Course Outline

0. Revision

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Part I

1. Sets, Real Numbers and Inequalities
   - Set notations and Real Number System
   - Solving Polynomial Inequalities

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   - Graphs help us visualize behavior of functions

     Advanced calculus is called analysis – to analyse functions

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3. Limits
   - Definition and example of *different types* of limits
   - Behavior of $f(x)$ as x approaches certain value
   - Used in definition of *derivative* (differentiation) and *integral* (integration)

4. Differentiation
    - Definition and *meaning* of derivative
    - Rate of change
    - Simple rules for differentiation

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6. Integration
   - Definite Integral (*area*)
   - Indefinite Integral (*anti differentiation*)
   - Fundamental Theorem of Integral Calculus

Part II

7. Trigonometric Functions
   - Definition
   - Differentiation

8. Exponential & Logarithmic Functions
   - Definition
   - Differentiation

9. More Differentiation
   - More rules for differentiation
   - More application examples

10. More Integration
    - More formulas
    - Technique (substitution method)

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Problem The demand equation for a certain product is

$$q - 90 + 2p = 0, \qquad 0 < q \leq 90,$$

where q is the number of units and p is the price per unit, and the average cost function is

$$C_{\text{av}} = q^2 - 8q + 57 + \frac{2}{q} \qquad 0 < q \leq 90$$

At what value of q will there be *maximum profit*? What is the maximum profit?

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From the demand equation, eg. when $q = 10$, we have $p = 40$ (units of money).

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- Find the cost function $C(q)$.
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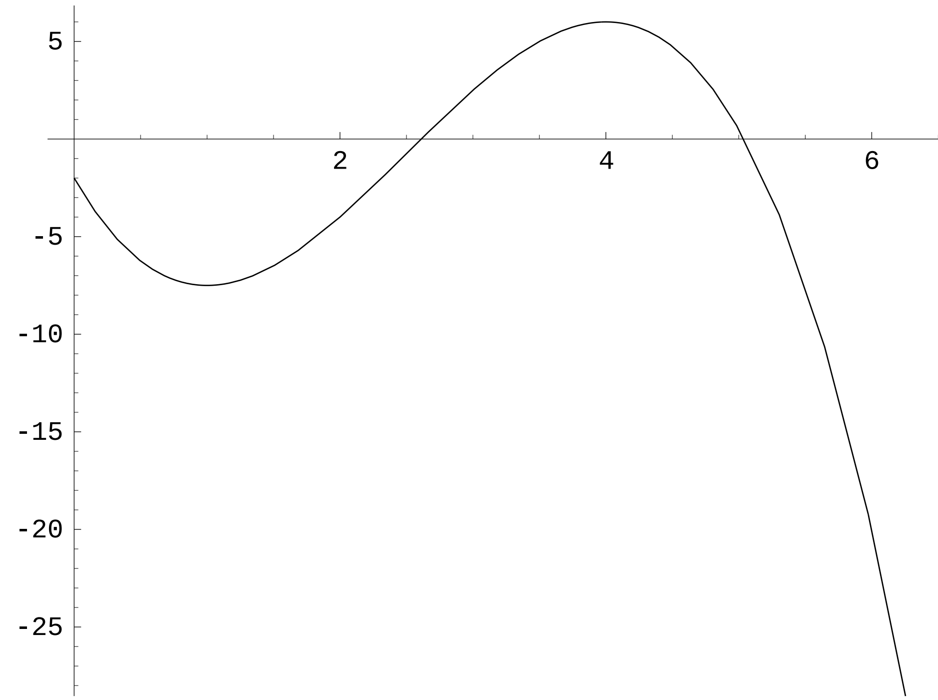
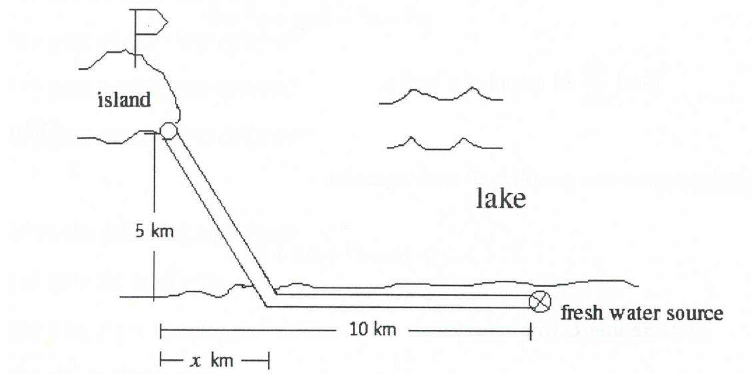


Problem A fresh water pipeline is to be run from a source on the edge of a lake to a small island $5\,km$ off-shore (see figure). If it costs 1.1 times as much to lay the pipe in the lake as it does on land, what should x be (in kilometers) to minimize the total cost of the project?



