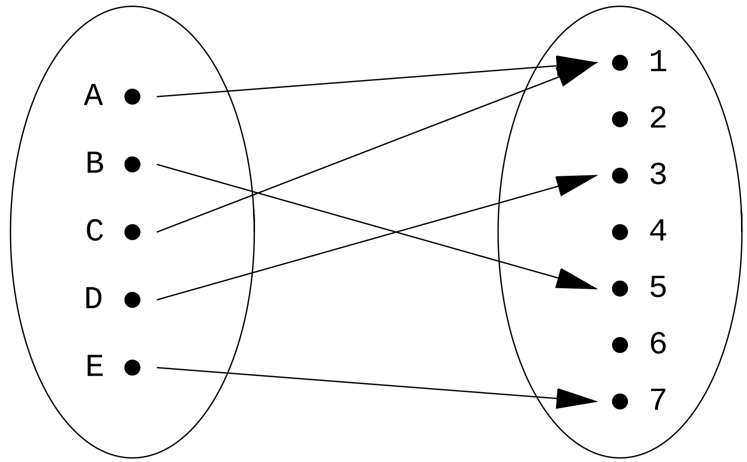

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Solution Need $(g \circ f)(A) \quad = \quad A$

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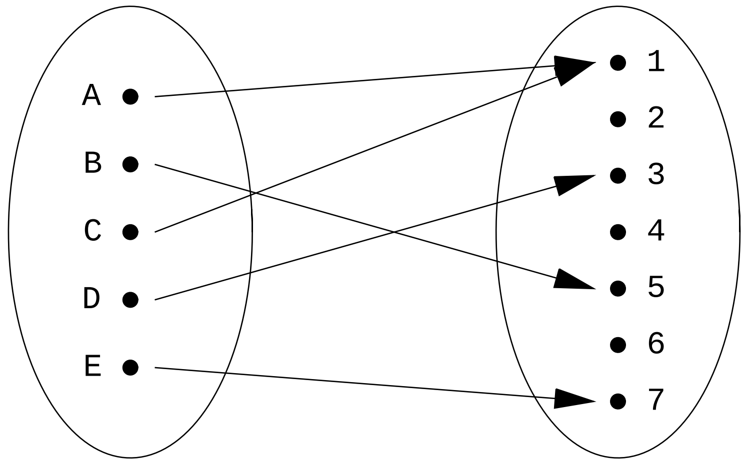

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Solution Need

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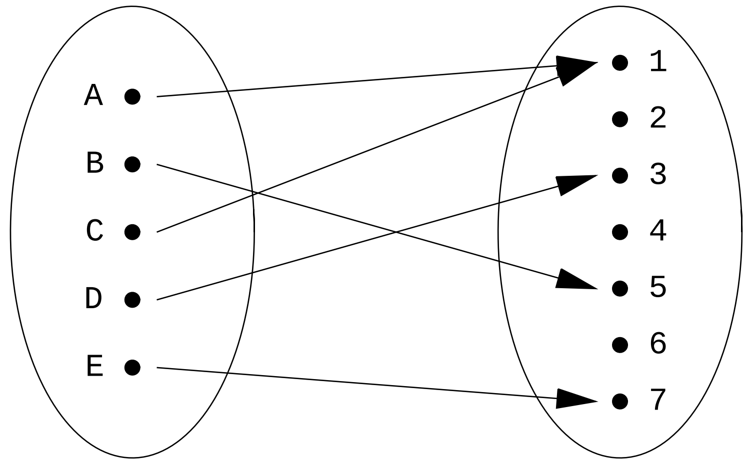

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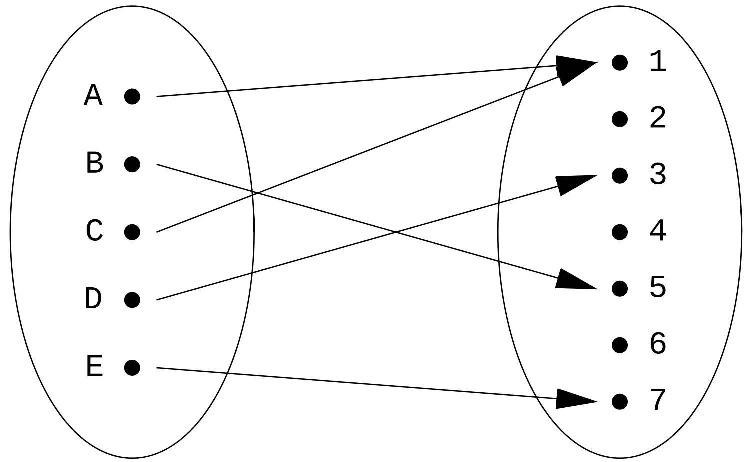