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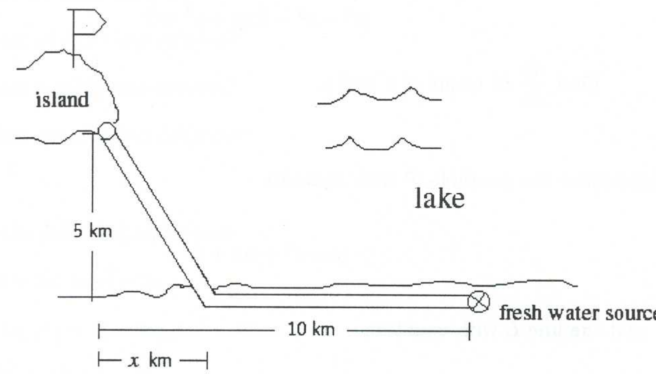

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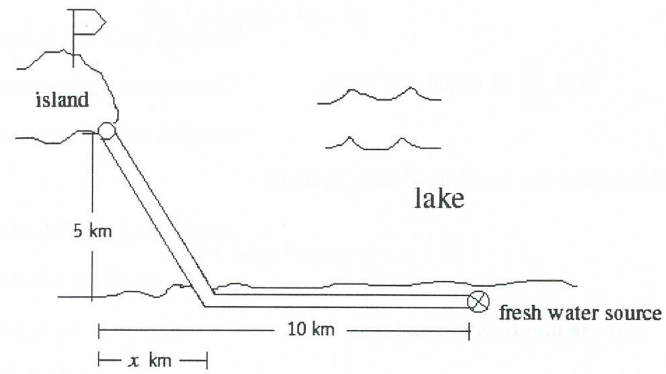

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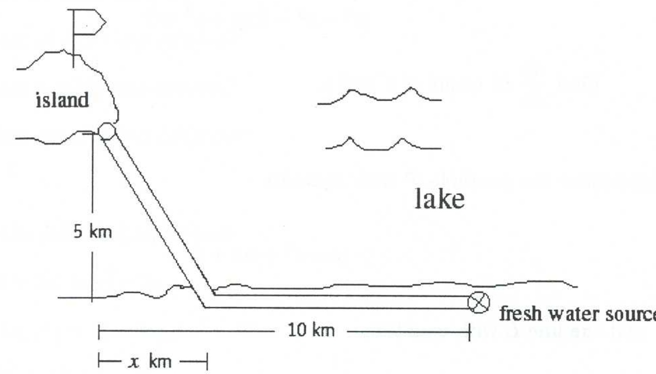

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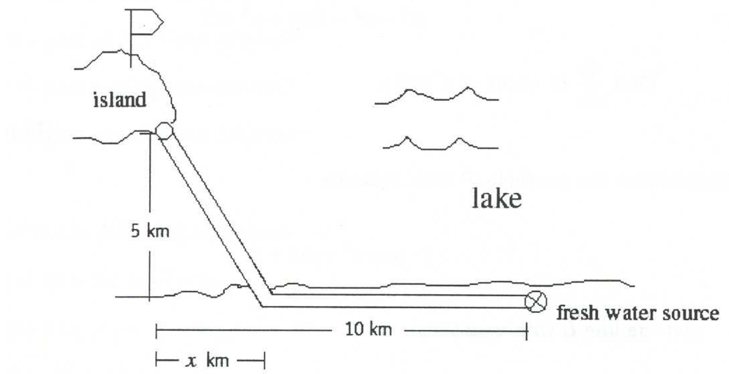

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- Length (in km) of pipe on land is $(10 - x)$ and length in water $\sqrt{x^2 + 5^2}$.
- Therefore cost is

$$C(x) = 1 \times (10 - x) + 1.1 \times \sqrt{x^2 + 25}$$

Want to *minimize* $C(x)$ for $0 \le x \le 10$.

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A ridicule or a compliment?

Solving Linear Equations

Properties of real numbers Let a, b and c be real numbers. Then

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(2) $a = b \Longrightarrow ac = bc$ and $ac = bc \Longrightarrow a = b$ *if* $c \neq 0$.

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Pronunciation http://www.m-w.com/dictionary/iff

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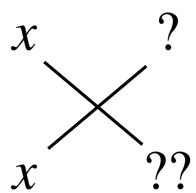

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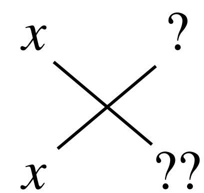

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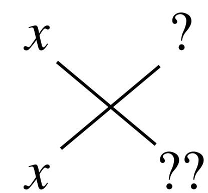

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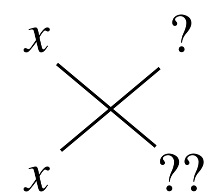

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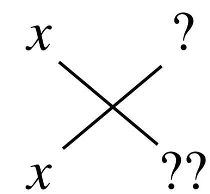

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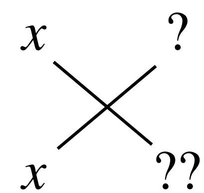

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Warning $(a + b)^2 \neq a^2 + b^2$ *(equality holds if and only if $a = 0$ or $b = 0$)*

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The quadratic equation

$$ax^2 + bx + c = 0 \qquad (1)$$

where a, b, and c are constants and $a \neq 0$, has solutions given by

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

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- ◇ If $b^2 - 4ac < 0$, then (1) has *no (real) solution*.

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Factor Theorem $(x - c)$ *is a factor of a polynomial* $p(x)$ *if and only if* $p(c) = 0$.

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$$\begin{aligned} p(2) &= 2^3 + 3(2^2) - 2(2) + 2 \\ &= 18 \end{aligned}$$

Factor Theorem $(x - c)$ *is a factor of a polynomial $p(x)$ if and only if $p(c) = 0$.*

Proof This follows immediately from the remainder theorem because

$(x - c)$ *is a factor* means that the *remainder is* 0.

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$p(x)$ can't be factorized with integer coefficients.

Theorem *Let*

$$p(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

be a polynomial of degree n where $c_0, c_1, \ldots, c_n$ are integers. Suppose $(ax - b)$ is a factor of $p(x)$ where a, b are integers. Then a divides *c_n and b divides c_0.*

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- Therefore $p(x) = 2\left(x - \frac{3 + \sqrt{17}}{4}\right)\left(x - \frac{3 - \sqrt{17}}{4}\right)$

  where the factor 2 is obtained by comparing the *leading term* *(the x^2 term)*.

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