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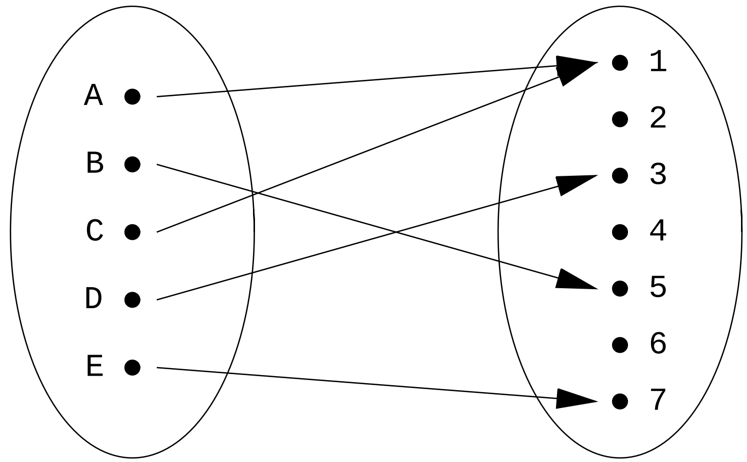

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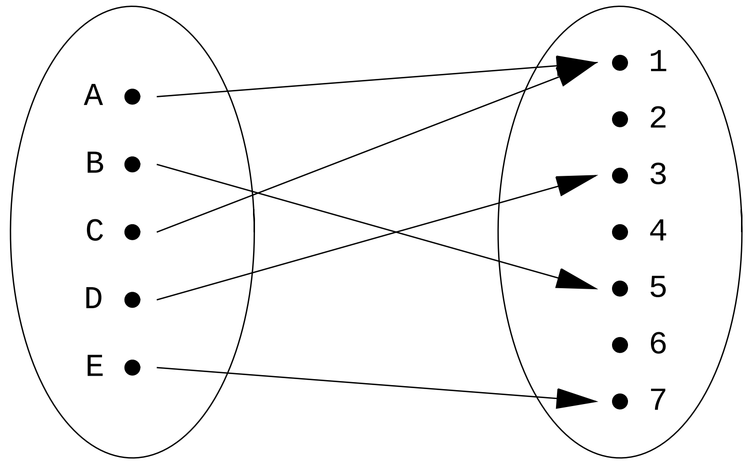

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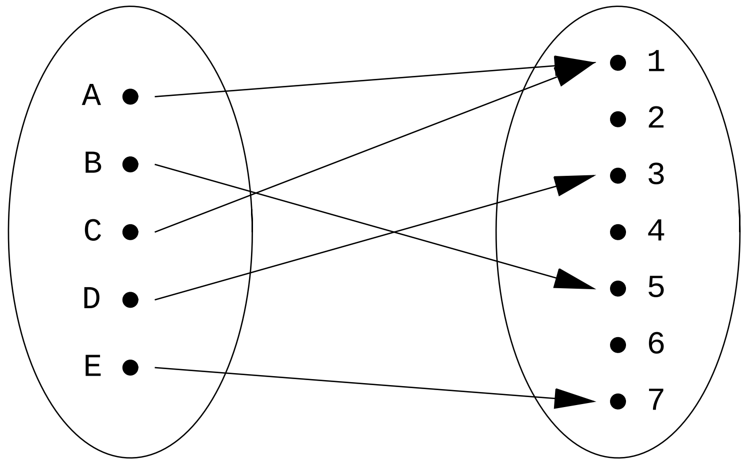
In particular, $g(1) = A$

Example Let $f : X \longrightarrow Y$ be represented by



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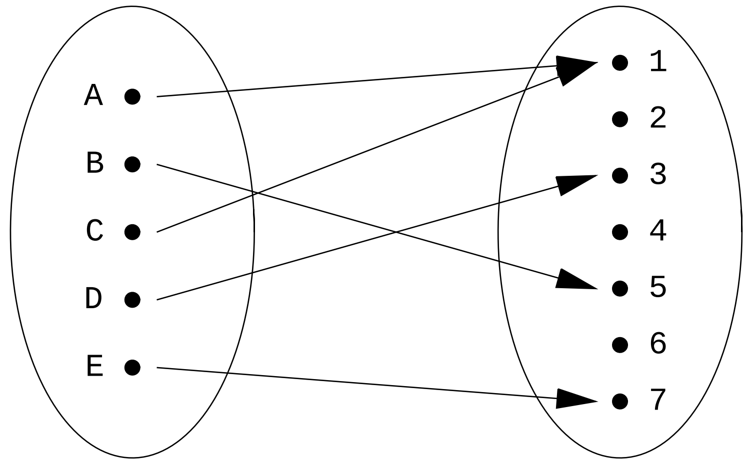

Is there any $g : Y \longrightarrow X$ satisfying $(g \circ f)(x) = x$ for all $x \in X$?

Solution Need

$$\begin{aligned}(g \circ f)(A) &= A \\ (g \circ f)(B) &= B \\ (g \circ f)(C) &= C \\ (g \circ f)(D) &= D \\ (g \circ f)(E) &= E\end{aligned}$$

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No such function g.

Definition A function $f : X \longrightarrow Y$ is said to be *injective* if

$$x_1 \neq x_2 \Longrightarrow f(x_1) \neq f(x_2)$$

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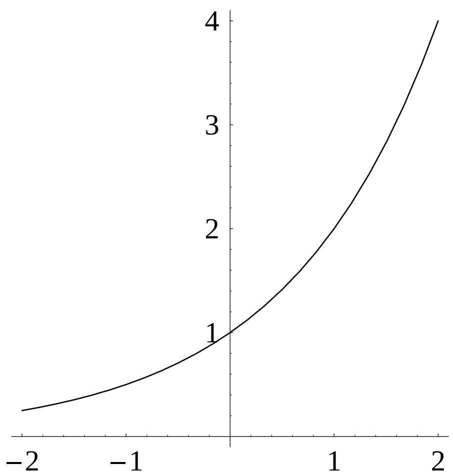

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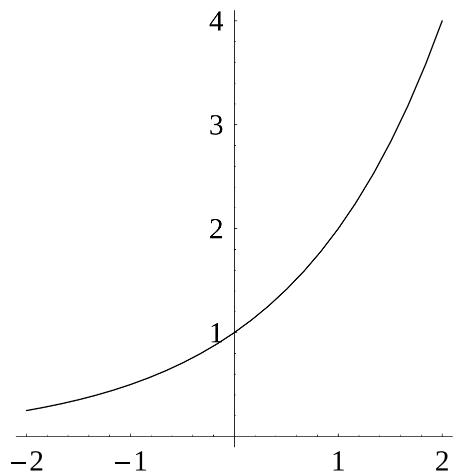

$$x_1 \neq x_2 \Longrightarrow 2^{x_1} \neq 2^{x_2}$$

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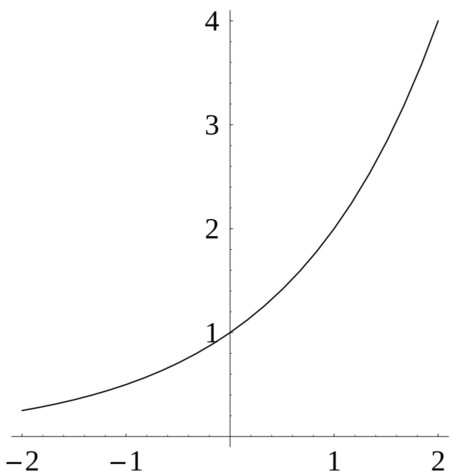

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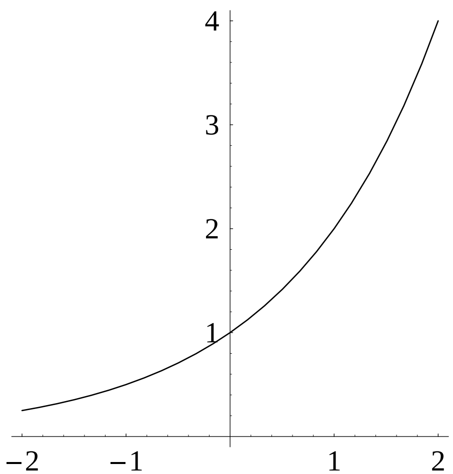

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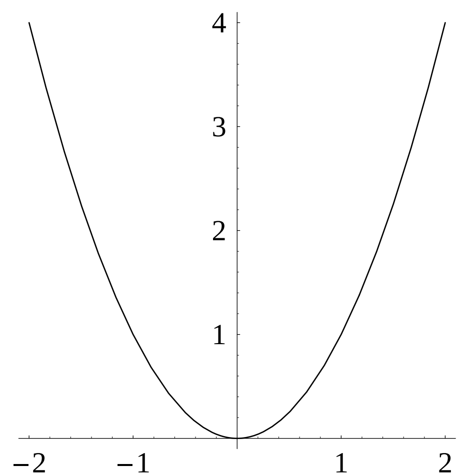

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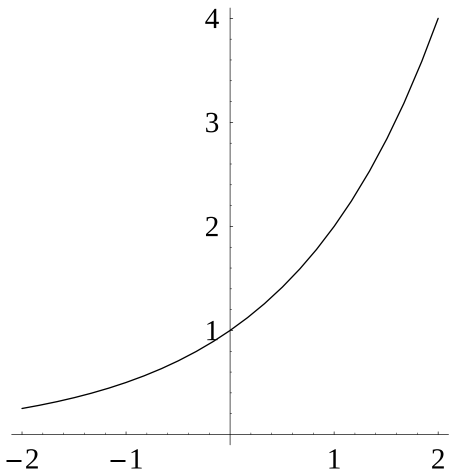

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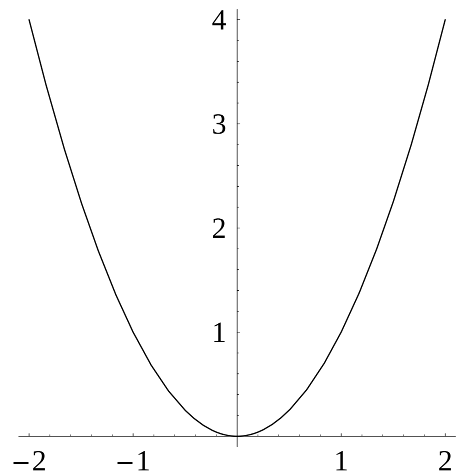

$-1 \neq 1$ but $(-1)^2 = 1^2$

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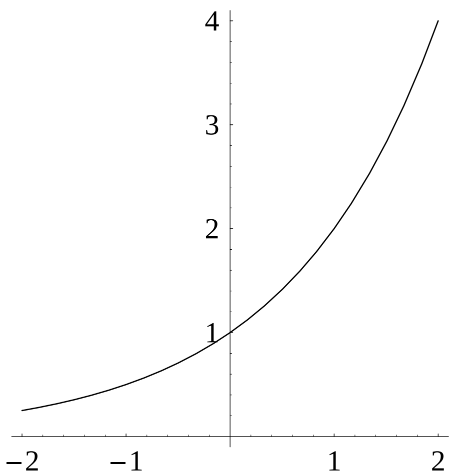

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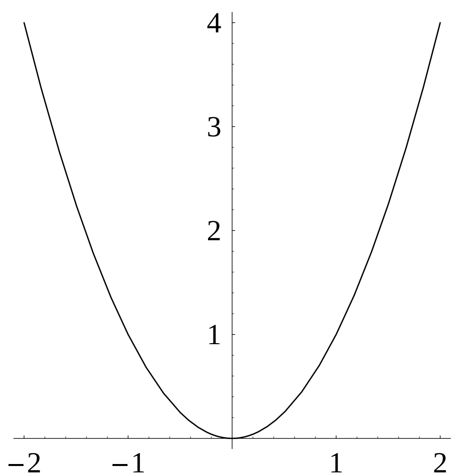

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Remark Geometrically, f is injective means that *graph of f intersects every horizontal line in at most one point*.